ON QUASI-NORMABLE ECHELON SPACES

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In this paper we prove that if the strong dual of an echelon space fulfils the Mackey convergence condition the echelon space is quasi-normable. Also we give a characterisation of the quasi-normable echelon spaces and we deduce that every non-quasi-normable echelon space is the strong dual of a non-complete (LB)-space.

The vector spaces used here are defined on the field K of real or complex numbers. By the word "space" we mean a "locally convex space". Given a dual pair $\langle E, F \rangle$, $\sigma(E, F)$, $\mu(E, F)$ and $\beta(E, F)$ are the weak, Mackey and strong topologies on E, respectively. Given a space H, H' is its topological dual. If B is a bounded absolutely convex subset of H, H_B is the normed space on the linear hull of B normed by the Minkowski functional of B. A sequence (x_n) in H converges to x in the sense of Mackey if there is an absolutely convex and bounded subset of H, A, such that x and x_n belong to H_A, $n = 1, 2, \ldots$, and (x_n) converges to x in H_A. H verifies the Mackey convergence condition if every convergent sequence in H converges in the sense of Mackey. H verifies the strict Mackey convergence condition if for every bounded set M in H there exists a bounded absolutely convex set B in H such that M is bounded in H_B and the topology of H_B coincides with topology of H on M. If H is quasi-barrelled, H is quasi-normable, (**1**, p. 106), if $H'[\beta(H', H)]$ verifies the strict Mackey convergence condition.

In the sequel λ will denote an echelon space and $\lambda^{\times}$ its α-dual. Let us suppose that the echelons which define λ

$$\alpha^{(n)} = (a_1^{(n)}, a_2^{(n)}, \ldots, a_p^{(n)}, \ldots), \quad n = 1, 2, \ldots,$$

are positive, form an increasing sequence of vectors and for each index p there is an index q such that $a_p^{(q)} \neq 0$, (see (**3**)). Let

$$B_n = \{\alpha = (a_1, a_2, \ldots, a_p, \ldots) : |a_p| \leq a_p^{(n)}, p = 1, 2, \ldots\}.$$

We put $\lambda_{\alpha^{(n)}}^{\times}$ for the linear hull of B_n endowed with the closed unit ball B_n. The norm of $\lambda_{\alpha^{(n)}}^{\times}$ is $\|\cdot\|$.

φ is the space of all sequences with only finitely many non-zero coordinates. By $\varphi^{(1)}$ we represent the subspace of $\lambda^{\times}[\beta(\lambda^{\times}, \lambda)]$ of all the limits in the sense of Mackey of sequences in φ.

If P is a subset of the set $\mathbb{N}$ of positive integers, let $\lambda^{\times}(P)$ be the subspace of $\lambda^{\times}[\beta(\lambda^{\times}, \lambda)]$ of the vectors with vanishing coordinates corresponding to indices outside P.

Lemma. *The space $\varphi^{(1)}$ is complete if and only if there is for each $p \in \mathbb{N}$ an integer $q > p$ such that, if P is an arbitrary subset of $\mathbb{N}$ which verifies*

$$a_n^{(p)} \neq 0, \quad n \in P, \quad \inf \left\{ \frac{a_n^{(p)}}{a_n^{(q)}} : n \in P \right\} \neq 0$$

then

$$\inf \left\{ \frac{a_n^{(p)}}{a_n^{(r)}} : n \in P \right\} \neq 0, \quad r = q+1, q+2, \ldots.$$

Proof. Suppose that there is a $p \in \mathbb{N}$ such that, for each $q > p$, we can find an integer $r, r > q$, and a subset P of $\mathbb{N}$, such that

$$a_n^{(p)} \neq 0, \quad n \in P, \quad \inf \left\{ \frac{a_n^{(p)}}{a_n^{(q)}} : n \in P \right\} \neq 0, \quad \inf \left\{ \frac{a_n^{(p)}}{a_n^{(r)}} : n \in P \right\} = 0.$$

We can select in P a sequence $n_1 < n_2 < \ldots < n_s < \ldots$, so that

$$\lim \frac{a_{n_s}^{(p)}}{a_{n_s}^{(q)}} = a_q \neq 0, \tag{1}$$

$$\lim \frac{a_{n_s}^{(p)}}{a_{n_s}^{(r)}} = 0. \tag{2}$$

The vector

$$\gamma = (c_1, c_2, \ldots, c_n, \ldots)$$

of $\lambda^\times$ defined by $c_n = 0$, if $n \neq n_s$, $c_n = a_n^{(p)}$, if $n = n_s$, $s = 1, 2, \ldots$, lies in $\lambda_{\alpha^{(p)}}^\times$.

Let φ_n be the closure of $\varphi \cap \lambda_{\alpha^{(n)}}^\times$ in $\lambda_{\alpha^{(n)}}^\times$ with the topology induced by $\lambda_{\alpha^{(n)}}^\times$. According to (1), γ does not belong to φ_q, and by (2), γ lies in φ_r. Therefore

$$\varphi_r \cap \lambda_{\alpha^{(p)}}^\times \neq \varphi_q \cap \lambda_{\alpha^{(p)}}^\times$$

and, consequently,

$$(\varphi_n \cap \lambda_{\alpha^{(p)}}^\times)_{n=p}^\infty$$

is an increasing non-stationary sequence of closed subspaces of $\lambda_{\alpha^{(p)}}^\times$. Applying the theorem of Baire to the Banach space $\lambda_{\alpha^{(p)}}^\times$ it results that

$$\cdot \cup \{ \varphi_n \cap \lambda_{\alpha^{(p)}}^\times : n = 1, 2, \ldots \} = \varphi^{(1)} \cap \lambda_{\alpha^{(p)}}^\times$$

is a non-closed subspace of $\lambda_{\alpha^{(p)}}^\times$ and therefore $\varphi^{(1)}$ is not complete.

Suppose now that the conditions of the lemma are verified. If $\varphi^{(1)} = \lambda^\times$ then $\varphi^{(1)}$ is complete. If $\varphi^{(1)}$ is different from $\lambda^\times$ we take a vector $\beta = (b_1, b_2, \ldots, b_n, \ldots)$ in $\lambda^\times$ which is not in $\varphi^{(1)}$. Let $p \in \mathbb{N}$ be such that $\beta \in \lambda_{\alpha^{(p)}}^\times$. We obtain an integer $q > p$ so that, if P is an arbitrary subset of $\mathbb{N}$ which satisfies

$$a_n^{(p)} \neq 0, \quad n \in P, \quad \inf \left\{ \frac{a_n^{(p)}}{a_n^{(q)}} : n \in P \right\} \neq 0$$

then

$$\inf\left\{\frac{a_n^{(p)}}{a_n^{(r)}}:n\in P\right\}\neq 0, \quad r=q+1, q+2,\dots.$$

Since $\beta\in\lambda_{\alpha^{(p)}}^{\times}$ and $\beta\notin\varphi_q$ we can find a sequence of positive integers

$$n_1<n_2<\dots<n_s<\dots$$

so that

$$a_{n_s}^{(p)}\neq 0, \quad s=1, 2,\dots$$

and

$$\inf\left\{\frac{|b_{n_s}|}{a_{n_s}^q}:s=1, 2,\dots\right\}\neq 0.$$

There exists a positive real number b such that

$$|b_{n_s}|\leqq ba_{n_s}^{(p)}, \quad s=1, 2,\dots.$$

Then

$$\inf\left\{\frac{a_{n_s}^{(p)}}{a_{n_s}^{(q)}}:s=1, 2,\dots\right\}\geqq\inf\left\{\frac{|b_{n_s}|}{ba_{n_s}^{(q)}}:s=1, 2,\dots\right\}\neq 0,$$

and therefore

$$\inf\left\{\frac{a_{n_s}^{(p)}}{a_{n_s}^{(r)}}:s=1, 2,\dots\right\}\neq 0, \quad r=q+1, q+2,\dots.$$

Using a diagonal process we can obtain a subsequence (m_s) of (n_s) so that

$$\lim_{s\to\infty}\frac{a_{m_s}^{(p)}}{a_{m_s}^{(r)}}=a_r\neq 0, \quad r=q, q+1,\dots \tag{3}$$

$$\lim_{s\to\infty}\frac{b_{m_s}}{a_{m_s}^{(q)}}=c\neq 0. \tag{4}$$

Let $\lambda_1^{\times}$ be the subspace of $\lambda^{\times}[\beta(\lambda^{\times}, \lambda)]$ consisting of all the sequences with the nth coordinate vanishing, for $n\in\mathbb{N}$, $n\neq m_s$, $s=1, 2,\dots$. Let $\lambda_2^{\times}$ be the subspace of $\lambda^{\times}[\beta(\lambda^{\times}, \lambda)]$ consisting of all the sequences with the m_sth coordinate vanishing, $s=1, 2,\dots$. Then $\lambda^{\times}[\beta(\lambda^{\times}, \lambda)]=\lambda_1^{\times}\oplus\lambda_2^{\times}$.

From (3) and (4) we obtain that $\lambda_1^{\times}$ is topologically isomorphic to l^{∞} and the projection of β onto $\lambda_1^{\times}$ along $\lambda_2^{\times}$ is not in the closure of $\varphi\cap\lambda_1^{\times}$ in $\lambda_1^{\times}$. On the other hand, since $\varphi^{(1)}$ is a normal space it follows in a straightforward way that the projection of the completion of $\varphi^{(1)}$ onto $\lambda_1^{\times}$ along $\lambda_2^{\times}$ coincides with the closure of $\varphi\cap\lambda_1^{\times}$ in $\lambda_1^{\times}$ and therefore β does not belong to the completion of $\varphi^{(1)}$ and so $\varphi^{(1)}$ is complete.

Theorem 1. *If $\varphi^{(1)}$ is complete, then $\lambda[\mu(\lambda, \lambda^{\times})]$ is quasi-normable.*

Proof. Given a bounded absolutely convex subset A of $\lambda^{\times}[\beta(\lambda^{\times}, \lambda)]$, we can find a positive integer p such that A is a bounded set in $\lambda_{\alpha^{(p)}}^{\times}$. Let q be a positive integer, $q>p$, which verifies the conditions of the lemma.

Let us suppose that $\beta(\lambda^x, \lambda)$ and the topology of $\lambda^x_{\alpha^{(a)}}$ do not coincide on A. Therefore there is a positive number c and a net in A

$$\{(x_i^{(j)}): j \in J, \geqq\} \tag{5}$$

which converges to the origin in $\lambda^x[\beta(\lambda^x, \lambda)]$ such that

$$\|(x_i^{(j)})\|_q > c, \quad j \in J.$$

If $j \in J$, we have that

$$\|(x_i^{(j)})\|_q = \sup\left\{\frac{|x_i^{(j)}|}{a_i^{(q)}} : a_i^{(q)} \neq 0\right\}.$$

Consequently, there exists a positive integer m_j so that

$$a_{m_j}^{(q)} \neq 0, \quad \frac{|x_{m_j}^{(j)}|}{a_{m_j}^{(q)}} > c. \tag{6}$$

For a suitable positive real number b

$$|x_{m_j}^{(j)}| \leqq b a_{m_j}^{(p)}, \quad j \in J,$$

hence

$$b \geqq \frac{b a_{m_j}^{(p)}}{a_{m_j}^{(q)}} \geqq \frac{|x_{m_j}^{(j)}|}{a_{m_j}^{(q)}} > c,$$

and therefore

$$\inf\left\{\frac{a_{m_j}^{(p)}}{a_{m_j}^{(q)}} : j \in J\right\} \neq 0.$$

Then, by the lemma,

$$\inf\left\{\frac{a_{m_j}^{(p)}}{a_{m_j}^{(r)}} : j \in J\right\} \neq 0, \quad r = q, q+1, \ldots. \tag{7}$$

If M is the set $\{m_j : j \in J\}$, the space $\lambda^x[\beta(\lambda^x, \lambda)]$ is the topological direct sum of $\lambda^x(M)$ and $\lambda^x(N \sim M)$. Condition (7) shows that $\lambda^x(M)$ is topologically isomorphic to a Banach space. The projection of the net (5) on $\lambda^x(M)$ along $\lambda^x(N \sim M)$ converges to the origin in $\lambda^x(M)$ and, therefore, given a positive number $\varepsilon < c$, there exists a $j_0 \in J$ such that, if $j \geqq j_0$,

$$\sup\left\{\frac{|x_i^{(j)}|}{a_i^{(q)}} : i \in M\right\} < \varepsilon < c,$$

and this contradicts (6).

If $\varphi^{(1)}$ is not complete it follows from the lemma that we can find a positive integer p such that, for each $q > p$, $q \in \mathbb{N}$, there exists $r \in \mathbb{N}$, $r > q$, and a subset H of $\mathbb{N}$ such that

$$a_n^{(p)} \neq 0, \quad n \in H,$$

$$\inf\left\{\frac{a_n^{(p)}}{a_n^q} : n \in H\right\} \neq 0$$

$$\inf\left\{\frac{a_n^{(p)}}{a_n^{(r)}} : n \in H\right\} = 0.$$

We put $q = q_1$, $r = q_2$. We determine an increasing sequence of natural numbers

$$n_{11} < n_{12} < \ldots < n_{1i} < \ldots$$

such that

$$\lim_{i \to \infty} \frac{a_{n_{1i}}^{(p)}}{a_{n_{1i}}^{(q_1)}} = c_1 \neq 0, \quad \lim_{i \to \infty} \frac{a_{n_{1i}}^{(p)}}{a_{n_{1i}}^{(q_2)}} = 0.$$

Let us suppose that, for $s = 1, 2, \ldots, m$, sequences of positive integers $n_{s1} < n_{s2} < \ldots < n_{si} < \ldots$ and numbers q_s, q_{s+1} in $\mathbb{N}$ have already been constructed such that

$$\lim_{i \to \infty} \frac{a_{n_{si}}^{(p)}}{a_{n_{si}}^{(q_s)}} = c_s \neq 0, \tag{8}$$

$$\lim_{i \to \infty} \frac{a_{n_{si}}^{(p)}}{a_{n_{si}}^{(q_{s+1})}} = 0. \tag{9}$$

Starting from q_{m+1} we can find a positive integer $q_{m+2} > q_{m+1}$ and a subset P of $\mathbb{N}$ such that

$$\inf \left\{ \frac{a_n^{(p)}}{a_n^{(q_{m+1})}} : n \in P \right\} \neq 0$$

$$\inf \left\{ \frac{a_n^{(p)}}{a_n^{(q_{m+2})}} : n \in P \right\} = 0.$$

By (9), we can assume that P has been chosen disjoint from

$$\{ n_{si} : s = 1, 2, \ldots, m; i = 1, 2, \ldots \}.$$

Now we select a sequence

$$n_{(m+1)1} < n_{(m+1)2} < \ldots < n_{(m+1)i} < \ldots$$

in P such that

$$\lim_{i \to \infty} \frac{a_{n_{(m+1)i}}^{(p)}}{a_{n_{(m+1)i}}^{(q_{m+1})}} = c_{m+1} \neq 0,$$

$$\lim_{i \to \infty} \frac{a_{n_{(m+1)i}}^{(p)}}{a_{n_{(m+1)i}}^{(q_{m+2})}} = 0.$$

Let Λ be the sectional subspace of λ defined by the system of steps

$$B^{(m)} = (b_{ij}^{(m)}; i, j = 1, 2, \ldots) \quad m = 1, 2, \ldots$$

where

$$b_{ij}^{(m)} = a_{n_{ij}}^{(q_{m+1})}, \quad m = 2, 3, \ldots$$
$$b_{ij}^{(1)} = a_{n_{ij}}^{(p)}.$$

Theorem 2. *If $\varphi^{(1)}$ is not complete, $\lambda^{\times}[\beta(\lambda^{\times}, \lambda)]$ does not fulfil the Mackey convergence condition.*

Proof. Obviously, it is enough to show that $\Lambda^x[\beta(\lambda^x, \Lambda)]$ does not verify the Mackey convergence condition. In Λ^x we consider the sequence $((x_{ij}^{(n)}))_{n=1}^\infty$ defined in the following way:

$$x_{1j}^{(n)} = 0, \qquad j = 1, 2, \ldots; \qquad n = 1, 2, \ldots$$

$$x_{ij}^{(n)} = 0, \qquad j = 1, 2, \ldots n; \qquad i = 2, 3, \ldots$$

$$x_{ij}^{(n)} = \frac{1}{i} b_{ij}^{(1)}, \quad j = n+1, n+2, \ldots; \quad i = 2, 3, \ldots.$$

Let us see first that the sequence $((x_{ij}^{(n)}))_{n=1}^\infty$ converges to the origin in $\Lambda^x[\beta(\Lambda^x, \Lambda)]$. If B is a bounded set in $\Lambda[\mu(\Lambda, \Lambda^x)]$ there exists a sequence (h_m) of positive numbers such that, for each $(u_{ij}) \in B$.

$$\sum_{i,j=1}^\infty |u_{ij}| \, b_{ij}^{(m)} \leqq h_m, \quad m = 1, 2, \ldots.$$

Given a positive number ε, we can find $i_0 \in \mathbb{N}$, $i_0 \geqq 2$, such that

$$\frac{h_1}{i_0} < \frac{\varepsilon}{2}$$

Then, for each $(u_{ij}) \in B$ and $n \in N$, it follows that

$$\left| \sum_{i,j=1}^\infty u_{ij} x_{ij}^{(n)} \right| \leqq \left| \sum_{i=2}^{i_0} \sum_{j=1}^\infty u_{ij} x_{ij}^{(n)} \right| + \sum_{i=i_0+1}^\infty \sum_{j=1}^\infty |u_{ij}| \frac{1}{i} b_{ij}^{(1)}$$

$$\leqq \left| \sum_{i=2}^{i_0} \sum_{j=1}^\infty u_{ij} x_{ij}^{(n)} \right| + \frac{1}{i_0} h_1 \leqq \left| \sum_{i=2}^{i_0} \sum_{j=1}^\infty u_{ij} x_{ij}^{(n)} \right| + \frac{\varepsilon}{2}.$$

We can put (9) in the form

$$\lim_{j \to \infty} \frac{b_{ij}^{(1)}}{b_{ij}^{(i)}} = 0, \quad i = 2, 3, \ldots$$

and, therefore, there exists $j_0 \in \mathbb{N}$ such that

$$\frac{b_{ij}^{(1)}}{b_{ij}^{(i)}} < \frac{\varepsilon}{2i_0 h_i}, \quad j \geqq j_0, \quad i = 2, 3, \ldots, i_0.$$

Then, if $n > j_0$ it follows for $(u_{ij}) \in B$,

$$\left| \sum_{i,j=1}^\infty u_{ij} x_{ij}^{(n)} \right| \leqq \left| \sum_{i=2}^{i_0} \sum_{j=1}^\infty u_{ij} x_{ij}^{(n)} \right| + \frac{\varepsilon}{2}$$

$$\leqq \sum_{i=2}^{i_0} \sum_{j=j_0+1}^\infty |u_{ij}| \, |x_{ij}^{(n)}| + \frac{\varepsilon}{2} \leqq \sum_{i=2}^{i_0} \sum_{j=j_0+1}^\infty |u_{ij}| \frac{b_{ij}^{(1)}}{i} + \frac{\varepsilon}{2}$$

$$\leqq \sum_{i=2}^{i_0} \frac{\varepsilon}{2i_0 h_i} \sum_{j=j_0+1}^\infty |u_{ij}| \, b_{ij}^{(i)} + \frac{\varepsilon}{2} \leqq \sum_{i=2}^{i_0} \frac{\varepsilon}{2i_0 h_i} h_i + \frac{\varepsilon}{2} \leqq \varepsilon,$$

hence $((x_{ij}^{(n)}))_{n=1}^\infty$ converges to the origin in $\Lambda^x[\beta(\Lambda^x, \Lambda)]$.

On the other hand, given the natural number m, by (8) it follows that

$$\lim_{j \to \infty} \frac{x^{(n)}_{(m+1)j}}{b^{(m)}_{(m+1)j}} = \frac{1}{m+1} \lim_{j \to \infty} \frac{b^{(1)}_{(m+1)j}}{b^{(m)}_{(m+1)j}} = \frac{1}{m+1} c_{m+1} \neq 0, \quad n = 1, 2, \ldots$$

so that $((x^{(n)}_{ij}))^{\infty}_{n=1}$ does not converge to the origin in $\Lambda^{x}_{\beta^{(m)}}$.

From the former results we obtain the following theorem:

Theorem 3. *In an echelon λ, the following conditions are equivalent:*
1. $\varphi^{(1)}$ *is complete.*
2. $\lambda[\mu(\lambda, \lambda^{x})]$ *is quasi-normable.*
3. $\lambda^{x}[\beta(\lambda^{x}, \lambda)]$ *verifies the Mackey convergence condition.*

In (**1**) A. Grothendieck asks if there exists a space with a countable fundamental system of bounded sets which verifies the Mackey convergence condition but not the strict Mackey convergence condition. In (**5**) S. Warner gives an example answering this question. In the following proposition we give a class of spaces with this property.

Proposition. *If E is a Banach space such that $E'[\sigma(E', E)]$ is not separable there is a topology $\mathfrak{E}$ on E, compatible with the dual pair $\langle E, E' \rangle$, such that $E[\mathfrak{E}]$ is a (DF)-space which verifies the Mackey convergence condition but not the strict Mackey convergence condition.*

Proof. Let $\mathfrak{E}$ be the topology on E of the uniform convergence on every countable bounded subset of $E'[\sigma(E', E)]$. Obviously $\mathfrak{E}$ is compatible with the dual pair $\langle E, E' \rangle$. If B is the closed unit ball of E, a fundamental system of bounded set of $E[\mathfrak{E}]$ is given by

$$\{nB : n = 1, 2, \ldots\}.$$

Every locally convex space E_{nB} coincides with E_{B}, $n = 1, 2, \ldots$ and, since $E'[\sigma(E', E)]$ is not separable, $\mathfrak{E}$ is different of the topology of E, hence $\mathfrak{E}$ does not coincide with the topology of E_{nB} on B and, therefore, $E[\mathfrak{E}]$ does not verify the strict Mackey convergence condition.

Since $E[\mathfrak{E}]$ is, obviously, a (DF)-space, $\mathfrak{E}$ and $\beta(E, E')$ (which coincides with the topology of E), coincide on every countable subset of $E[\mathfrak{E}]$, [3, p. 392] and therefore if (x_{n}) is a sequence in $E[\mathfrak{E}]$ which converges to x, it also converges to x in $E = E_{B}$.

Question. Is it true that every separable Fréchet space, such that its strong dual verifies the Mackey convergence condition, is quasi-normable?

Theorem 4. *If $\lambda[\mu(\lambda, \lambda^{x})]$ is not quasi-normable, it is the strong dual of a (LB)-space which is not complete.*

Proof. The space $\lambda[\mu(\lambda, \lambda^{x})]$ is the strong dual of the subspace φ of $\lambda^{x}[\beta(\lambda^{x}, \lambda)]$ (**2**), hence $\lambda[\mu(\lambda, \lambda^{x})]$ is the strong dual of $\varphi^{(1)}$. On the other hand, $\varphi^{(1)}$ is a (LB)-space, (**4**), and, by Theorem 1, $\varphi^{(1)}$ is not complete.

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Corollary. *If* $\lambda[\mu(\lambda, \lambda^{\times})]$ *is not distinguished, it is the strong dual of a non-complete* *(LB)-space.*

Proof. It is straightforward considering Theorem 4 and the fact that every quasi-normable Fréchet space is distinguished. (**3**, p. 399).

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