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Rational Integer Invariants of Regular Cyclic Actions

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Abstract. Let $g\colon M^{2n} \to M^{2n}$ be a smooth map of period $m > 2$ which preserves orientation. Suppose that the cyclic action defined by g is regular and that the normal bundle of the fixed point set F has a g-equivariant complex structure. Let $F \pitchfork F$ be the transverse self-intersection of F with itself. If the g-signature $\operatorname{Sign}(g, M)$ is a rational integer and $n < \phi(m)$, then there exists a choice of orientations such that $\operatorname{Sign}(g, M) = \operatorname{Sign} F = \operatorname{Sign}(F \pitchfork F)$.

1 Introduction

Let M^{2n} be a smooth, closed, oriented $2n$-manifold. Let G_m denote the cyclic group of order m. Let $g\colon M^{2n} \to M^{2n}$ be a diffeomorphism of period m which preserves the preferred orientation of M^{2n}. Suppose that the smooth G_m action defined by g has fixed point set F and that ν is the normal bundle of F in M^{2n}. *We will assume throughout this paper that ν admits a complex structure compatible with the g-action.* This assumption is automatically fulfilled for m odd. *We will also assume that the orientation of ν is the one determined by its complex structure. This orientation, together with the preferred orientation of M^{2n}, determines an orientation of F.*

Let $\operatorname{Sign}(g, M)$ be the g-signature of the action [2]. The g-signature is an algebraic integer, that is $\operatorname{Sign}(g, M) \in \mathbb{Z}[\lambda]$ where $\lambda = \exp(2\pi i/m)$. If $\operatorname{Sign}(g, M)$ is a rational integer, that is $\operatorname{Sign}(g, M) \in \mathbb{Z}$, then it is related to the signatures of F and the transverse self-intersection of F with itself, $F \pitchfork F$, if the action is regular. The action is *regular* if there is a fixed irreducible representation of G_m which determines every normal slice type (Definition 2.4). Let F_{even} (F_{odd}) be the union of all components of F where the restriction of ν has even (odd) complex dimension. Let $\phi(m)$ be the number of integers smaller than m and relatively prime to m.

Theorem 1 *Suppose that $g\colon M^{2n} \to M^{2n}$ is an orientation preserving diffeomorphism of period $m > 2$. If the G_m action defined by g is regular and* $\operatorname{Sign}(g, M) \in \mathbb{Z}$ *and $n < \phi(m)$, then*

$$\operatorname{Sign}(g, M) = \operatorname{Sign} F_{\text{even}} = \operatorname{Sign}(F \pitchfork F)$$

and $\operatorname{Sign} F_{\text{odd}} = 0$. *In particular* $\operatorname{Sign}(g, M) = \operatorname{Sign} F$.

Theorem 1 strengthens an earlier result that if $m = p$ an odd prime, $\operatorname{Sign}(g, M) \in \mathbb{Z}$ and $n < p - 1$, then $\operatorname{Sign}(g, M) = \operatorname{Sign} F$ [11, Theorem A]. The assertion in Theorem 1 about $\operatorname{Sign}(F \pitchfork F)$ is new even in the odd primary case. If M^{2n} admits an orientation preserving involution $T\colon M^{2n} \to M^{2n}$, then $\operatorname{Sign}(T, M) = \operatorname{Sign}(F \pitchfork F)$

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[7], [2, Proposition 6.15], [8, p. 27]. Theorem 1 shows that for regular actions with $\mathrm{Sign}(g, M) \in \mathbb{Z}$ and $n < \phi(m)$, $\mathrm{Sign}(g, M)$ behaves like the signature of an involution.

If the intersection form underlying the g-signature is definite, then $\mathrm{Sign}(g, M) \in \mathbb{Z}$ [3, Lemma 3.1]. If g^* is the identity on $H^*(M; \mathbb{Q})$, then $\mathrm{Sign}(g, M) = \mathrm{Sign}\, M$ [1, p. 329], [3, Section 1] and so our next result is an immediate consequence of Theorem 1.

Theorem 2 *Suppose that $g\colon M^{2n} \to M^{2n}$ is an orientation preserving diffeomorphism of period $m > 2$. If the G_m action defined by g is regular and g^* is the identity on $H^n(M; \mathbb{Q})$ and $n < \phi(m)$, then*

$$\mathrm{Sign}\, M = \mathrm{Sign}\, F_{\mathrm{even}} = \mathrm{Sign}(F \pitchfork F)$$

and $\mathrm{Sign}\, F_{\mathrm{odd}} = 0$. *In particular* $\mathrm{Sign}\, M = \mathrm{Sign}\, F$.

Theorem 2 is related to results in the literature. If m is odd and M^{2n} admits any G_m action such that g^* is the identity on $H^n(M; \mathbb{Q})$, then $\mathrm{Sign}\, M \equiv \mathrm{Sign}\, F \pmod 4$ and if the action is regular, then $\mathrm{Sign}\, M \equiv \mathrm{Sign}\, F \pmod{2^{\phi(m)}}$ [1, Theorems 1 and 4]. If p is an odd prime and M^{2n} admits a regular G_p action and $n < p - 1$, then $\mathrm{Sign}\, M \equiv \mathrm{Sign}\, F \pmod p$ [9, Theorem 2.2]. It follows from these last two results that if M^{2n} admits a regular G_p action with g^* the identity on $H^n(M; \mathbb{Q})$ and $n < p - 1$, then $\mathrm{Sign}\, M \equiv \mathrm{Sign}\, F \pmod{2^{p-1}p}$. *Theorem 2 shows that this congruence is an equality.*

Our next theorem is a consequence of Theorems 1 and 2 and properties of the transverse self-intersection.

Theorem 3 *Suppose that $g\colon M^{2n} \to M^{2n}$ is an orientation preserving diffeomorphism of period $m > 2$ and that the G_m action defined by g is regular. If* $\mathrm{Sign}(g, M) \in \mathbb{Z}$ *and* $\mathrm{Sign}(g, M) \neq 0$ *and $n < \phi(m)$, then n is even and F contains a nonempty component of dimension at least n. If g^* is the identity on $H^*(M; \mathbb{Q})$ and* $\mathrm{Sign}\, M \neq 0$ *and $n < \phi(m)$, then n is even and F contains a nonempty component of dimension at least n.*

If p is an odd prime and M^{2n} admits a regular G_p action $\mathrm{Sign}\, M \not\equiv 0 \pmod p$ and $n < p - 1$, then F contains a nonempty component of dimension at least n [9, Corollary 2.7]. *Theorem 3 shows that if g^* is the identity on $H^n(M; \mathbb{Q})$, then* $\mathrm{Sign}\, M \neq 0$ *is enough to imply that F contains a nonempty component of dimension at least n if $n < p-1$.* If $I_{2n}(p)$ is the subgroup of Ω_{2n} consisting of classes all of whose Pontrjagin numbers are divisible by p and $[M] \neq 0$ in $\Omega_{2n}/I_{2n}(p)$, then F contains a nonempty component of dimension at least n [12, Theorem 1.3].

We offer a congruence for $\mathrm{Sign}(g, M)$ and $\mathrm{Sign}\, F$ for regular G_m actions, $m > 2$, and a congruence for $\mathrm{Sign}(g, M)$ and $\mathrm{Sign}(F \pitchfork F)$ for some values of m. The former congruence contains the congruence for m odd described above [1, Theorem 4]. Let $\rho(m) = \phi(m) - 1$ if $m = 2^e$ and $\rho(m) = \phi(m)$ if $m \neq 2^e$.

Theorem 4 *Suppose that $g\colon M^{2n} \to M^{2n}$ is an orientation preserving diffeomorphism of period $m > 2$. If the G_m action defined by g is regular and* $\mathrm{Sign}(g, M) \in \mathbb{Z}$, *then* $\mathrm{Sign}(g, M) \equiv \mathrm{Sign}\, F_{\mathrm{even}} \pmod{2^{\rho(m)}}$ *and* $\mathrm{Sign}\, F_{\mathrm{odd}} \equiv 0 \pmod{2^{\rho(m)}}$ *and so*

$$\mathrm{Sign}(g, M) \equiv \mathrm{Sign}\, F \pmod{2^{\rho(m)}}.$$

If p is an odd prime and $m = 2p^e$, then $\operatorname{Sign}(g, M) \equiv \operatorname{Sign}(F \pitchfork F) \pmod{p}$.

We will apply our results to cohomology complex projective n-space. We say that M^{2n} is a *cohomology complex projective n-space* if there is a class $x \in H^2(M; \mathbb{Z})$ such that $H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$. These manifolds are good candidates for applications since, for m odd, g^* is the identity on $H^n(M; \mathbb{Q})$ and so $\operatorname{Sign}(g, M) = \operatorname{Sign} M$. We say that a submanifold $i\colon K^{2n-2t} \subset M^{2n}$ has *degree d* if $i_*[K] \in H_{2n-2t}(M; \mathbb{Z})$ is dual to dx^t.

Theorem 5 *Suppose that M^{4q} is a cohomology complex projective $2q$-space and that p is an odd prime. If M^{4q} admits a regular G_p action and $2q < p - 1$, then F contains a nonempty connected $2r$-manifold such that $r \geq q$ and* $\operatorname{Sign}(F^{2r} \pitchfork F^{2r}) = 1$. *If d is the degree of F^{2r} and $r = 2q - 1$, then d^2 is an odd divisor of $(2q)!$ and if $r = q$, then $d^2 = 1$.*

Theorem 6 *Suppose that M^{4q} is a cohomology complex projective $2q$-space, $q = 1$ or 2, and that $p > 5$ is a prime. If M^{4q} admits a regular G_p action, then F has two components. If $q = 1$, then F is the union of a point and a 2-sphere of degree ± 1. If $q = 2$, then F is either the union of a point and a 6-manifold of degree ± 1 or the union of a 4-manifold of degree ± 1 and a 2-sphere.*

This paper is organized as follows. Section 2 contains a discussion of the Atiyah-Singer g-Signature Formula (ASgSF) as formulated by Berend and Katz [3, Theorem 2.2]. This version of the ASgSF expresses $\operatorname{Sign}(g, M)$ explicitly as an element in $\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_{m-1}]$, $\alpha_j = (\lambda^i + 1)(\lambda^j - 1)^{-1}$. The ASgSF for regular actions is also discussed. Section 3 describes the minimal polynomial of α_j over $\mathbb{Q}$. We prove Theorems 1, 2, 3, and 4 in Section 4 (Theorems 4.3, 4.12 and 4.19) and Theorems 5 and 6 in Section 5 (Theorem 5.1, Corollary 5.4 and Theorem 5.10).

2 The Atiyah-Singer g-Signature Formula

Suppose that M^{2n} admits an arbitrary G_m action generated by an orientation preserving diffeomorphism $g\colon M^{2n} \to M^{2n}$. We are not assuming regularity at this point and $m \geq 2$. Let ν be the normal bundle of F in M^{2n}. Over each connected component of F, ν splits into a sum of λ^j-eigen bundles ν_j where G_m acts on ν_j as multiplication by λ^j, $\lambda = \exp(2\pi i/m)$. Each component of F has a *normal slice type* $\mu = (\mu_1, \mu_2, \ldots, \mu_{m-1})$, $\mu_j = \dim_{\mathbb{C}} \nu_j$. Let F_μ be the union of all components of F with slice type μ and ν_μ the normal bundle of F_μ in M^{2n}. Note that $\dim_{\mathbb{C}} \nu_\mu = \sum_{j=1}^{m-1} \mu_j$.

Let $\mathbb{Z}_+$ be the set of nonnegative integers. If q is a positive integer, let $S(q)$ be the symmetric group on q letters and put $S(\mu) = \prod_{j=1}^{m-1} S(\mu_j)$. Let $\Omega(\mu) = \tilde{\Omega}(\mu)/S(\mu)$ where $\tilde{\Omega}(\mu) = \prod_{j=1}^{m-1} \mathbb{Z}_+^{\mu_j}$. If $\omega \in \Omega(\mu)$, let $\|\omega\|_j$ be the sum of the entries in ω from $\mathbb{Z}_+^{\mu_j}$ and $|\omega|_j$ the number of these entries which are not zero. Put $\|\omega\| = \sum_{j=1}^{m-1} \|\omega\|_j$. Let $\alpha_j = (\lambda^j + 1)(\lambda^j - 1)^{-1}$, $1 \leq j \leq m - 1$ and $\lambda = \exp(2\pi i/m)$.

Theorem 2.1 **(Berend-Katz** ASgSF**, [3, Theorem 2.2])** *Let M^{2n} be a smooth, closed, oriented $2n$-manifold and $g\colon M^{2n} \to M^{2n}$ an orientation preserving diffeomorphism of* *period $m \geq 2$. There exist rational integers $S_\omega(\nu_\mu) \in \mathbb{Z}$ for each normal slice type μ and $\omega \in \Omega(\mu)$ such that (2.2)*

$$\operatorname{Sign}(g, M) = \sum_{\mu} \sum_{\omega \in \Omega(\mu)} (-1)^{\|\omega\|} \Big(\prod_j \alpha_j^{\mu_j + \|\omega\|_j - 2|\omega|_j} (\alpha_j^2 - 1)^{|\omega|_j} \Big) S_\omega(\nu_\mu).$$

The rational integers $S_\omega(\nu_\mu)$ can be described as follows. Let $x_{j,\ell} \in H^2(F_\mu;\mathbb{Z})$, $1 \leq \ell \leq \mu_j$, $1 \leq j \leq m-1$, be classes such that the Chern classes of ν_j are the elementary symmetric polynomials in the variables $x_{j,\ell}$, $1 \leq \ell \leq \mu_j$ and let $Y_{j,\ell} \subset F_\mu$ be the Poincaré dual of $x_{j,\ell}$. If $\tilde{\omega} = (\omega_{j,\ell}) \in \tilde{\Omega}(\mu)$, put $Y^{(\tilde{\omega})} = \pitchfork_{j,\ell} Y_{j,\ell}^{(\omega_{j,\ell})}$, where $Y_{j,\ell}^{(\omega_{j,\ell})}$ is the transverse self-intersection of $\omega_{j,\ell}$ copies of $Y_{j,\ell}$ with itself. If $\omega \in \Omega(\mu)$ is covered by $\tilde{\omega}$, then

$$S_\omega(\nu_\mu) = \sum_{\sigma \in S(\mu)} |\operatorname{St}_{\tilde{\omega}}|^{-1} \operatorname{Sign} Y^{(\sigma\tilde{\omega})}, \tag{2.3}$$

where $|\operatorname{St}_{\tilde{\omega}}|$ is the order of the stabilizer of $\tilde{\omega}$ [3, Section 3].

Definition 2.4 A G_m action on M^{2n} is *regular* if there exists a j_0, $1 \leq j_0 \leq m-1$, such that j_0 is relatively prime to m and for *every* normal slice type $\mu = (\mu_1, \mu_2, \ldots, \mu_{m-1})$, $\mu_j = 0$ if $j \neq j_0$.

If $\mu = (\mu_1, \mu_2, \ldots, \mu_{m-1})$ and $\mu_j = 0$ if $j \neq j_0$, then $\mu_{j_0} = \dim_{\mathbb{C}} \nu_\mu$. The case $\mu_{j_0} = 0$ corresponds to an action which is trivial on at least one component of M^{2n}. If a regular action has s slice types, each can be identified with a complex codimension c_i, $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$, where F^{2n-2c} is the union of all components of F of dimension $2n-2c$. Let ν_c be the normal bundle of F^{2n-2c} in M^{2n} and note that if a slice type of a regular action μ is such that $c = \dim_{\mathbb{C}} \nu_\mu$ and $c \neq 0$, then $\Omega(\mu) = \mathbb{Z}_+^c/S(c)$ equipped with the norms $\|\cdot\|$ and $|\cdot|$.

Definition 2.5 If $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ is the fixed point set of a regular G_m action, then for each nonzero $c \in \{c_1, c_2, \ldots, c_s\}$ and integers j, k with $1 \leq j \leq c$ and $j \leq k \leq n-c$, let $s(c, j, k) = \{\omega \in \mathbb{Z}_+^c/S(c) : |\omega| = j, \|\omega\| = k\}$ and

$$S_c(j,k)(\nu_c) = \sum_{\omega \in s(c,j,k)} S_\omega(\nu_c). \tag{2.6}$$

Definition 2.7 If $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ is the fixed point set of a regular G_m action and $c \in \{c_1, c_2, \ldots, c_s\}$, then the polynomial $p_c(x) \in \mathbb{Z}[x]$ is defined by the conditions $p_0(x) = 0$ and if $c \neq 0$, then

$$p_c(x) = \sum_{j=1}^{c} \sum_{k=1}^{n-c} (-1)^k x^{c+k-2j} (x^2-1)^{j-1} S_c(j,k)(\nu_c). \tag{2.8}$$

The polynomials $p_c(x)$ play a role in the ASgSF for regular G_m actions. Our next proposition determines an upper bound on the degree of $p_c(x)$ and $p_c(0)$.

Proposition 2.9 *If $c \in \{c_1, c_2, \ldots, c_s\}$ and $c \neq 0$, then the degree of $p_c(x)$ is at most $n-2$ and*

$$p_c(0) = -\operatorname{Sign}(F^{2n-2c} \pitchfork F^{2n-2c}). \tag{2.10}$$

Proof The remark about the degree of $p_c(x)$ follows immediately from (2.8). Formula (2.10) follows by observing that (2.8) implies that

$$p_c(0) = -S_c(c,c)(\nu_c), \tag{2.11}$$

and then noting that (2.6) implies that

$$S_c(c,c)(\nu_c) = S_{[(1,1,\ldots,1)]}(\nu_c), \tag{2.12}$$

where $[(1,1,\ldots,1)] \in \mathbb{Z}_+^c/S(c)$ is the equivalence class of $(1,1,\ldots,1) \in \mathbb{Z}_+^c$. Formula (2.10) now follows from (2.11), (2.12) and Lemma 2.4 in [3]. ■

Definition 2.13 If $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ is the fixed point set of a regular G_m action, then the polynomial $p(x) \in \mathbb{Z}[x]$ is defined by

$$p(x) = \sum_{i=1}^{s} p_{c_i}(x). \tag{2.14}$$

Definition 2.15 If $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ is the fixed point set of a G_m action, then the polynomial $s(x) \in \mathbb{Z}[x]$ is defined by

$$s(x) = \sum_{i=1}^{s} \operatorname{Sign} F^{2n-2c_i} x^{c_i}. \tag{2.16}$$

Theorem 2.17 **(Berend-Katz** ASgSF **for Regular** G_m **Actions)** *Suppose that $g\colon M^{2n} \to M^{2n}$ is an orientation preserving diffeomorphism of period $m \geq 2$. If the G_m action defined by g is regular and $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$, then there exists an $\alpha \in \{\alpha_j : 1 \leq j \leq m-1, (j,m) = 1\}$ such that*

$$\operatorname{Sign}(g, M) = s(\alpha) + (\alpha^2 - 1)p(\alpha). \tag{2.18}$$

Proof There exists a j_0 such that $(j_0, m) = 1$ and $\mu_j = 0$, $j \neq j_0$, for every slice type $\mu = (\mu_1, \mu_2, \ldots, \mu_{m-1})$. It follows that if $c \in \{c_1, c_2, \ldots, c_s\}$, then $\mu_{j_0} = c$ and if $c \neq 0$, $\Omega(\mu) = \mathbb{Z}_+^c/S(c)$ with norms $\|\cdot\|$ and $|\cdot|$. If $\alpha = \alpha_{j_0}$, then it follows from (2.2) that

$$\operatorname{Sign}(g, M) = \operatorname{Sign} F^{2n} + \sum_{c \neq 0} \sum_{\omega \in \mathbb{Z}_+^c/S(c)} (-1)^{\|\omega\|} \alpha^{c+\|\omega\|-2|\omega|} (\alpha^2 - 1)^{|\omega|} S_\omega(\nu_c). \tag{2.19}$$

Formula (2.18) follows by putting $|\omega| = j$, $\|\omega\| = k$, and using (2.8), (2.14) and (2.16) together with $S_{[(0,0,\dots,0)]}(\nu_c) = \text{Sign}\, F^{2n-2c}$ [3, Lemma 2.4]. ∎

Corollary 2.20 **(Hirzebruch** ASgSF **for Involutions** [7]**)** *Suppose that* $T\colon M^{2n} \to M^{2n}$ *is an orientation preserving smooth involution. If* F *is the fixed point set of* T*, then*

$$\text{Sign}(T, M) = \text{Sign}(F \pitchfork F). \tag{2.21}$$

Proof The G_2 action defined by T is automatically regular and so it follows from (2.18) with $\alpha = \alpha_1 = 0$ and (2.10) that

$$\text{Sign}(T, M) = s(0) + \sum_{c \neq 0} \text{Sign}(F^{2n-2c} \pitchfork F^{2n-2c}). \tag{2.22}$$

The right hand side of (2.22) is $\text{Sign}(F \pitchfork F)$ and so (2.21) follows. ∎

Next we offer $p_c(x)$ for a few values of c. To make our results easier to state, we define the symbol $S_c(j,k)$, j and k arbitrary nonnegative integers, to be $S_c(j,k)(\nu_c)$ if $1 \le j \le c$ and $j \le k \le n-c$ and to be zero if j and k are outside of this range.

Lemma 2.23 *If* $n \ge 3$*, then*

$$(-1)^{n-1} p_1(x) = \begin{cases} \sum_{k=1}^{[n/2]} S_1(1, 2k-1) x^{2k-2}, & n \text{ even}, \\ \sum_{k=1}^{[n/2]} S_1(1, 2k) x^{2k-1}, & n \text{ odd}. \end{cases} \tag{2.24}$$

(2.25)

$$(-1)^n p_2(x) = \begin{cases} \displaystyle\sum_{k=1}^{[n/2]} \big(S_2(1, 2k-2) + S_2(2, 2k-2) - S_2(2, 2k)\big) x^{2k-2}, & n \text{ even}, \\ \displaystyle\sum_{k=1}^{[n/2]} \big(S_2(1, 2k-1) + S_2(2, 2k-1) - S_2(2, 2k+1)\big) x^{2k-1}, & n \text{ odd}. \end{cases}$$

$$p_{n-1}(x) = -S_{n-1}(1,1) x^{n-2}. \tag{2.26}$$

$$p_n(x) = 0. \tag{2.27}$$

Proof Formulas (2.24) and (2.25) follow from (2.8), the definition of $S_c(j,k)$, and the fact that $S_\omega(\nu_c)$ in (2.6) is zero unless $n - c - \|\omega\| = n - c - k$ is even and $0 \le k \le n - c$ ([11], Section 2). Formulas (2.26) and (2.27) follow from (2.8). ∎

3 The Algebraic Numbers α_j

This section contains some well known facts about the numbers $\alpha_j = (\lambda^j + 1) \times (\lambda^j - 1)^{-1}$, $\lambda = \exp(2\pi i/m)$.

Definition 3.1 If $m \geq 2$ and $\phi(m) = |\{j : j < m, (j, m) = 1\}|$ and $\Phi_m(x)$ is the m-th cyclotomic polynomial, then the polynomial $M_m(x) \in \mathbb{Z}[x]$ is defined by

$$M_m(x) = (x-1)^{\phi(m)} \Phi_m\big((x+1)(x-1)^{-1}\big). \tag{3.2}$$

Proposition 3.3 *If $m \geq 2$ and $(j, m) = 1$, then $M_m(x)$ is the minimal polynomial of α_j over $\mathbb{Q}$. The polynomials $M_m(x)$ satisfy the equations below where p is an odd prime.*

$$M_m(0) = \begin{cases} 0, & m = 2, \\ 2, & m = 2^e,\ e > 1, \\ p, & m = 2p^e,\ e \geq 1, \\ 1, & \textit{otherwise}. \end{cases} \tag{3.4}$$

$$M_m(\pm 1) = (\pm 1)^{\phi(m)} 2^{\phi(m)}. \tag{3.5}$$

Proof It follows from the identity $\lambda^j = (\alpha_j + 1)(\alpha_j - 1)^{-1}$ that $M_m(\alpha_j) = 0$. The fact that $\Phi_m(x)$ is irreducible over $\mathbb{Q}$ of degree $\phi(m)$ together with (3.2) implies that $M_m(x)$ is irreducible over $\mathbb{Q}$ of degree $\phi(m)$. Formula (3.4) follows from the values of $\Phi_m(-1)$ [3, Lemma 4.1] and (3.5) follows from the facts that the degree of $\Phi_m(x)$ is $\phi(m)$ and $\Phi_m(0) = 1$. ■

It follows immediately from (3.4) that $M_m(x)$ is primitive if $m \neq 2^e$ and $m \neq 2p^e$. We will investigate the cases $m = 2^e$ and $m = 2p^e$ after a definition and an elementary proposition stated without proof.

Definition 3.6 If t is a nonnegative integer, then

$$f_t^{\pm}(x) = \frac{1}{2}[(x+1)^t \pm (x-1)^t]. \tag{3.7}$$

Proposition 3.8 *If t is a nonnegative integer, then $f_t^+(x)$ is primitive. If t is odd, then $f_t^-(x)$ is primitive. The polynomials $f_t^{\pm}(x)$ satisfy the equations below.*

$$f_t^+(x) = \sum_{k=0}^{[t/2]} \binom{t}{2k} x^{t-2k}. \tag{3.9}$$

$$f_t^-(x) = \sum_{k=0}^{[t/2]} \binom{t}{2k+1} x^{t-2k-1}. \tag{3.10}$$

Proposition 3.11 *If p is an odd prime and $e \geq 1$, then*

$$M_m(x) = \begin{cases} f_{p^e}^-(x) f_{p^{e-1}}^-(x)^{-1}, & m = p^e, \\ 2 f_{2^e - 2^{e-1}}^+(x), & m = 2^e, \\ f_{p^e}^+(x) f_{p^{e-1}}^+(x)^{-1}, & m = 2p^e, \end{cases} \tag{3.12}$$

Proof The first formula in (3.12) follows from (3.2), (3.7) and the fact that $\Phi_{p^e}(x) = (x^{p^e} - 1)(x^{p^{e-1}} - 1)^{-1}$. The second formula follows from (3.2), (3.7) and the fact that $\Phi_{2^e}(x) = x^{2^e - 2^{e-1}} + 1$. The last formula follows from (3.2), (3.7) and the fact that $\Phi_{2p^e}(x) = (x^{p^e} + 1)(x^{p^{e-1}} + 1)^{-1}$. ∎

Proposition 3.13 *If m is not a power of 2, then $M_m(x)$ is primitive. If $e \geq 1$, then $2^{-1}M_{2^e}(x)$ is primitive.*

Proof If $m \neq 2^e$ and $m \neq 2p^e$, p an odd prime, then the proposition follows from (3.4). If $m = 2^e$ or $m = 2p^e$, then the proposition follows from Proposition 3.8 and (3.12). ∎

4 Regular Actions with Rational Integer g-Signature

The purpose of this section is to prove Theorems 1, 2, 3 and 4. Throughout this section, we will assume that M^{2n} admits an orientation preserving diffeomorphism $g\colon M^{2n} \to M^{2n}$ of period $m \geq 2$. We begin with (2.18) when $\mathrm{Sign}(g, M) \in \mathbb{Z}$.

Proposition 4.1 *If the G_m action defined by g is regular and $\mathrm{Sign}(g, M) \in \mathbb{Z}$, then there exists a polynomial with rational integer coefficients $a(x) \in \mathbb{Z}[x]$ such that the degree of $a(x)$ is at most n and $a(\alpha) = 0$ for some $\alpha \in \{\alpha_j : 1 \leq j \leq m-1, (j, m) = 1\}$.*

Proof If $p(x)$ and $s(x)$ are as in (2.14) and (2.16), put

$$a(x) = s(x) + (x^2 - 1)p(x) - \mathrm{Sign}(g, M). \tag{4.2}$$

If $\mathrm{Sign}(g, M) \in \mathbb{Z}$, then $a(x) \in \mathbb{Z}[x]$ since $p(x)$ and $s(x)$ are in $\mathbb{Z}[x]$. The degree of $s(x)$ is clearly at most n (2.16) and the degree of $p(x)$ is at most $n-2$ (Proposition 2.9 and (2.14)) and so the degree of $a(x)$ is at most n. The fact that there exists $\alpha \in \{\alpha_j : 1 \leq j \leq m-1, (j, m) = 1\}$ such that $a(x) = 0$ is (2.18). ∎

Proposition 4.1 and Section 3 will provide the tools to prove Theorems 1, 2, 3 and 4. We begin with Theorem 4. Recall that $\rho(m) = \phi(m) - 1$ if $m = 2^e$ and $\rho(m) = \phi(m)$ if $m \neq 2^e$.

Theorem 4.3 *Suppose that $m > 2$ and that the G_m action defined by g is regular. If $\mathrm{Sign}(g, M) \in \mathbb{Z}$, then $\mathrm{Sign}(g, M) \equiv \mathrm{Sign}\, F_{\mathrm{even}} \pmod{2^{\rho(m)}}$ and $\mathrm{Sign}\, F_{\mathrm{odd}} \equiv 0 \pmod{2^{\rho(m)}}$ and so*

$$\mathrm{Sign}(g, M) \equiv \mathrm{Sign}\, F \pmod{2^{\rho(m)}}. \tag{4.4}$$

If g^* is the identity on $H^n(M; \mathbb{Q})$, then $\mathrm{Sign}\, M \equiv \mathrm{Sign}\, F_{\mathrm{even}} \pmod{2^{\rho(m)}}$ and $\mathrm{Sign}\, F_{\mathrm{odd}} \equiv 0 \pmod{2^{\rho(m)}}$ and so

$$\mathrm{Sign}\, M \equiv \mathrm{Sign}\, F \pmod{2^{\rho(m)}}. \tag{4.5}$$

If $\operatorname{Sign}(g, M) \in \mathbb{Z}$ *and* $m = 2p^e$*,* p *an odd prime, then*

$$\operatorname{Sign}(g, M) \equiv \operatorname{Sign}(F \pitchfork F) \pmod{p}. \tag{4.6}$$

If g^* *is the identity on* $H^n(M; \mathbb{Q})$ *and* $m = 2p^e$*,* p *an odd prime, then*

$$\operatorname{Sign} M \equiv \operatorname{Sign}(F \pitchfork F) \pmod{p}. \tag{4.7}$$

Proof Note that if $m > 2$, then $M_m(x) \in \mathbb{Z}[x^2]$. This follows from the fact that the set $\{\alpha_j : 1 \le j \le m-1, (j, m) = 1\}$ is a complete set of roots of $M_m(x)$ and $\alpha_j = -\alpha_{m-j}$. If $a(x)$ is a polynomial in x, let $a(x)_{\text{even}}$ and $a(x)_{\text{odd}}$ be the parts of $a(x)$ with even and odd powers of x, respectively. To prove (4.4), note that if $a(x) \in \mathbb{Z}[x]$ is the polynomial in (4.2), then it follows from Proposition 3.3 that there exists $b(x) \in \mathbb{Q}[x]$ such that

$$a(x)_{\text{even}} = M_m(x) b(x)_{\text{even}}, \quad a(x)_{\text{odd}} = M_m(x) b(x)_{\text{odd}}. \tag{4.8}$$

If $m \neq 2^e$, then $M_m(x)$ is primitive by Proposition 3.13, so $b(x) \in \mathbb{Z}[x]$ since $a(x) \in \mathbb{Z}[x]$ and therefore (3.5) and (4.8) imply that

$$a(1)_{\text{even}} \equiv 0 \pmod{2^{\phi(m)}}, \quad a(1)_{\text{odd}} \equiv 0 \pmod{2^{\phi(m)}}. \tag{4.9}$$

If $m \neq 2^e$, formula (4.9) implies that $\operatorname{Sign}(g, M) \equiv \operatorname{Sign} F_{\text{even}} \pmod{2^{\phi(m)}}$ and $\operatorname{Sign} F_{\text{odd}} \equiv 0 \pmod{2^{\phi(m)}}$. If $m = 2^e$, then $2^{-1} M_m(x)$ is primitive by Proposition 3.13 and so $2b(x) \in \mathbb{Z}[x]$ since $a(x) \in \mathbb{Z}[x]$ and therefore (3.5) and (4.8) imply that

$$a(1)_{\text{even}} \equiv 0 \pmod{2^{\phi(m)-1}}, \quad a(1)_{\text{odd}} \equiv 0 \pmod{2^{\phi(m)-1}}. \tag{4.10}$$

Formula (4.10) implies that $\operatorname{Sign}(g, M) \equiv \operatorname{Sign} F_{\text{even}} \pmod{2^{\phi(m)-1}}$ and $\operatorname{Sign} F_{\text{odd}} \equiv 0 \pmod{2^{\phi(m)-1}}$ if $m = 2^e$ and so the proof of the first two assertions in Theorem 4.3 is complete.

The next two assertions in Theorem 4.3 follow since $\operatorname{Sign}(g, M) = \operatorname{Sign} M$ if g^* is the identity on $H^n(M; \mathbb{Q})$.

To prove (4.6), note that if $\operatorname{Sign}(g, M) \in \mathbb{Z}$ and $m = 2p^e$, p an odd prime, then $M_m(x)$ is primitive by Proposition 3.13 and so $b(x) \in \mathbb{Z}[x]$, and so (3.4) and (4.8) imply that

$$a(0) \equiv 0 \pmod{p}. \tag{4.11}$$

Formula (4.11) is (4.6) in view of (2.10). Formula (4.7) follows immediately from (4.6) since $\operatorname{Sign}(g, M) = \operatorname{Sign} M$ if g^* is the identity on $H^n(M; \mathbb{Q})$. ■

Theorem 4.1 contains Theorem 4 in the introduction. We now turn to Theorems 1 and 2. We will observe that (4.9) and (4.11) are equalities if $n < \phi(m)$.

Theorem 4.12 *Suppose that* $m > 2$ *and that the* G_m *action defined by* g *is regular. If* $\operatorname{Sign}(g, M) \in \mathbb{Z}$ *and* $n < \phi(m)$*, then* $\operatorname{Sign} F_{\text{odd}} = 0$ *and*

$$\operatorname{Sign}(g, M) = \operatorname{Sign} F_{\text{even}} = \operatorname{Sign}(F \pitchfork F). \tag{4.13}$$

If g^* is the identity on $H^n(M;\mathbb{Q})$ and $n < \phi(m)$, then $\operatorname{Sign} F_{\text{odd}} = 0$ *and*

$$\operatorname{Sign} M = \operatorname{Sign} F_{\text{even}} = \operatorname{Sign}(F \pitchfork F). \tag{4.14}$$

Proof Note that (3.2) implies that the degree of $M_m(x)$ is $\phi(m)$ and so if $\operatorname{Sign}(g, M) \in \mathbb{Z}$ and $a(x) \in \mathbb{Z}[x]$ is as in (4.2), then Proposition 3.3 implies that if $n < \phi(m)$, then because the degree of $a(x)$ is at most n, $a(x)$ is *identically zero*,

$$a(x) \equiv 0. \tag{4.15}$$

It follows that $\operatorname{Sign} F_{\text{odd}} = a(1)_{\text{odd}} = 0$ and that the first equality in (4.13) holds since $a(1)_{\text{even}} = 0$. The second equality in (4.13) follows by putting $x = 0$ in (4.15) and (2.10), (2.14), (2.16) and (4.2). Formula (4.14) follows immediately from (4.13) because $\operatorname{Sign}(g, M) = \operatorname{Sign} M$ if g^* is the identity on $H^n(M;\mathbb{Q})$. ■

Theorem 4.12 contains Theorems 1 and 2. Theorem 2 was stated separately to highlight its relationship to the literature [1, Theorem 4], [9, Theorem 2.2]. Our next task is to note that (4.15) implies that $p(x)$ (2.14) has a special form if the hypotheses of Theorem 4.12 are enforced.

Proposition 4.16 *Suppose that $m > 2$ and that the G_m action defined by g is regular. If* $\operatorname{Sign}(g, M) \in \mathbb{Z}$*, $n < \phi(m)$, and p_k is the coefficient of x^k in $p(x)$ (2.14), $0 \le k \le n-2$, then*

$$p_k = \begin{cases} \sum_{j=0}^{\ell} \operatorname{Sign} F^{2n-4j} - \operatorname{Sign}(g, M), & k = 2\ell, \\ \sum_{j=0}^{\ell} \operatorname{Sign} F^{2n-4j-2}, & k = 2\ell + 1. \end{cases} \tag{4.17}$$

Proof The hypotheses guarantee the identity (4.15) and so (4.2) implies that for $0 \le k \le n-2$, the derivatives of $s(x)$ and $p(x)$ satisfy

$$s^{(k)}(x) + k(k-1)p^{(k-2)}(x) + 2kxp^{(k-1)}(x) + (x^2 - 1)p^{(k)}(x) = 0. \tag{4.18}$$

The derivatives of $s(x)$ are easily determined (2.16) and (4.17) then follows easily from (4.18). ■

We now prove Theorem 3. Let $\dim F$ denote the largest dimension of the components of F.

Theorem 4.19 *Suppose that $m > 2$ and that the G_m action defined by g is regular. If* $\operatorname{Sign}(g, M) \in \mathbb{Z}$*, $n < \phi(m)$, and* $\dim F < n$*, then* $\operatorname{Sign}(g, M) = \operatorname{Sign} F = \operatorname{Sign}(F \pitchfork F) = 0$*. If* $\operatorname{Sign}(g, M) \in \mathbb{Z} - \{0\}$ *and $n < \phi(m)$, then n is even and* $\dim F \ge n$*. If g^* is the identity on $H^n(M;\mathbb{Q})$, $n < \phi(m)$, and* $\dim F < n$*, then* $\operatorname{Sign} M = \operatorname{Sign} F = \operatorname{Sign}(F \pitchfork F) = 0$*. If g^* is the identity on $H^n(M;\mathbb{Q})$,* $\operatorname{Sign} M \ne 0$*, and $n < \phi(m)$, then n is even and* $\dim F \ge n$*.*

Proof If $\dim F < n$, then $\operatorname{Sign}(F \pitchfork F) = 0$, and so the first assertion in the theorem follows from (4.13). The second assertion follows from the first and the observation that n must be even because $\operatorname{Sign}(g, M) = \operatorname{Sign} F_{\text{even}} \ne 0$. The third and fourth statements follow because $\operatorname{Sign}(g, M) = \operatorname{Sign} M$ if g^* is the identity on $H^n(M;\mathbb{Q})$. ■

5 Regular G_p Actions on Cohomology Complex Projective Space

In this section, we apply our results to G_p action on cohomology complex projective n-space.

Theorem 5.1 *Suppose that M^{4q} is a cohomology complex projective $2q$-space admitting a diffeomorphism $g\colon M^{4q} \to M^{4q}$ of odd prime period p. If the G_p action defined by g is regular and $2q < p-1$, then F contains a nonempty connected $2r$-manifold F^{2r} such that $r \geq q$. All other components of F have dimension less than $2q$ and $\mathrm{Sign}(F^{2r} \pitchfork F^{2r}) = 1$. If d is the degree of F^{2r} and $r = 2q-1$, then d^2 is an odd divisor of $(2q)!$ and if $r = q$, then $d^2 = 1$.*

Proof We choose as preferred orientation of M^{4q} the one such that $\mathrm{Sign}\, M = 1$. It follows from (4.14) that

$$\mathrm{Sign}(F \pitchfork F) = 1 \tag{5.2}$$

and so Theorem 4.19 implies that $\dim F \geq 2q$ and so F contains a connected manifold F^{2r} with $r \geq q$. Since M^{4q} is a cohomology complex projective $2q$-space, all other components of F have dimension strictly less than $2q$ [4, p. 378] and so (5.2) becomes

$$\mathrm{Sign}(F^{2r} \pitchfork F^{2r}) = 1. \tag{5.3}$$

If d is the degree of F^{2r} and $r = 2q-1$, then (5.3) implies that d^2 is an odd divisor of $(2q)!$ [10, Theorem 1.1] and if $r = q$, then $F^{2q} \pitchfork F^{2q}$ is the union of d^2 points with a common orientation and so (5.3) implies that $d^2 = 1$. ■

Corollary 5.4 *Suppose that M^4 is a cohomology complex projective 2-space and that $p > 3$ is a prime. If M^4 admits a regular G_p action, then F is the union of a 2-sphere of degree ± 1 and a point.*

Proof It follows from Theorem 5.1 that F contains a 2-sphere of degree ± 1, S^2 and so F is the union of S^2 and a point [4, p. 378]. ■

Theorem 5.1 is Theorem 5 and Corollary 5.4 is the assertion in Theorem 6 about cohomology complex projective 2-space. Theorem 6 will be proved when we establish the assertions about cohomology complex projective 4-space.

Lemma 5.5 *Suppose that M^{4q} is a cohomology projective $2q$-space and that p is an odd prime. If M^{4q} admits a regular G_p action and $2q < p-1$, then F has at most $q+1$ components.*

Proof We know that $F = \bigcup_{i=1}^{s} F^{2n_i}$, F^{2n_i} connected, $s \leq p$ and $\sum_{i=1}^{s}(n_i+1) = 2q+1$ [4, p. 378]. By Theorem 5.1 there is an i_0 such that $n_{i_0} \geq q$ and so $s \leq q+1$. ■

Lemma 5.6 *Suppose that M^8 is a cohomology complex projective 4-space and that $p > 5$ is a prime. If M^8 admits a regular G_p action, then F has two components.*

Proof By Lemma 5.5, it is enough to show that F can not have three components. If F has three components, then F is the union of a 4-manifold F^4 and two points [4, p. 378]. It follows from (2.14), (2.25), and (2.27) together with the fact that $p > 5$ and (4.17), that

$$S_2(2,2) = 1, \tag{5.7}$$

$$S_2(1,2) + S_2(2,2) = \text{Sign } F^4 - 1. \tag{5.8}$$

Since F^4 is a cohomology complex projective 2-space mod p [4, p. 378], Sign $F^4 = \pm 1$, and so (5.7) and (5.8) imply that $S_2(1,2) = -1, -3$. If ν is the normal bundle of F^4 in M^8, that is, $\nu = \nu_2$ in the notation of Section 2, then (2.3) and (2.6) imply that

$$S_2(1,2) = \left(c_1^2(\nu) - 2c_2(\nu)\right)[F^4]. \tag{5.9}$$

If d is the degree of F^4, then $d^2 = 1$ by Theorem 5.1 and so $c_2(\nu)[F^4] = 1$. If follows from (5.9) that $c_1^2(\nu)[F^4] = \pm 1$. This leads to a contradiction.

If $H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/(x^5)$, $x \in H^2(M;\mathbb{Z})$, let $\hat{x} = x|F^4$. If $x_i \in H^2(F^4;\mathbb{Z})$, $i = 1,2$, are classes such that $c_1(\nu) = x_1 + x_2$ and $c_2(\nu) = x_1 x_2$, then $x_i = a_i \hat{x} \pmod{\text{torsion}}$, $a_i \in \mathbb{Z}$, $i = 1,2$ [5, Lemma 3.1]. Since $c_2(\nu)[F^4] = 1$, $a_i = \pm 1$, $i = 1,2$, and so $c_1^2(\nu)[F^4] \equiv 0 \pmod 2$. This contradicts $c_1^2(\nu)[F^4] = \pm 1$ and so F has two components. ■

Theorem 5.10 *Suppose that M^8 is a cohomology complex projective 4-space and that $p > 5$ is a prime. If M^8 admits a regular G_p action, then F has two components and either F is the union of a 6-manifold of degree ± 1 and a point or F is the union of a 4-manifold of degree ± 1 and a 2-sphere.*

Proof Lemma 5.6 says that F has two components and so either $F = F^6 \cup \{\text{point}\}$ or $F = F^4 \cup S^2$ [4, p. 378]. In either case, if d is the degree of F^{2r}, $r = 2$ or 3, then Theorem 5.1 implies that $d^2 = 1$. ■

Theorem 6 is the sum of Corollary 5.4 and Theorem 5.10. Strengthened versions of parts of Theorem 6 can be found in the literature. Any G_p action on M^8 such that $F = F^6 \cup \{\text{point}\}$ must be regular and, if d is the degree of F^6, then $d^2 = 1$ [5, Theorem 4(ii), $p \geq 5$], [6, Theorem E, $p = 3$]. Theorems 5.1 and 5.10 show that if it is assumed that the action is regular and $p > 5$, then the ASgSF can be used to retrieve the fact that a fixed F^6 has degree one and establish the two results that F has two components and that a fixed F^4 has degree one.

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