

Dimension reduction and homogenization of random degenerate operators. Part I

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ABSTRACT

Our aim in this paper is to identify the limit behavior of the solutions of random degenerate equations of the form $-\operatorname{div} A_\varepsilon(x', \nabla U_\varepsilon) + \rho_\omega^\varepsilon(x')U_\varepsilon = F$ with mixed boundary conditions on Ω_ε whenever $\varepsilon \rightarrow 0$, where Ω_ε is an N -dimensional thin domain with a small thickness $h(\varepsilon)$, $\rho_\omega^\varepsilon(x') = \rho_\omega(x'/\varepsilon)$, where ρ_ω is the realization of a random function $\rho(\omega)$, and $A_\varepsilon(x', \xi) = a(T_{x'/\varepsilon}\omega, \xi)$, the map $a(\omega, \xi)$ being measurable in ω and satisfying degenerated structure conditions with weight ρ in ξ . As usual in dimension reduction problems, we focus on the rescaled equations and we prove that under the condition $h(\varepsilon)/\varepsilon \rightarrow 0$, the sequence of solutions of them converges to a limit u_0 , where u_0 is the solution of an $(N - 1)$ -dimensional limit problem with homogenized and auxiliary equations.

1. Introduction

The present work is concerned with stationary heat diffusion problems taking general form, which cover in particular the following Poisson equation:

$$-\operatorname{div}(\rho_\omega^\varepsilon(x')\nabla U_\varepsilon) + \rho_\omega^\varepsilon(x')U_\varepsilon = F \quad \text{in } \Omega_\varepsilon \quad (1.1)$$

with mixed boundary data in a microscopically heterogeneous thin plate filling the cylinder

$$\Omega_\varepsilon = \Sigma \times]-h(\varepsilon), h(\varepsilon)[,$$

Σ being an open bounded connected subset of \mathbb{R}^{N-1} , $N \geq 2$, with Lipschitz boundary $\partial\Sigma$, and $h(\varepsilon)$ being a real positive number which goes to zero as $\varepsilon \rightarrow 0$ and such that

$$l := \lim_{\varepsilon \rightarrow 0} \frac{h(\varepsilon)}{\varepsilon} = 0. \quad (1.2)$$

The local characteristics of the body are represented by the randomly rapidly oscillating function

$$\rho_\omega^\varepsilon(x') = \rho_\omega\left(\frac{x'}{\varepsilon}\right), \quad x' \in \mathbb{R}^{N-1},$$

ρ_ω being a weight, that is, a function that allows it to approach zero or infinity. Here, the variable ω belongs to a probability space (X, \mathcal{F}, μ) and the function F is defined on Ω_ε .

In the homogenization context, such problems were considered by Engström *et al.* [15], where the authors have investigated random nonlinear monotone operators in divergence form which satisfy weighted structure conditions by assuming that the function ρ_ω fulfils some uniform integrability condition of Muckenhoupt type [15, Definition 2.2], and they used a compensated compactness lemma adapted to the framework of weighted spaces.

In this paper, we consider more precisely the following boundary value problem:

$$\begin{cases} -\operatorname{div} a(T_{x'/\varepsilon}\omega, \nabla U_\varepsilon) + \rho_\omega^\varepsilon(x')U_\varepsilon = F & \text{in } \Omega_\varepsilon, \\ U_\varepsilon|_{\Gamma_\varepsilon^{\text{lat}}} = 0, a(T_{x'/\varepsilon}\omega, \nabla U_\varepsilon) \cdot \nu_\varepsilon|_{(\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon^{\text{lat}})} = 0, \end{cases} \quad (1.3)$$

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where $x = (x', x_N) = (x_1, \dots, x_{N-1}, x_N)$ will denote a generic point of \mathbb{R}^N , ν_ε is the unitary outward normal vector to Ω_ε , $\partial\Omega_\varepsilon$ is the boundary of Ω_ε and

$$\Gamma_\varepsilon^{\text{lat}} = \partial\Sigma \times [-h(\varepsilon), h(\varepsilon)]$$

its lateral boundary, T_z is a dynamical system associated to the probability space (X, \mathcal{F}, μ) , that is, a group of measurable maps $T_z : X \rightarrow X$ such that:

- $T_{z_1+z_2} = T_{z_1} \cdot T_{z_2}$, $z_1, z_2 \in \mathbb{R}^{N-1}$, $T_0 = \text{Id}$;
- $\mu((T_z)^{-1}A) = \mu(A)$ for every $z \in \mathbb{R}^{N-1}$, $A \in \mathcal{F}$;
- $T_z(\omega)$ is a measurable map from $(\mathbb{R}^{N-1} \times X, \mathcal{B} \times \mathcal{F})$ to (X, \mathcal{F}) , where \mathcal{B} is the Borel σ -algebra on \mathbb{R}^{N-1}

and we assume furthermore that X is a compact metric space, \mathcal{F} is a Borel σ -algebra on X and T_z is a map from $\mathbb{R}^{N-1} \times X$ to X continuous in this metric, and also that T_z is ergodic, that is, for every subset $A \in \mathcal{F}$,

$$T_z(A) = A \quad \forall z \in \mathbb{R}^{N-1} \Rightarrow \mu(A) = 0 \text{ or } \mu(A) = 1.$$

In the following, for a random field $g : X \rightarrow \mathbb{R}$ and a fixed $\omega \in X$, the function

$$g_\omega(z) = g(T_z\omega), \quad z \in \mathbb{R}^{N-1}$$

will be called a realization of g . Then we will suppose that the weight ρ_ω is a realization of a random field $\rho : X \rightarrow \mathbb{R}$ which is a measurable function such that $\rho > 0$ almost surely and

$$\rho \in L^1(X, \mu), \quad \rho^{-1} \in L^1(X, \mu). \tag{1.4}$$

By Fubini's theorem, for almost every $\omega \in X$, $\rho_\omega(z) = \rho(T_z\omega)$ is such that $\rho_\omega(z) > 0$ almost everywhere on \mathbb{R}^{N-1} and

$$\rho_\omega \in L^1_{\text{loc}}(\mathbb{R}^{N-1}), \quad \rho_\omega^{-1} \in L^1_{\text{loc}}(\mathbb{R}^{N-1}). \tag{1.5}$$

The vector-valued function $a = (a', a_N) = (a_1, \dots, a_{N-1}, a_N) : X \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfies the random degenerate structure conditions:

- (H₁) for every $\xi \in \mathbb{R}^N$, $a(\cdot, \xi)$ is \mathcal{F} -measurable;
- (H₂) for almost every $\omega \in X$, for every $\xi' \in \mathbb{R}^{N-1}$,

$$a'(\omega, 0) = 0 = a_N(\omega, \xi', 0); \tag{1.6}$$

- (H₃) there exist constants $c_1 > 0$ and $0 < \alpha \leq 1$ such that

$$|a(\omega, \xi_1) - a(\omega, \xi_2)| \leq c_1 \rho(\omega) (1 + |\xi_1| + |\xi_2|)^{1-\alpha} |\xi_1 - \xi_2|^\alpha \tag{1.7}$$

for almost every $\omega \in X$, for every $\xi_1, \xi_2 \in \mathbb{R}^N$;

- (H₄) there exist constants $c_2 > 0$ and $2 \leq \beta < \infty$ such that

$$(a(\omega, \xi_1) - a(\omega, \xi_2)) \cdot (\xi_1 - \xi_2) \geq c_2 \rho(\omega) (1 + |\xi_1| + |\xi_2|)^{2-\beta} |\xi_1 - \xi_2|^\beta \tag{1.8}$$

for almost every $\omega \in X$, for every $\xi_1, \xi_2 \in \mathbb{R}^N$.

Condition (H₂) implies in particular that

$$a(\omega, 0) = (a'(\omega, 0), a_N(\omega, 0)) = 0.$$

Furthermore, as consequences of conditions (H₂)–(H₄) there are constants $c_3, c_4, c_5 > 0$ such that the following growth condition holds:

$$|a(\omega, \xi)| \leq c_3 \rho(\omega) (1 + |\xi|), \tag{1.9}$$

and also the coercivity condition

$$a(\omega, \xi) \cdot \xi \geq c_5 \rho(\omega) (|\xi|^2 - c_4). \tag{1.10}$$

We are interested in the asymptotic behavior of the solutions of the original problem (1.3) whenever the parameter of thickness of the plate $h(\varepsilon)$ and the one of oscillation of

degeneracy ε go simultaneously to zero. Actually, we will consider only the case described in (1.2) to simplify analysis, and the cases $0 < l < \infty$ and $l = \infty$ will be addressed in a forthcoming paper (there is usually a distinction between the three cases).

As usual in the study of structures with thin thickness, it is better to work in a domain with fixed thickness by rescaling the original problem (1.3) (see for example [7, 17]), which permits us to simplify the derivation of *a priori* estimates. The resulting problem takes the form

$$\begin{cases} -\operatorname{div}_\varepsilon a(T_{x'/\varepsilon}\omega, \nabla_\varepsilon u_\varepsilon) + \rho_\omega^\varepsilon(x')u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon|_{\Gamma^{\text{lat}}} = 0, a(T_{x'/\varepsilon}\omega, \nabla_\varepsilon u_\varepsilon) \cdot \nu|_{(\partial\Omega \setminus \Gamma^{\text{lat}})} = 0, \end{cases} \tag{1.11}$$

where

$$\Omega = \Sigma \times]-1, +1[\tag{1.12}$$

with boundary $\partial\Omega$ and lateral boundary

$$\Gamma^{\text{lat}} = \partial\Sigma \times [-1, 1],$$

ν being the unitary outward normal vector to Ω , and

$$\operatorname{div}_\varepsilon A = \sum_{i=1}^{N-1} \left(\frac{\partial A_i}{\partial x_i} \right) + \frac{1}{h(\varepsilon)} \frac{\partial A_N}{\partial x_N}, \quad A = (A_i)_{1 \leq i \leq N}$$

being any function for which the weak derivatives here make sense,

$$\nabla_\varepsilon = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{N-1}}, \frac{1}{h(\varepsilon)} \frac{\partial}{\partial x_N} \right) \quad \text{and} \quad f \in L^\infty(\Omega).$$

From now on, our purpose will be to characterize the asymptotic behavior of the sequence of solutions of (1.11) as $\varepsilon \rightarrow 0$ and to identify the limiting equations by means of the stochastic version of the singular measure method available in Zhikov and Piatnitski [28] (see [9, 20, 24, 25] for the periodic case), taking advantage of this approach to make the so-called Muckenhoupt condition not necessary (as it is known in the periodic setting). In this context, we extend some properties from the linear stochastic case (cf. [28]) and those from the monotone periodic case (cf. [20]) to pass to the monotone stochastic one.

The other subject of this work consists in coupling simultaneously a dimension reduction analysis together with the stochastic homogenization process (only the latter is considered in [15]), which will be more complicated. It is proved that the sequence (u_ε) of solutions of (1.11) converges in a sense which will be described later to a function u_0 which solves the following homogenized limit problem:

$$\begin{cases} -\operatorname{div}_{x'} B(\nabla_{x'} u_0) + 2P(X)u_0 = g & \text{in } \Sigma, \\ u_0 = 0 & \text{on } \partial\Sigma, \end{cases} \tag{1.13}$$

where

$$\nabla_{x'} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{N-1}} \right), \quad \operatorname{div}_{x'} = \sum_{i=1}^{N-1} \frac{\partial}{\partial x_i},$$

the function $B(\nabla_{x'} u_0)$ having been determined (see equations (3.9) and (3.10)), P is the measure on X defined as follows:

$$dP(\omega) = \rho(\omega) d\mu(\omega) \tag{1.14}$$

and g is the function defined in (3.11). This means that the dimension reduction occurs in the limit problem (compare with problem (1.11), which is N dimensional). Moreover, we obtain an auxiliary problem in which the probability space plays the role of the cell (see equation (3.10)).

For various dimension reduction problems, see for example [6, 7, 16–18], and for the homogenization of partial differential equations in the random context, see for example [4, 5, 11, 12, 27, 28] and the references therein.

The outline of the paper is the following: in Section 2, some definitions and preliminary results are formulated, namely, we discuss the notion of derivation in the stochastic sense, we invoke some useful ergodic theorems and we introduce the stochastic two-scale convergence with respect to random measures. The principal theorem of the paper will be given in Section 3, and the final section will contain the proof through several steps. In Step 1 we derive some *a priori* estimates and we use them in Step 2 to get compactness results by means of the stochastic two-scale convergence. In Step 3 we prove the convergence of energies and make use of it in Step 4 together with a Minty argument. Step 5 will accomplish the proof of Theorem 3.1 by an identification of the homogenized and auxiliary problems.

2. Preliminaries

The purpose of this section is to state some concepts and results that are used throughout this work. Some of these results are given without proofs as they can be readily found in the references given below.

2.1. Lebesgue and Sobolev spaces with respect to a measure defined on \mathbb{R}^d

Let $d \geq 2$, G be a bounded domain in \mathbb{R}^d with Lipschitz boundary and λ be a measure in \mathbb{R}^d . We denote by $L^2(G, \lambda)$ the set of functions defined by

$$L^2(G, \lambda) := \left\{ f : G \rightarrow \overline{\mathbb{R}} : f \text{ is measurable and } \int_G |f|^2 d\lambda(x) < \infty \right\}. \quad (2.1)$$

We give now a brief review of Sobolev spaces with respect to a measure; for more details the reader can see the Zhikov bibliography [23–26, 28] and also Bouchitté *et al.* [8–10].

DEFINITION 2.1. Let S be a closed subset of ∂G with Lipschitz boundary. We say that $u \in L^2(G, \lambda)$ belongs to $H^1(G, S, \lambda)$ and $z \in L^2(G, \lambda)^d$ is a *gradient* of this function if there exist a sequence of functions $u_k \in C_0^\infty(\overline{G} \setminus S)$ such that

$$u_k \rightarrow u \text{ strongly in } L^2(G, \lambda), \quad (2.2)$$

$$\nabla u_k \rightarrow z \text{ strongly in } L^2(G, \lambda)^d \quad (2.3)$$

as $k \rightarrow +\infty$. We shall denote a gradient of u by ∇u . The space $H(G, S, \lambda)$ will be the set of pairs (u, z) endowed with the natural norm

$$\|(u, z)\|_{H(G, S, \lambda)}^2 := \|u\|_{L^2(G, \lambda)}^2 + \|z\|_{L^2(G, \lambda)^d}^2.$$

In other words, it is the closure of the set of vector-valued functions $\{(\varphi, \nabla \varphi) : \varphi \in C_0^\infty(\overline{G} \setminus S)\}$ in the norm of $L^2(G, \lambda)^{d+1}$.

REMARK 2.1. The gradient of a function $u \in L^2(G, \lambda)$ is not necessarily unique. It is defined to within the set of gradients of the zero function. Nevertheless, there are exceptions. For instance, if the measure λ is absolutely continuous with respect to the Lebesgue measure, that is, $d\lambda(x) = w(x) dx$, where $w > 0$ almost everywhere in \mathbb{R}^d with $w, w^{-1} \in L^1(G)$, then $H^1(G, S, \lambda) \subset W^{1,1}(G)$ and any gradient of a function $u \in H^1(G, S, \lambda)$ in the sense of Definition 2.1 coincides with the usual gradient in the $W^{1,1}$ sense. To show this, let us take a sequence of functions $u_k \in C_0^\infty(\overline{G} \setminus S)$ such that (2.2) and (2.3) hold true for $u \in H^1(G, S, \lambda)$ and a gradient z ; because of the Hölder inequality,

$$\int_G |u_k - u| dx \leq \left(\int_G w^{-1} dx \right)^{1/2} \left(\int_G |u_k - u|^2 d\lambda(x) \right)^{1/2}, \quad (2.4)$$

$$\xrightarrow{k \rightarrow 0} 0 \quad (2.5)$$

and, in the same manner, we obtain that $\int_G |\nabla u_k - z| dx \xrightarrow{k} 0$. Hence, we have proved that $\|u_k - u\|_{W^{1,1}(G)} \rightarrow 0$ as $k \rightarrow \infty$. The uniqueness of z follows as the gradient of u in the $W^{1,1}$ sense. This is important if one needs $H^1(G, S, \lambda)$ to be a Banach space, equipped with the norm

$$\|u\|_{H^1(G,S,\lambda)}^2 := \|u\|_{L^2(G,\lambda)}^2 + \|\nabla u\|_{L^2(G,\lambda)^d}^2.$$

Moreover, the separability and reflexivity are also fulfilled for the space $H^1(G, S, \lambda)$ in that particular case, and can be proved exactly as for the classical Sobolev spaces.

2.2. *The derivation in a stochastic sense*

For the remainder of this section, (X, \mathcal{F}, μ) is the probability space with dynamical system T_z as introduced in the beginning of Section 1 but with $z \in \mathbb{R}^d$, $d \geq 2$, and P the measure defined in (1.14).

In the stochastic homogenization framework, the probability space X may be compared with the periodicity cell Y in the periodic homogenization and, likewise, the derivation with respect to $\omega \in X$ plays the same role as for a point y of Y . So, also here it is fundamental that the differentiation with respect to a variable $\omega \in X$ makes sense (for more about the stochastic differentiation, see [3, 12, 28]). Let us consider the set of continuous functions defined on X such that the limits

$$(\partial_i u)(\omega) = \lim_{\delta \rightarrow 0} \frac{u(T_{\delta e_i} \omega) - u(\omega)}{\delta}$$

exist for every $\omega \in X$ and $i \in \{1, 2, \dots, d\}$ and are continuous on X , where (e_1, \dots, e_d) is the Euclidean canonical basis of \mathbb{R}^d . We denote this set by $C^1(X)$ and we use the notation

$$\nabla_\omega u = (\partial_i u)_{1 \leq i \leq d}.$$

REMARK 2.2. The set $C^1(X)$ is dense in $L^2(X, P)$, since $\mathcal{C}(X)$ is dense in $L^2(X, P)$ and, for every $\varphi \in \mathcal{C}(X)$, we take the family of functions

$$\varphi^\delta(\omega) = \delta^{-d} \int_{\mathbb{R}^d} K\left(\frac{z}{\delta}\right) \varphi(T_z \omega) dz, \tag{2.6}$$

where K is a C_0^∞ -function with integral equal to 1. The properties of the function φ^δ are derived from the continuity of the dynamical system T_z .

DEFINITION 2.2. $L_{\text{pot}}^2(X, P)$ is the space of potential vectors defined to be the closure of the set $\{\nabla_\omega b : b \in C^1(X)\}$ in $L^2(X, P)^d$.

2.3. *Ergodic theorems*

Let $g \in L_{\text{loc}}^1(\mathbb{R}^d)$. The number $M\{g\}$ is called *the mean value* of g if

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|K|} \int_K g\left(\frac{z}{\varepsilon}\right) dz = M\{g\}$$

for any Lebesgue measurable bounded set $K \in \mathbb{R}^d$; here and in all the following, $|K|$ stands for the Lebesgue measure of K .

Let us now recall the well-known Birkhoff ergodic theorem (see [27]).

THEOREM 2.1. *Let $g \in L^p(X, \mu)$, $p \geq 1$. Then, for almost all $\omega \in X$, the realization $g(T_z \omega)$ possesses a mean value in the following sense:*

$$g(T_{z/\varepsilon} \omega) \rightharpoonup M\{g(T_z \omega)\} \quad \text{weakly in } L_{\text{loc}}^p. \tag{2.7}$$

Moreover, the mean value $M\{g(T_z\omega)\}$, considered as a function of $\omega \in X$, is invariant and

$$\langle g \rangle := \int_X g(\omega) d\mu = \int_X M\{g(T_z\omega)\} d\mu.$$

Since the system T_z is ergodic,

$$M\{g(T_z\omega)\} = \langle g \rangle \quad \text{for almost all } \omega \in X.$$

There is also a generalized version of the Birkhoff ergodic theorem in terms of stationary random measures with finite intensity (see [14] and [28, Theorem 1.1]).

Let $\sigma : X \rightarrow \mathbb{R}$ be a measurable function such that $\sigma > 0$ almost surely in X and

$$\sigma \in L^1(X, \mu), \quad \sigma^{-1} \in L^1(X, \mu). \quad (2.8)$$

For every $\omega \in X$, let us consider the random function defined for every $z \in \mathbb{R}^d$ by

$$\sigma_\omega(z) = \sigma(T_z\omega), \quad (2.9)$$

and define the measures F and τ_ω respectively on X and \mathbb{R}^d by

$$dF(\omega) = \sigma(\omega) d\mu(\omega), \quad (2.10)$$

$$d\tau_\omega(z) = \sigma_\omega(z) dz. \quad (2.11)$$

REMARK 2.3. The measure F is called *the Palm measure* associated to the random measure τ_ω . Conversely, the last one is considered as a realization of the first. Palm theory is concerned in particular with stationary random measures; a family of Radon measures μ_ω on \mathbb{R}^d , $\omega \in X$, is called a *stationary random measure* if for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ the random function

$$F_\varphi(z, \omega) = \int_{\mathbb{R}^d} \varphi(y - z) d\mu_\omega(y)$$

is measurable and stationary, that is,

$$F_\varphi(z, \omega) = F_\varphi(T_z\omega), \quad (2.12)$$

where $F_\varphi(\omega) = \int_{\mathbb{R}^d} \varphi(y) d\mu_\omega(y)$. In general, a stationary random measure μ_ω and its Palm measure P are related by the Campbell formula

$$\int_X \int_{\mathbb{R}^d} f(z, T_z\omega) d\mu_\omega(z) d\mu(\omega) = \int_{\mathbb{R}^d} \int_X f(z, \omega) dP(\omega) dz \quad (2.13)$$

for all functions $f = f(z, \omega)$ which are integrable with respect to $dz \times P$ or non-negative measurable. A detailed exposition of all these notions is given in [28].

THEOREM 2.2 (The ergodic theorem). *We have*

$$\lim_{t \rightarrow +\infty} \frac{1}{t^d |A|} \int_{tA} g(T_z\tilde{\omega}) d\tau_{\tilde{\omega}}(z) = \int_X g(\omega) dF(\omega), \quad \text{a.s. with respect to } \mu, \quad (2.14)$$

for all bounded Borel sets $A \subset \mathbb{R}^d$, $|A| > 0$, and all $g \in L^1(X, F)$.

2.4. Stochastic two-scale convergence with respect to random measures

The concept of two-scale convergence was introduced firstly in the periodic setting by Nguetseng [21] and developed by Allaire (see [1, 2]; see also [19]). An extension of this method in the stochastic framework has been elaborated by Bourgeat *et al.* [12] and generalized in spaces with random measures by Zhikov and Piatnitski [28]. The present paragraph will be formulated by collecting some results from [28] and completing them by adapting others from the periodic two-scale convergence setting [19, 20].

The two-scale convergence associated to a random measure as exploited in [28] depends on the notion of *typical trajectory*, which is related to the validity of the ergodic theorem (Theorem 2.2). A point $\omega \in X$ is said to be *typical* if (2.14) holds at this point for all $g \in \mathcal{C}(X)$. The set of typical points will be denoted by \tilde{X} and it satisfies $\mu(\tilde{X}) = 1$. As pointed out in [28], every function $g \in L^1(X, F)$ can be changed on a set of F-measure zero in such a way as to be defined on \tilde{X} , and (2.14) holds for all $\tilde{\omega} \in \tilde{X}$ and then almost surely in X . So, in what follows, we identify functions belonging to $L^2(X, F)$ with modifications of them for which (2.14) holds.

Let $\varepsilon > 0$ and $\sigma_\omega^\varepsilon$ be the function defined on \mathbb{R}^{N-1} by

$$\sigma_\omega^\varepsilon(z) = \sigma_\omega\left(\frac{z}{\varepsilon}\right),$$

where σ_ω is the random function defined in (2.9), and τ_ω^ε and ν_ω^ε be the measures defined respectively on \mathbb{R}^{N-1} and \mathbb{R}^N by

$$d\tau_\omega^\varepsilon(z) = \sigma_\omega^\varepsilon(z) dz, \tag{2.15}$$

$$d\nu_\omega^\varepsilon(x) = d\tau_\omega^\varepsilon(x') \otimes dx_N. \tag{2.16}$$

Let G be a Lipchitz domain in \mathbb{R}^N and $\tilde{\omega} \in \tilde{X}$. A family of functions (v_ε) in $L^2(G, \nu_\omega^\varepsilon)$ is bounded if

$$\limsup_{\varepsilon \rightarrow 0} \int_G |v_\varepsilon|^2 d\nu_\omega^\varepsilon(x) < +\infty. \tag{2.17}$$

DEFINITION 2.3. Let (v_ε) be a bounded sequence of functions in $L^2(G, \nu_\omega^\varepsilon)$. Then we say that v_ε two-scale converges weakly to a limit $v \in L^2(G \times X, dx \times dF(\omega))$, and we write $v_\varepsilon \xrightarrow{2:\varepsilon} v$ if

$$\lim_{\varepsilon \rightarrow 0} \int_G v_\varepsilon(x) \varphi(x) b(T_{x'/\varepsilon} \tilde{\omega}) d\nu_\omega^\varepsilon(x) = \int_{G \times X} v(x, \omega) \varphi(x) b(\omega) dF(\omega) dx \tag{2.18}$$

for every $\varphi \in \mathcal{C}_0^\infty(G)$ and $b \in \mathcal{C}^1(X)$. If (V_ε) is a bounded sequence of functions in $L^2(G, \nu_\omega^\varepsilon)^N$ with $V_\varepsilon = (V_\varepsilon^i)$, $i = 1, \dots, N$, then we say that V_ε two-scale converges weakly to a limit $V = (V^i) \in L^2(G \times X, dx \times dF(\omega))^N$, and we write $V_\varepsilon \xrightarrow{2:\varepsilon} V$ if $V_\varepsilon^i \xrightarrow{2:\varepsilon} V^i$ for every $i = 1, \dots, N$.

REMARK 2.4. In the above definition, we can take any test function $\phi(x, \omega) \in \mathcal{R}$, where \mathcal{R} is the space of functions defined as

$$\mathcal{R} := \left\{ \phi(x, \omega) = \sum_{j \in J} \varphi_j(x) b_j(\omega) : J \text{ finite, } \varphi_j \in \mathcal{C}_0^\infty(G), b_j \in \mathcal{C}^1(X) \forall j \in J \right\}, \tag{2.19}$$

endowed with the norm

$$\|\phi\|_{\mathcal{R}} = \sup_{G \times X} |\phi(x, \omega)|.$$

We can also define strong two-scale convergence in the variable space $L^2(G, \nu_\omega^\varepsilon)$.

DEFINITION 2.4. We say that a bounded sequence (v_ε) in $L^2(G, \nu_\omega^\varepsilon)$ two-scale converges strongly to a function $v \in L^2(G \times X)$ as $\varepsilon \rightarrow 0$ and we denote it by $v_\varepsilon \xrightarrow{2:e} v$ if

$$\lim_{\varepsilon \rightarrow 0} \int_G v_\varepsilon(x) \cdot w_\varepsilon(x) d\nu_\omega^\varepsilon(x) = \int_{G \times X} v(x, \omega) \cdot w(x, \omega) dF(\omega) dx \tag{2.20}$$

for every sequence (w_ε) for which we have weak two-scale convergence to $w(x, \omega)$ in $L^2(G, \nu_\omega^\varepsilon)$. If (V_ε) is a bounded sequence of functions in $L^2(G, \nu_\omega^\varepsilon)^N$ with $V_\varepsilon = (V_\varepsilon^i)$, $i = 1, \dots, N$, then we say that V_ε two-scale converges strongly to a limit $V = (V^i) \in L^2(G \times X, dx \times dF(\omega))^N$, and we write $V_\varepsilon \xrightarrow{2:e} V$ if $V_\varepsilon^i \xrightarrow{2:e} V^i$ for every $i = 1, \dots, N$.

Now, we want to incorporate a more general class of test functions in Definition 2.3. Namely, the set \mathcal{B}_2 in the following definition.

DEFINITION 2.5. We say that a function $v : G \times X \rightarrow \mathbb{R}$ belongs to the set $\mathcal{B}(\mathbb{F})$ if:

- (a) the function $x \mapsto v(x, \omega)$ is continuous for μ -almost every $\omega \in X$;
- (b) the function $\omega \mapsto v(x, \omega)$ is \mathcal{F} -measurable for every $x \in G$;
- (c) the function $\omega \mapsto \sup_{x \in G} |v(x, \omega)|$ is in $L^1(X, \mathbb{F})$.

We say that a function $v : G \times X \rightarrow \mathbb{R}$ belongs to the set $\mathcal{B}_2(\mathbb{F})$ if $v \in \mathcal{B}(\mathbb{F})$ and $\omega \mapsto \sup_{x \in G} |v(x, \omega)| \in L^2(X, \mathbb{F})$.

To this aim, the ergodic theorem (Theorem 2.2) must be extended as follows.

THEOREM 2.3. Let $v \in \mathcal{B}(\mathbb{F})$. Then, almost surely in X ,

$$\lim_{\varepsilon \rightarrow 0} \int_G v(x, T_{x'/\varepsilon} \tilde{\omega}) d\nu_{\tilde{\omega}}^\varepsilon(x) = \int_{G \times X} v(x, \omega) d\mathbb{F}(\omega) dx. \tag{2.21}$$

Proof. Let $\omega \in X$ and set

$$b_\omega(x, z) = v(x, T_z \omega), \quad x \in G, \quad z \in \mathbb{R}^{N-1}.$$

It is clear by assertions (a) and (b) satisfied by the function v and the measurability of the map $z \in (\mathbb{R}^{N-1}, \mathcal{B}) \mapsto T_z \omega \in (X, \mathcal{F})$ that $b_\omega(\cdot, z)$ is continuous for almost every $z \in \mathbb{R}^{N-1}$ and $b_\omega(x, \cdot)$ is measurable for every $x \in G$, which asserts that the function b_ω is of Caratheodory type and the measurability of the function $x \mapsto b_\omega(x, x'/\varepsilon)$ is then ensured. Now, we divide the remaining of the proof into two steps.

Step 1. We prove (2.21) for step functions. Let $\square = [0, 1]^N$ and fix an integer $n \in \mathbb{N}$. For every $k \in \mathbb{Z}^N$, we set $\square_{n,k} = 1/n(\square + k)$ and we consider the following partition of \mathbb{R}^N :

$$\mathbb{R}^N = \bigsqcup_{k \in \mathbb{Z}^N} \square_{n,k}. \tag{2.22}$$

Let $v_n : G \times X \rightarrow \mathbb{R}$ be a step function defined by

$$v_n(x, \omega) = \sum_{\{k: \square_{n,k} \subset G\}} v(x_k, \omega) \mathcal{X}_k(x),$$

where x_k is an arbitrary point in $\square_{n,k}$ and $\mathcal{X}_k(x) = \mathcal{X}_{\square_{n,k}}(x)$. Then, for every $\tilde{\omega} \in \tilde{X}$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_G v_n(x, T_{x'/\varepsilon} \tilde{\omega}) d\nu_{\tilde{\omega}}^\varepsilon(x) = \int_{G \times X} v_n(x, \omega) d\mathbb{F}(\omega) dx. \tag{2.23}$$

Indeed, the condition (c) which v satisfies implies that $v(x_k, \cdot) \in L^1(X, \mathbb{F})$ for each $x_k \in G$. So, by the change of scale $z = x'/\varepsilon$ and the ergodic theorem (Theorem 2.2),

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\square_{n,k}} v_n(x, T_{x'/\varepsilon} \tilde{\omega}) d\nu_{\tilde{\omega}}^\varepsilon(x) &= |\square_{n,k}^N| \lim_{\varepsilon \rightarrow 0} \int_{\square'_{n,k}} v(x_k, T_{x'/\varepsilon} \tilde{\omega}) d\tau_{\tilde{\omega}}^\varepsilon(x') \\ &= |\square_{n,k}| \lim_{\varepsilon \rightarrow 0} \frac{1}{|(1/\varepsilon)\square'_{n,k}|} \int_{(1/\varepsilon)\square'_{n,k}} v(x_k, T_z \tilde{\omega}) d\tau_{\tilde{\omega}}(z) \\ &= |\square_{n,k}| \int_X v(x_k, \omega) d\mathbb{F}(\omega) \\ &= \int_{\square_{n,k}} \int_X v_n(x, \omega) d\mathbb{F}(\omega) dx, \end{aligned} \tag{2.24}$$

where by notation, if $A \subset \mathbb{R}^N$,

$$A' = \{x' \in \mathbb{R}^{N-1} : (x', x_N) \in A\},$$

$$A^N = \{x_N \in \mathbb{R} : (x', x_N) \in A\}.$$

We get (2.23) by summing in (2.24) over the set of numbers $\{k : \square_{n,k} \subset G\}$ (which is finite).

Step 2. We write

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_G v(x, T_{x'/\varepsilon} \tilde{\omega}) \, d\nu_{\tilde{\omega}}^\varepsilon(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_G (v(x, T_{x'/\varepsilon} \tilde{\omega}) - v_n(x, T_{x'/\varepsilon} \tilde{\omega})) \, d\nu_{\tilde{\omega}}^\varepsilon(x) \\ & \quad + \lim_{\varepsilon \rightarrow 0} \int_G v_n(x, T_{x'/\varepsilon} \tilde{\omega}) \, d\nu_{\tilde{\omega}}^\varepsilon(x) - \int_{G \times X} v_n(x, \omega) \, dF(\omega) \, dx \\ & \quad + \int_{G \times X} (v_n(x, \omega) - v(x, \omega)) \, dF(\omega) \, dx \\ & \quad + \int_{G \times X} v(x, \omega) \, dF(\omega) \, dx = L_1^n + L_2^n + L_3^n + L_4, \end{aligned} \tag{2.25}$$

where, for each i , L_i^n and L_4 are the terms corresponding to each line of the right-hand side of (2.25). According to Step 1,

$$L_2^n = 0. \tag{2.26}$$

In what concerns the terms L_1^n and L_3^n , let us define a function $C^n : X \rightarrow \mathbb{R}$ by

$$C^n(\omega) = \sup_{x \in G} |v(x, \omega) - v_n(x, \omega)|.$$

Since the function $v(x, \omega)$ is continuous in x for almost every $\omega \in X$, $C^n(\omega) \rightarrow 0$ for almost every $\omega \in X$. Moreover, $C^n(\omega) \leq 2 \sup_{x \in G} |v(x, \omega)| \in L^1(X, F)$. So, applying the Lebesgue dominated convergence theorem,

$$C^n \rightarrow 0 \text{ strongly in } L^1(X, F).$$

On the one hand, we have

$$|L_3^n| \leq |G| \int_X C^n(\omega) \, dF(\omega) \rightarrow 0 \tag{2.27}$$

and, on the other, like in (2.24), by the ergodic theorem (Theorem 2.2),

$$\begin{aligned} |L_1^n| &\leq \limsup_{\varepsilon} \int_G C^n(T_{x'/\varepsilon} \tilde{\omega}) \, d\nu_{\tilde{\omega}}^\varepsilon(x) \\ &= |G| \int_X C^n(\omega) \, dF(\omega) \rightarrow 0, \end{aligned} \tag{2.28}$$

(2.21) is deduced by gathering (2.25)–(2.28). □

REMARK 2.5. If $v \in \mathcal{B}(F)$, then, for every $\varphi \in C_0^\infty(G)$ and $b \in C^1(X)$, the function $(x, \omega) \mapsto v(x, \omega) \cdot \varphi(x) \cdot b(\omega) \in \mathcal{B}(F)$. So, Theorem 2.3 implies that almost surely in X ,

$$\lim_{\varepsilon \rightarrow 0} \int_G v(x, T_{x'/\varepsilon} \tilde{\omega}) \cdot \varphi(x) \cdot b(T_{x'/\varepsilon} \tilde{\omega}) \, d\nu_{\tilde{\omega}}^\varepsilon(x) = \int_{G \times X} v(x, \omega) \cdot \varphi(x) \cdot b(\omega) \, dF(\omega) \, dx \tag{2.29}$$

and then the sequence $v(x, T_{x'/\varepsilon} \tilde{\omega})$ weakly two-scale converges to v .

A consequence of the strong two-scale convergence is the convergence of the norms

$$\lim_{\varepsilon \rightarrow 0} \int_G |v_\varepsilon(x)|^2 \, d\nu_{\tilde{\omega}}^\varepsilon(x) = \int_{G \times X} |v(x, \omega)|^2 \, dF(\omega) \, dx.$$

In fact, this relation can be used instead of (2.20) in the definition of strong two-scale convergence. This is shown by the following theorem.

THEOREM 2.4. *The weak two-scale convergence of the sequence (v_ε) in $L^2(G, \nu_\omega^\varepsilon)$ to $v \in L^2(G \times X)$ together with*

$$\lim_{\varepsilon \rightarrow 0} \int_G |v_\varepsilon(x)|^2 d\nu_\omega^\varepsilon(x) = \int_{G \times X} |v(x, \omega)|^2 dF(\omega) dx \quad (2.30)$$

is equivalent to strong two-scale convergence of (v_ε) to v .

The proof of this theorem is similar to that of [20, Theorem 5], and uses the following theorem.

THEOREM 2.5. *Let (v_ε) be a sequence in $L^2(G, \nu_\omega^\varepsilon)$ which two-scale converges weakly to $v \in L^2(G \times X)$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{L^2(G, \nu_\omega^\varepsilon)} \geq \|v\|_{L^2(G \times X)}.$$

Proof. Let (ϕ_m) be a sequence in \mathcal{R} (\mathcal{R} defined as in (2.19)) such that ϕ_m converges to v strongly in $L^2(G \times X)$ (by density of \mathcal{R} in $L^2(G \times X)$). The Young inequality for real numbers a and b states that $ab \leq |a|^2/2 + |b|^2/2$, which implies that

$$\int_G |v_\varepsilon|^2 d\nu_\omega^\varepsilon(x) \geq 2 \int_G v_\varepsilon(x) \cdot \phi_m(x, T_{x'/\varepsilon}\tilde{\omega}) d\nu_\omega^\varepsilon(x) - \int_G |\phi_m(x, T_{x'/\varepsilon}\tilde{\omega})|^2 d\nu_\omega^\varepsilon(x).$$

By passing to the limit in ε , we get

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{L^2(G, \nu_\omega^\varepsilon)}^2 &\geq 2 \int_{G \times X} v(x, \omega) \cdot \phi_m(x, \omega) dF(\omega) dx \\ &\quad - \int_{G \times X} |\phi_m(x, \omega)|^2 dF(\omega) dx. \end{aligned}$$

Hence, by passing to the limit in m , we obtain

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{L^2(G, \nu_\omega^\varepsilon)}^2 &\geq 2 \int_{G \times X} |v(x, \omega)|^2 dF(\omega) dx - \int_{G \times X} |v(x, \omega)|^2 dF(\omega) dx \\ &= \|v\|_{L^2(G \times X)}^2. \quad \square \end{aligned}$$

In the next proof and also in the others throughout this paper, C will stand for any constant independent of ε and it may be different from line to line.

Proof of Theorem 2.4. (i) We start by proving that weak two-scale convergence together with (2.30) imply strong two-scale convergence. Let (ϕ_m) be a sequence in \mathcal{R} such that ϕ_m converges to v strongly in $L^2(G \times X)$ and (w_ε) be a sequence in $L^2(G, \nu_\omega^\varepsilon)$ which two-scale converges weakly to a function $w \in L^2(G \times X)$. Then

$$\lim_{m \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int_G \phi_m(x, T_{x'/\varepsilon}\tilde{\omega}) \cdot w_\varepsilon(x) d\nu_\omega^\varepsilon(x) = \int_{G \times X} v(x, \omega) \cdot w(x, \omega) dF(\omega) dx. \quad (2.31)$$

However,

$$\begin{aligned} & \left| \int_G v_\varepsilon(x) \cdot w_\varepsilon(x) \, d\nu_\omega^\varepsilon(x) - \int_{G \times X} v(x, \omega) \cdot w(x, \omega) \, dF(\omega) \, dx \right| \\ & \leq \left| \int_G [v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})] \cdot w_\varepsilon(x) \, d\nu_\omega^\varepsilon(x) \right| \\ & \quad + \left| \int_G \phi_m(x, T_{x'/\varepsilon}\tilde{\omega}) \cdot w_\varepsilon(x) \, d\nu_\omega^\varepsilon(x) \right. \\ & \quad \left. - \int_{G \times X} v(x, \omega) \cdot w(x, \omega) \, dF(\omega) \, dx \right|. \end{aligned} \tag{2.32}$$

Equations (2.31) and (2.32) imply that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left| \int_G v_\varepsilon(x) \cdot w_\varepsilon(x) \, d\nu_\omega^\varepsilon(x) - \int_{G \times X} v(x, \omega) \cdot w(x, \omega) \, dF(\omega) \, dx \right| \\ & \leq \limsup_{m \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_G [v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})] \cdot w_\varepsilon(x) \, d\nu_\omega^\varepsilon(x) \right|. \end{aligned} \tag{2.33}$$

By this last result, it follows that to get (2.20) it is sufficient to prove that

$$\limsup_{m \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_G [v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})] \cdot w_\varepsilon(x) \, d\nu_\omega^\varepsilon(x) \right| = 0. \tag{2.34}$$

The Hölder inequality and the fact that each weakly two-scale convergent sequence is bounded imply

$$\begin{aligned} & \left| \int_G [v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})] \cdot w_\varepsilon(x) \, d\nu_\omega^\varepsilon(x) \right| \\ & \leq \left(\int_G |v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})|^2 \, d\nu_\omega^\varepsilon(x) \right)^{1/2} \times \left(\int_G |w_\varepsilon(x)|^2 \, d\nu_\omega^\varepsilon(x) \right)^{1/2} \\ & \leq C \|v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})\|_{L^2(G, \nu_\omega^\varepsilon)}. \end{aligned} \tag{2.35}$$

The triangle inequality gives

$$\|v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})\|^2 \leq 2\|v_\varepsilon\|^2 + 2\|\phi_m(x, T_{x'/\varepsilon}\tilde{\omega})\|^2 - \|v_\varepsilon(x) + \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})\|^2,$$

where all norms are the usual norm in $L^2(G, \nu_\omega^\varepsilon)$. However, since $\phi_m \in \mathcal{B}(F)$ and by Remark 2.5,

$$\phi_m(x, T_{x'/\varepsilon}\tilde{\omega}) \stackrel{2-\varepsilon}{\rightharpoonup} \phi_m.$$

This, together with Remark 2.4, yield

$$\lim_{\varepsilon \rightarrow 0} \|\phi_m(x, T_{x'/\varepsilon}\tilde{\omega})\|_{L^2(G, \nu_\omega^\varepsilon)} = \|\phi_m(x, \omega)\|_{L^2(G \times X)}. \tag{2.36}$$

Therefore, thanks to (2.30), (2.36) and Theorem 2.5 applied to the sequence $v_\varepsilon(x) + \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})$, which two-scale converges weakly to $v + \phi_m$, it follows that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \|v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})\|_{L^2(G, \nu_\omega^\varepsilon)}^2 \\ & \leq 2\|v\|_{L^2(G \times X)}^2 + 2\|\phi_m\|_{L^2(G \times X)}^2 - \|v + \phi_m\|_{L^2(G \times X)}^2. \end{aligned} \tag{2.37}$$

Passing to the limit $m \rightarrow \infty$ in (2.37) and taking into account that ϕ_m converges to v strongly in $L^2(G \times X)$, we obtain

$$\limsup_{m \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \|v_\varepsilon(x) - \phi_m(x, T_{x'/\varepsilon}\tilde{\omega})\|^2 = 0. \tag{2.38}$$

We deduce (2.34) by letting $\varepsilon \rightarrow 0$ and $m \rightarrow \infty$ in (2.35) and making use of (2.38).

(ii) Conversely, if a sequence v_ε two-scale converges strongly to $v(x, \omega)$, then taking $w_\varepsilon(x) = \varphi(x) \cdot b(T_{x'/\varepsilon}\tilde{\omega})$ and $w(x, \omega) = \varphi(x) \cdot b(\omega)$ in (2.20), where $\varphi \in C_0^\infty(G)$ and $b \in C^1(X)$, we see

that v_ε also weakly two-scale converges to v . Equation (2.30) follows directly from (2.20) by taking $v_\varepsilon = w_\varepsilon$ and $v = w$, which ends the proof of Theorem 2.4. \square

Let us recall our subject, that is, to include the set $\mathcal{B}_2(\mathbb{F})$ as a space of test functions in Definition 2.3. This is done through the following corollary.

COROLLARY 2.1. *If a sequence (v_ε) in $L^2(G, \nu_\omega^\varepsilon)$ two-scale converges weakly to $v \in L^2(G \times X)$, then*

$$\lim_{\varepsilon \rightarrow 0} \int_G v_\varepsilon(x) \cdot \phi(x, T_{x'/\varepsilon} \tilde{\omega}) d\nu_\omega^\varepsilon(x) = \int_{G \times X} v(x, \omega) \cdot \phi(x, \omega) d\mathbb{F}(\omega) dx \tag{2.39}$$

for all $\phi \in \mathcal{B}_2(\mathbb{F})$.

Proof. If $\phi \in \mathcal{B}_2(\mathbb{F})$, then $|\phi| \in \mathcal{B}_2(\mathbb{F})$, and hence $|\phi|^2 \in \mathcal{B}(\mathbb{F})$. So, by Theorem 2.3,

$$\lim_{\varepsilon \rightarrow 0} \int_G |\phi(x, T_{x'/\varepsilon} \tilde{\omega})|^2 d\nu_\omega^\varepsilon(x) = \int_{G \times X} |\phi(x, \omega)|^2 d\mathbb{F}(\omega) dx. \tag{2.40}$$

Moreover, Remark 2.5 implies that $\phi(x, T_{x'/\varepsilon} \tilde{\omega})$ two-scale converges weakly to ϕ . Hence, taking into account this last result together with (2.40) and applying Theorem 2.4, (2.39) follows. \square

Below are some important properties of stochastic two-scale convergence with respect to random measures collected from [28].

PROPOSITION 2.1. (i) *For every bounded sequence (v_ε) in $L^2(G, \nu_\omega^\varepsilon)$, there exist a subsequence and a function $v \in L^2(G \times X)$ such that the subsequence two-scale converges to v .*

(i) *Let S be a closed subset of ∂G with Lipschitz boundary and let (v_ε) be a sequence in $H^1(G, S, \nu_\omega^\varepsilon)$ such that*

$$\|v_\varepsilon\|_{L^2(G, \nu_\omega^\varepsilon)} \leq C(\tilde{\omega}), \quad \|\nabla v_\varepsilon\|_{L^2(G, \nu_\omega^\varepsilon)^N} \leq C(\tilde{\omega}). \tag{2.41}$$

Then, for a subsequence

$$v_\varepsilon \xrightarrow{2-\text{e}} v_0(x), \tag{2.42}$$

$$\nabla v_\varepsilon \xrightarrow{2-\text{e}} \nabla v_0(x) + V_1(x, \omega), \tag{2.43}$$

where v_0 belongs to the space

$$H^1(G, S) := \{v \in H^1(G) : v = 0 \text{ on } S\},$$

$V_1 = (v_1, 0)$ with $v_1 \in L^2(G; L^2_{\text{pot}}(X, \mathbb{F}))$, which is the space of measurable functions $u : x \in G \rightarrow u(x) \in L^2_{\text{pot}}(X, \mathbb{F})$ such that

$$\|u(x)\|_{L^2(X, \mathbb{F})} \in L^2(G), \tag{2.44}$$

where $L^2_{\text{pot}}(X, \mathbb{F})$ is the space given in Definition 2.2 for $d = N - 1$.

REMARK 2.6. We may also define $H^1(G, S)$ as the closure of $\mathcal{C}_0^\infty(\overline{G} \setminus S)$ in $H^1(G)$.

REMARK 2.7. $L^2(G; L^2_{\text{pot}}(X))$ is a Banach space endowed with the norm

$$\|u\|_{L^2(G; L^2_{\text{pot}}(X, \mathbb{F}))} = \left(\int_\Omega \|u(x)\|_{L^2(X, \mathbb{F})}^2 dx \right)^{\frac{1}{2}}.$$

REMARK 2.8. Assertion (ii) of this proposition is nothing else than (5.12) and (5.13) in [28, Corollary 5.1], which we can use also here since it was proved in [28, Section 8] that the measure $d\tau_\omega(z) = \sigma(T_z\omega) dz$ is non-degenerate. Let us remark also that the structure of the function V_1 comes from the fact that the dynamical system T_z is independent of the variable x_N .

3. Main result

For a fixed $\varepsilon > 0$ and $\omega \in X$, let $\mu_\omega^\varepsilon, \lambda_\omega^\varepsilon$ be the measures defined by

$$d\mu_\omega^\varepsilon(x') = \rho(T_{x'/\varepsilon}\omega) dx', \quad x' \in \mathbb{R}^{N-1}, \tag{3.1}$$

$$d\lambda_\omega^\varepsilon(x) = d\mu_\omega^\varepsilon(x') \otimes dx_N, \quad x = (x', x_N) \in \mathbb{R}^N, \tag{3.2}$$

with ρ satisfying (1.4).

In the following, a weak solution of problem (1.11) is a solution of the variational formulation

$$\left\{ \begin{array}{l} \text{find } u_\varepsilon \in H^1(\Omega, \Gamma^{\text{lat}}, \lambda_\omega^\varepsilon) \text{ such that} \\ \int_\Omega a_\omega^\varepsilon(x', \nabla_\varepsilon u_\varepsilon) \cdot \nabla_\varepsilon v dx + \int_\Omega u_\varepsilon \cdot v d\lambda_\omega^\varepsilon(x) = \int_\Omega f \cdot v dx, \quad \forall v \in H^1(\Omega, \Gamma^{\text{lat}}, \lambda_\omega^\varepsilon), \end{array} \right. \tag{3.3}$$

where, for a fixed $\varepsilon > 0$ and $\omega \in X$, the function a_ω^ε is defined for every $z \in \mathbb{R}^{N-1}$ and $\xi \in \mathbb{R}^N$ by

$$a_\omega^\varepsilon(z, \xi) = a(T_{z/\varepsilon}\omega, \xi).$$

REMARK 3.1. As $H^1(\Omega, \Gamma^{\text{lat}}, \lambda_\omega^\varepsilon)$ is a reflexive Banach space (see Remark 2.1) and under hypotheses (H_1) – (H_4) we can apply the Minty–Browder theorem (see [22, Theorem 26.A, p. 557]) to prove existence and uniqueness of the solution $u_\varepsilon \in H^1(\Omega, \Gamma^{\text{lat}}, \lambda_\omega^\varepsilon)$ of problem (3.3) for each $\varepsilon > 0$ and $\omega \in X$ (see also [20, proof of Theorem 13]).

Before the statement of the main theorem, it will be convenient to notice at first that we shall identify the space $H_0^1(\Sigma)$ with the space of functions $u \in H^1(\Omega, \Gamma^{\text{lat}})$ such that $\partial u / \partial x_N = 0$, and also that all the convergence results in the stochastic sense contained in the theorem below are given with respect to the measure $\lambda_\omega^\varepsilon$.

The main result is stated as follows.

THEOREM 3.1. Assume hypotheses (H_1) – (H_4) and let u_ε be the weak solution of problem (1.11). Then, μ -almost surely in X , as $\varepsilon \rightarrow 0$,

$$\frac{\partial u_\varepsilon}{\partial x_N} \xrightarrow{2.e} 0 \tag{3.4}$$

and, up to a subsequence,

$$u_\varepsilon \xrightarrow{2.e} u_0, \tag{3.5}$$

$$\nabla_{x'} u_\varepsilon \xrightarrow{2.e} \nabla_{x'} u_0(x') + u_1(x, \omega), \tag{3.6}$$

$$\frac{1}{h(\varepsilon)} \frac{\partial u_\varepsilon}{\partial x_N} \xrightarrow{2.e} 0, \tag{3.7}$$

where $u_0 \in H_0^1(\Sigma)$ is the weak solution of the homogenized problem

$$\begin{cases} -\text{div}_{x'} B(\nabla_{x'} u_0) + 2P(X)u_0 = g & \text{in } \Sigma, \\ u_0 = 0 & \text{on } \partial\Sigma. \end{cases} \tag{3.8}$$

Here B is given by

$$B(\nabla_{x'} u_0(x')) = \int_{]-1,1[\times X} a'_0(\omega, \nabla_{x'} u_0(x') + u_1(x, \omega)) d\mu(\omega) dx_N, \tag{3.9}$$

with, by notation,

$$a'_0(\omega, \xi) := a'(\omega, \xi, 0), \quad \omega \in X, \xi \in \mathbb{R}^{N-1}$$

and $u_1 \in L^2(\Omega; L^2_{\text{pot}}(X, P))$ is the unique solution of the following equation:

$$\int_X \left(\int_{]-1,1[} a'_0(\omega, \nabla_{x'} u_0(x') + u_1(x, \omega)) dx_N \right) \cdot \phi(\omega) d\mu(\omega) = 0, \quad \forall \phi \in L^2_{\text{pot}}(X, P). \tag{3.10}$$

The function g is defined on Σ as follows:

$$g(x') = \int_{]-1,1[} f(x) dx_N. \tag{3.11}$$

REMARK 3.2. Under hypotheses (H_1) – (H_4) on the function a and by applying the Browder–Minty theorem, equation (3.10) admits a unique solution u_1 . Hence, the function B is well defined. Its properties and the uniqueness of the solution of (3.8) can be derived as in [20] (see Theorem 17), and so we have convergence of the whole sequence.

4. Proof of Theorem 3.1

It will be given through several steps.

4.1. Step 1. The a priori estimates

For a fixed $\varepsilon > 0$ and $\omega \in X$, define the measures $[\mu_\omega^\varepsilon]^{-1}$, $[\lambda_\omega^\varepsilon]^{-1}$ as follows:

$$d[\mu_\omega^\varepsilon]^{-1}(x') = \rho^{-1}(T_{x'/\varepsilon}\omega) dx', \quad x' \in \mathbb{R}^{N-1}, \tag{4.1}$$

$$d[\lambda_\omega^\varepsilon]^{-1}(x) = d[\mu_\omega^\varepsilon]^{-1}(x') \otimes dx_N, \quad x = (x', x_N) \in \mathbb{R}^N, \tag{4.2}$$

with ρ^{-1} defined as in (1.4), and also recall the measure $\lambda_\omega^\varepsilon$ given in (3.2). The main estimates on the solution of (3.3) are contained in the following proposition.

PROPOSITION 4.1. Assume (H_1) – (H_4) and let u_ε be the solution of (3.3). Then, almost surely in X ,

$$\|u_\varepsilon\|_{H^1(\Omega, \Gamma^{\text{lat}}, \lambda_\omega^\varepsilon)} \leq C, \tag{4.3}$$

$$\frac{1}{h(\varepsilon)} \left\| \frac{\partial u_\varepsilon}{\partial x_N} \right\|_{L^2(\Omega, \lambda_\omega^\varepsilon)} \leq C, \tag{4.4}$$

$$\|a_\omega^\varepsilon(\cdot, \nabla_\varepsilon u_\varepsilon)\|_{[L^2(\Omega, [\lambda_\omega^\varepsilon]^{-1})]^N} \leq C. \tag{4.5}$$

Proof. We start by proving estimates (4.3) and (4.4). If we take $v = u_\varepsilon$ in (3.3) and use hypothesis (1.10), we get

$$\begin{aligned} \int_\Omega [|\nabla_\varepsilon u_\varepsilon|^2 + |u_\varepsilon|^2] d\lambda_\omega^\varepsilon(x) &\leq \int_\Omega a_\omega^\varepsilon(x', \nabla_\varepsilon u_\varepsilon) \cdot \nabla_\varepsilon u_\varepsilon dx + \int_\Omega |u_\varepsilon|^2 d\lambda_\omega^\varepsilon(x) + c_4 \int_\Omega d\lambda_\omega^\varepsilon(x) \\ &= \int_\Omega f \cdot u_\varepsilon dx + c_4 \int_\Omega d\lambda_\omega^\varepsilon(x). \end{aligned} \tag{4.6}$$

The Hölder inequality gives

$$\int_\Omega f \cdot u_\varepsilon dx \leq \|f\|_{L^\infty(\Omega)} \cdot \|u_\varepsilon\|_{L^1(\Omega)}. \tag{4.7}$$

Applying it again this time with respect to the measure $\lambda_\omega^\varepsilon$,

$$\int_\Omega |u_\varepsilon| dx \leq \left(\int_\Omega |u_\varepsilon|^2 d\lambda_\omega^\varepsilon \right)^{1/2} \cdot \left(\int_\Omega \rho^{-1}(T_{x'/\varepsilon}\tilde{\omega}) dx \right)^{1/2}. \tag{4.8}$$

By the change of scale $z = x'/\varepsilon$ and the Birkhoff theorem (Theorem 2.1) applied to the function ρ^{-1} , we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho^{-1}(T_{x'/\varepsilon} \tilde{\omega}) \, dx &= 2 \lim_{\varepsilon \rightarrow 0} \int_{\Sigma} \rho^{-1}(T_{x'/\varepsilon} \tilde{\omega}) \, dx' \\ &= 2|\Sigma| \lim_{\varepsilon \rightarrow 0} \frac{1}{|1/\varepsilon\Sigma|} \int_{1/\varepsilon\Sigma} \rho^{-1}(T_z \tilde{\omega}) \, dz \\ &= 2|\Sigma| \int_X \rho^{-1}(\omega) \, d\mu(\omega) < +\infty. \end{aligned} \tag{4.9}$$

Making use of (4.7)–(4.9), it follows that

$$\int_{\Omega} f \cdot u_{\varepsilon} \, dx \leq C \cdot \|u_{\varepsilon}\|_{L^2(\Omega, \lambda_{\tilde{\omega}}^{\varepsilon})}. \tag{4.10}$$

As in (4.9), we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} d\lambda_{\tilde{\omega}}^{\varepsilon}(x) &= 2 \lim_{\varepsilon \rightarrow 0} \int_{\Sigma} \rho(T_{x'/\varepsilon} \tilde{\omega}) \, dx' \\ &= 2|\Sigma| \int_X \rho(\omega) \, d\mu(\omega) < +\infty. \end{aligned} \tag{4.11}$$

Hence, (4.6), (4.10) and (4.11) provide

$$\int_{\Omega} [|\nabla_{\varepsilon} u_{\varepsilon}|^2 + |u_{\varepsilon}|^2] \, d\lambda_{\tilde{\omega}}^{\varepsilon}(x) \leq C(\|u_{\varepsilon}\|_{L^2(\Omega, \lambda_{\tilde{\omega}}^{\varepsilon})} + 1), \tag{4.12}$$

which implies necessarily (4.3) and (4.4). It remains now to estimate the function

$$A_{\varepsilon}(x) = a_{\tilde{\omega}}^{\varepsilon}(x', \nabla_{\varepsilon} u_{\varepsilon}(x)).$$

By definition, $d[\lambda_{\tilde{\omega}}^{\varepsilon}]^{-1}(x) = \rho^{-1}(T_{x'/\varepsilon} \tilde{\omega}) \, dx$; hence, from (1.9), (4.3) and (4.4), it is straightforward that

$$\begin{aligned} \int_{\Omega} |A_{\varepsilon}(x)|^2 \, d[\lambda_{\tilde{\omega}}^{\varepsilon}]^{-1}(x) &= \int_{\Omega} |A_{\varepsilon}(x)|^2 \rho^{-1}(T_{x'/\varepsilon} \tilde{\omega}) \, dx \\ &\leq C \int_{\Omega} (1 + |\nabla_{\varepsilon} u_{\varepsilon}|^2) \rho^2(T_{x'/\varepsilon} \tilde{\omega}) \rho^{-1}(T_{x'/\varepsilon} \tilde{\omega}) \, dx \\ &= C \int_{\Omega} (1 + |\nabla_{\varepsilon} u_{\varepsilon}|^2) \, d\lambda_{\tilde{\omega}}^{\varepsilon}(x), \\ &\leq C, \end{aligned}$$

which gives (4.5) and ends the proof of this proposition. □

4.2. Step 2. Compactness results

In the following, we suppose that the convergence results hold true for the same subsequence (otherwise we pass to a smaller one), and to simplify we use the same notation for the sequence and its subsequence.

Firstly, let us prove (3.4)–(3.7). We proceed by proving (3.4). Let u_{ε} be the solution of (3.3). Then, according to (4.4) and the Hölder inequality, for every function w_{ε} which two-scale converges weakly, we have

$$\begin{aligned} \left| \int_{\Omega} \frac{\partial u_{\varepsilon}}{\partial x_N}(x) \cdot w_{\varepsilon}(x) \, d\lambda_{\tilde{\omega}}^{\varepsilon}(x) \right| &\leq \left\| \frac{\partial u_{\varepsilon}}{\partial x_N} \right\|_{L^2(\Omega, \lambda_{\tilde{\omega}}^{\varepsilon})} \cdot \|w_{\varepsilon}\|_{L^2(\Omega, \lambda_{\tilde{\omega}}^{\varepsilon})} \\ &\leq \text{Ch}(\varepsilon). \end{aligned}$$

Hence, we obtain (3.4) by taking the limit as $\varepsilon \rightarrow 0$ in the above inequality (3.5) and (3.6) are straightforward consequences of (4.3) and assertion (ii) of Proposition 2.1 applied to $\nu_{\tilde{\omega}}^{\varepsilon} = \lambda_{\tilde{\omega}}^{\varepsilon}$

and $F = P$, but with the function u_0 dependent on x_N . To show the independence with respect to this variable, from (3.5) and applying the two-scale convergence definition to a test function $\varphi \in C_0^\infty(\Omega)$ and $b(\omega) = C$, where C is a constant independent of ω , we get

$$\begin{aligned} P(X) \int_{\Omega} u_0(x) \cdot \frac{\partial \varphi(x)}{\partial x_N} dx &= \int_{\Omega \times X} u_0(x) \cdot \frac{\partial \varphi(x)}{\partial x_N} dP(\omega) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \cdot \frac{\partial \varphi(x)}{\partial x_N} d\lambda_\omega^\varepsilon(x) \\ &= -\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{\partial u_\varepsilon(x)}{\partial x_N} \cdot \varphi(x) d\lambda_\omega^\varepsilon(x), \end{aligned} \tag{4.13}$$

with P defined as in (1.14). By the Hölder inequality and estimate (4.4), we obtain

$$\begin{aligned} \int_{\Omega} \left| \frac{\partial u_\varepsilon(x)}{\partial x_N} \cdot \varphi(x) \right| d\lambda_\omega^\varepsilon(x) &\leq \left(\int_{\Omega} \left| \frac{\partial u_\varepsilon(x)}{\partial x_N} \right|^2 d\lambda_\omega^\varepsilon(x) \right)^{1/2} \cdot \left(\int_{\Omega} |\varphi(x)|^2 d\lambda_\omega^\varepsilon(x) \right)^{1/2} \\ &\leq h(\varepsilon) \left(\int_{\Omega} |\varphi(x)|^2 d\lambda_\omega^\varepsilon(x) \right)^{1/2}. \end{aligned}$$

With a majoration of the function φ and use of (4.11), after letting $\varepsilon \rightarrow 0$ in the above inequality the right-hand side of (4.13) will be equal to zero, that is,

$$\int_{\Omega} u_0(x) \cdot \frac{\partial \varphi(x)}{\partial x_N} dx = 0 \tag{4.14}$$

for every $\varphi \in C_0^\infty(\Omega)$, which means that the function u_0 is independent of x_N . To show (3.7), let us set

$$v_\varepsilon = \frac{1}{h(\varepsilon)} \frac{\partial u_\varepsilon}{\partial x_N}. \tag{4.15}$$

By virtue of (4.4), the sequence v_ε is bounded in $L^2(\Omega, \lambda_\omega^\varepsilon)$. Hence, by assertion (i) of Proposition 2.1, up to a subsequence there exists a function $v_0 \in L^2(G \times X, dx \times dP(\omega))$ so that

$$v_\varepsilon \xrightarrow{2-\varepsilon} v_0(x, \omega) \quad \text{with respect to } \lambda_\omega^\varepsilon. \tag{4.16}$$

To complete the proof of (3.7), it remains to show that $v_0(x, \omega) = 0$, which is carried out in Step 5 (see the proof of (4.38)).

REMARK 4.1. It can also be proved that the function u_1 in (3.6) is independent of x_N , using (3.7) and [28, Theorem 2.2] applied to the measure $\mu_\omega(x') = \rho(T_{x'}\omega) dx'$. The details of the proof will be given in a forthcoming paper when dealing with $l \in]0, +\infty]$, l defined in (1.2).

Let us set

$$d[P]^{-1}(\omega) = \rho^{-1}(\omega) d\mu(\omega), \tag{4.17}$$

the function ρ^{-1} being defined as in (1.4).

PROPOSITION 4.2. Assume (H_1) – (H_4) and let u_ε be the solution of (3.3). Then, almost surely in X , up to a subsequence there exists a function $A = (A', A_N) \in L^2(G \times X, dx \times d[P]^{-1}(\omega))^N$ such that

$$a_\omega^\varepsilon(x', \nabla_\varepsilon u_\varepsilon(x)) \xrightarrow{2-\varepsilon} A(x, \omega) \quad \text{with respect to } [\lambda_\omega^\varepsilon]^{-1}. \tag{4.18}$$

Moreover, for almost every $x \in \Omega$ and almost every $\omega \in X$,

$$A_N(x, \omega) = 0, \tag{4.19}$$

and, for almost every $x' \in \Sigma$,

$$\int_{]-1,1[\times X} A'(x, \omega) \cdot b(\omega) \, d\mu(\omega) \, dx_N = 0, \quad \forall b \in L^2_{\text{pot}}(X, \mathbb{P}). \tag{4.20}$$

Before we proceed with the proof of this proposition, we need the following lemmas.

LEMMA 4.1 (Ciarlet [13, p. 37]). *Let $w \in L^p(\Omega)$, $p \geq 1$, be such that*

$$\int_{\Omega} w(x) \cdot \frac{\partial}{\partial x_N} v(x) \, dx = 0, \quad \forall v \in C^\infty(\bar{\Omega}) \text{ with } v = 0 \text{ on } \Gamma^{\text{lat}};$$

then $w = 0$.

Proof. Let $\varphi \in C^\infty_0(\Omega)$ and $v : \bar{\Omega} \rightarrow \mathbb{R}$ be the function defined by

$$v(x', x_N) := \int_{-1}^{x_N} \varphi(x', t) \, dt.$$

Then $v \in C^\infty(\bar{\Omega})$ and $v = 0$ on Γ^{lat} . Hence,

$$\int_G w(x) \cdot \varphi(x) \, dx = \int_{\Omega} w(x) \cdot \frac{\partial}{\partial x_N} v(x) \, dx = 0,$$

and consequently $w = 0$. □

LEMMA 4.2. *If a function v belongs to $\mathcal{B}_2(\mathbb{P})$ (cf. Definition 2.5), then the function $(x, \omega) \mapsto v(x, \omega) \cdot \rho(\omega)$ belongs to $\mathcal{B}_2(\mathbb{P}^{-1})$. Likewise, if v belongs to $\mathcal{B}_2(\mathbb{P}^{-1})$, then the function $(x, \omega) \mapsto v(x, \omega) \cdot \rho^{-1}(\omega)$ belongs to $\mathcal{B}_2(\mathbb{P})$.*

Proof. It is sufficient to prove the first part of this lemma. Let us set

$$w(x, \omega) = v(x, \omega) \cdot \rho(\omega).$$

Firstly, assertions (a) and (b) of Definition 2.5 are clear for the function w . Now, it remains to show that the function

$$g(\omega) = \sup_{x \in \Omega} |w(x, \omega)|$$

belongs to $L^2(X, \mathbb{P}^{-1})$, which is true because we have

$$\int_X g(\omega)^2 \, d\mathbb{P}^{-1}(\omega) = \int_X \sup_{x \in \Omega} |v(x, \omega)|^2 \, d\mathbb{P}(\omega).$$

□

Proof of Proposition 4.2. Let us denote

$$A_\varepsilon = (A'_\varepsilon, A_{\varepsilon N}) = a^\varepsilon_\omega(x', \nabla_\varepsilon u_\varepsilon(x)). \tag{4.21}$$

Estimation (4.5) and assertion (i) of Proposition 2.1 ensure the existence of a subsequence and a function A in $[L^2(G \times X, dx \times d[P]^{-1}(\omega))]^N$ such that (4.18) holds true. In order to prove (4.19), let $\phi \in C^\infty(\bar{\Omega})$ be such that $\phi = 0$ on Γ^{lat} , $b \in C^1(X)$ and let us take $v(x) = h(\varepsilon)\phi(x)b(T_{x'/\varepsilon}\tilde{\omega})$ as a test function in (3.3), which reads as follows:

$$\begin{aligned} & \int_{\Omega} A'_\varepsilon \cdot \left[h(\varepsilon)\nabla_{x'}\phi(x) \cdot b(T_{x'/\varepsilon}\tilde{\omega}) + \frac{h(\varepsilon)}{\varepsilon}\phi(x) \cdot \nabla_\omega b(T_{x'/\varepsilon}\tilde{\omega}) \right] dx \\ & + \int_{\Omega} A_{\varepsilon, N} \cdot \frac{\partial}{\partial x_N}\phi(x) \cdot b(T_{x'/\varepsilon}\tilde{\omega}) \, dx + h(\varepsilon) \int_{\Omega} u_\varepsilon(x) \cdot \phi(x) \cdot b(T_{x'/\varepsilon}\tilde{\omega}) \, d\lambda^\varepsilon_\omega(x) \\ & = h(\varepsilon) \int_{\Omega} f(x) \cdot \phi(x) \cdot b(T_{x'/\varepsilon}\tilde{\omega}) \, dx. \end{aligned} \tag{4.22}$$

Taking into account that $dx = \rho(T_{x'/\varepsilon}\tilde{\omega}) d[\lambda_{\tilde{\omega}}^\varepsilon]^{-1}(x)$ and recalling Lemma 4.2, Corollary 2.1, (3.5) and (4.18), we pass to the two-scale limit with respect to $[\lambda_{\tilde{\omega}}^\varepsilon]^{-1}$ in the first and second integral terms in the left-hand side of (4.22), and to the two-scale limit with respect to $\lambda_{\tilde{\omega}}^\varepsilon$ in the third integral term. For the integral term in the right-hand side of equation (4.22), we apply the Birkhoff theorem (Theorem 2.1). As a result, we obtain

$$\int_{\Omega \times X} A_N(x, \omega) \frac{\partial}{\partial x_N} \phi(x) \cdot b(\omega) d\mu(\omega) dx = 0$$

for every $b \in C^1(X)$. So, almost surely in X ,

$$\int_{\Omega} A_N(x, \omega) \frac{\partial}{\partial x_N} \phi(x) dx = 0$$

for every function $\phi \in C^\infty(\bar{\Omega})$ such that $\phi = 0$ on Γ^{lat} . Therefore, (4.19) follows in view of Lemma 4.1. As regards (4.20), we take $v(x) = \varepsilon \phi(x') b(T_{x'/\varepsilon}\tilde{\omega})$ as a test function in (3.3), where $\phi \in C_0^\infty(\Sigma)$ and $b \in C^1(X)$, which entails

$$\begin{aligned} & \int_{\Omega} A'_\varepsilon \cdot [\varepsilon \nabla_{x'} \phi(x') \cdot b(T_{x'/\varepsilon}\tilde{\omega}) + \phi(x') \cdot \nabla_\omega b(T_{x'/\varepsilon}\tilde{\omega})] dx \\ & + \varepsilon \int_{\Omega} u_\varepsilon(x) \cdot \phi(x') \cdot b(T_{x'/\varepsilon}\tilde{\omega}) d\lambda_{\tilde{\omega}}^\varepsilon(x) \\ & = \varepsilon \int_{\Omega} f(x) \cdot \phi(x') \cdot b(T_{x'/\varepsilon}\tilde{\omega}) dx. \end{aligned}$$

After letting $\varepsilon \rightarrow 0$ in the above equation as before, we get

$$\int_{\Omega \times X} A'(x, \omega) \cdot \phi(x') \cdot \nabla_\omega b(\omega) d\mu(\omega) dx = 0$$

for every function $\phi \in C_0^\infty(\Sigma)$. Consequently, almost everywhere in Σ ,

$$\int_{]-1,1[\times X} A'(x, \omega) \cdot \nabla_\omega b(\omega) d\mu(\omega) dx_N = 0$$

for every $b \in C^1(X)$. Hence, (4.20) is obtained by density of the set $\{\nabla_\omega b : b \in C^1(X)\}$ in $L^2_{\text{pot}}(X, P)$ (by definition).

4.3. Step 3. Convergence of energies

Let A_ε be defined in (4.21) and A be the function given in Proposition 4.2. Define the vector-valued function

$$U_0(x, \omega) = (\nabla_{x'} u_0(x') + u_1(x, \omega), v_0(x, \omega)), \quad x \in \Omega, \omega \in X \tag{4.23}$$

with v_0 defined as in (4.16). At this step, we prove that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon \cdot \nabla_\varepsilon u_\varepsilon dx + \int_{\Omega} u_\varepsilon^2 d\lambda_{\tilde{\omega}}^\varepsilon(x) \\ & = \int_{\Omega \times X} A(x, \omega) \cdot U_0(x, \omega) d\mu(\omega) dx + 2P(X) \int_{\Sigma} u_0(x')^2 dx'. \end{aligned} \tag{4.24}$$

To do so, we take $v = u_\varepsilon$ in (3.3) and we pass to the limit making use of (3.5) and the fact that

$$f(x) \cdot \rho^{-1}(T_{x'/\varepsilon}\tilde{\omega}) \xrightarrow{2.e} f(x) \cdot \rho^{-1}(\omega) \quad \text{with respect to } \lambda_{\tilde{\omega}}^\varepsilon$$

(we apply Theorem 2.4). Hence,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon \cdot \nabla_\varepsilon u_\varepsilon dx + \int_{\Omega} u_\varepsilon^2 d\lambda_{\tilde{\omega}}^\varepsilon(x) & = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} f(x) \cdot u_\varepsilon(x) dx \\ & = \int_{\Omega} f(x) \cdot u_0(x') dx. \end{aligned} \tag{4.25}$$

Let φ be a function in $C_0^\infty(\Sigma)$. If we take $v(x) = \varphi(x')$ in (3.3) and we pass to the two-scale limit as before, by virtue of (3.5) and (4.18) it follows that

$$\int_{\Omega} f(x) \cdot \varphi(x') \, dx = \int_{\Omega \times X} A'(x, \omega) \cdot \nabla \varphi(x') \, d\mu(\omega) \, dx + 2P(X) \int_{\Sigma} u_0(x') \cdot \varphi(x') \, dx'. \tag{4.26}$$

A density argument yields

$$\int_{\Omega} f(x) \cdot u_0(x') \, dx = \int_{\Omega \times X} A'(x, \omega) \cdot \nabla u_0(x') \, d\mu(\omega) \, dx + 2P(X) \int_{\Sigma} u_0(x')^2 \, dx'. \tag{4.27}$$

Now, taking into account (4.19) and (4.20), it turns out that

$$\int_{\Omega \times X} A'(x, \omega) \cdot \nabla u_0(x') \, d\mu(\omega) \, dx = \int_{\Omega \times X} A(x, \omega) \cdot U_0(x, \omega) \, d\mu(\omega) \, dx. \tag{4.28}$$

Applying (4.27) and (4.28) in (4.25) gives us the convergence of energies (4.24).

4.4. Step 4. Identification of the function A

This step aims at showing that

$$A(x, \omega) = a(\omega, U_0(x, \omega)) \quad \text{a.e. in } \Omega \times X, \tag{4.29}$$

and the tool will be the well-known Minti argument. Let $\eta(x, \omega) \in \mathcal{R}^N$, the space \mathcal{R} being defined by (2.19), and let us denote $\eta_\varepsilon(x) = \eta(x, T_{x'/\varepsilon}\tilde{\omega})$. Since $\mathcal{R} \subset \mathcal{B}_2(\mathbb{P})$, by Corollary 2.1 and Lemma 4.2,

$$\eta_\varepsilon(x) \xrightarrow{2\text{-}e} \eta(x, \omega) \quad \text{with respect to } \lambda_{\tilde{\omega}}^\varepsilon, \tag{4.30}$$

$$\eta_\varepsilon(x) \cdot \rho(T_{x'/\varepsilon}\tilde{\omega}) \xrightarrow{2\text{-}e} \eta(x, \omega) \cdot \rho(\omega) \quad \text{with respect to } [\lambda_{\tilde{\omega}}^\varepsilon]^{-1}. \tag{4.31}$$

According to the monotonicity condition (H_4) , we have

$$\int_{\Omega} [A_\varepsilon(x) - a_{\tilde{\omega}}^\varepsilon(x', \eta_\varepsilon)] \cdot (\nabla_\varepsilon u_\varepsilon - \eta_\varepsilon) \, dx \geq 0 \tag{4.32}$$

and equivalently

$$\int_{\Omega} A_\varepsilon \cdot \nabla_\varepsilon u_\varepsilon \, dx - \int_{\Omega} A_\varepsilon \cdot \eta_\varepsilon \, dx - \int_{\Omega} a_{\tilde{\omega}}^\varepsilon(x', \eta_\varepsilon) \cdot (\nabla_\varepsilon u_\varepsilon - \eta_\varepsilon) \, dx \geq 0. \tag{4.33}$$

Letting $\varepsilon \rightarrow 0$ in the second integral term of the left-hand side of (4.33) and by virtue of (4.18) and (4.31), we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon(x) \cdot \eta_\varepsilon(x) \, dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon(x) \cdot \eta_\varepsilon(x) \cdot \rho(T_{x'/\varepsilon}\tilde{\omega}) \, d[\lambda_{\tilde{\omega}}^\varepsilon]^{-1}(x) \\ &= \int_{\Omega \times X} A(x, \omega) \cdot \eta(x, \omega) \, d\mu(\omega) \, dx. \end{aligned} \tag{4.34}$$

On the other hand, we want to prove that the third integral term of the left-hand side of inequality (4.33) converges toward

$$\int_{\Omega \times X} a(\omega, \eta(x, \omega)) \cdot (U_0(x, \omega) - \eta(x, \omega)) \, d\mu(\omega) \, dx \tag{4.35}$$

as $\varepsilon \rightarrow 0$. To do so, set

$$v(x, \omega) = a(\omega, \eta(x, \omega)), \quad x \in \Omega, \omega \in X.$$

Then $v \in \mathcal{B}_2(\mathbb{P}^{-1})$. Indeed, the conditions (H_1) and (H_3) on a and the continuity of η provide immediately (a) and (b) in Definition 2.5. Moreover, since $a(\omega, \cdot)$ is continuous and satisfies (H_2) and $\eta(\cdot, \omega)$ is continuous on $\bar{\Omega}$ (it vanishes outside Ω), the $\sup_{x \in \Omega} |v(x, \omega)|$ is attained.

Therefore, making use of (1.9), there exists a point $x_0 \in \Omega$ such that

$$\sup_{x \in \Omega} |v(x, \omega)| = |v(x_0, \omega)| = |a(\omega, \eta(x_0, \omega))| \leq c_3 \rho(\omega) \cdot (1 + |\eta(x_0, \omega)|) \in L^2(X, P^{-1})$$

(since $\rho \in L^2(X, P^{-1})$ and η is bounded). So, $v \in \mathcal{B}_2(P^{-1})$ and hence, by Corollary 2.1 and Lemma 4.2,

$$a_{\tilde{\omega}}^\varepsilon(x', \eta_\varepsilon(x)) \cdot \rho^{-1}(T_{x'/\varepsilon}\tilde{\omega}) \xrightarrow{2\text{-}e} a(\omega, \eta(x, \omega)) \cdot \rho^{-1}(\omega) \quad \text{with respect to } \lambda_{\tilde{\omega}}^\varepsilon.$$

Since, by (3.6) and (4.16),

$$\nabla_\varepsilon u_\varepsilon \xrightarrow{2\text{-}e} U_0(x, \omega) \quad \text{with respect to } \lambda_{\tilde{\omega}}^\varepsilon,$$

the last two results together with (4.30) and the fact that $dx = \rho^{-1}(T_{x'/\varepsilon}\tilde{\omega}) d\lambda_{\tilde{\omega}}^\varepsilon(x)$ provide (4.35). However, as we know by (4.24) and Theorem 2.5 applied to the sequence (u_ε) , which by (3.5) weakly two-scale converges to u_0 with respect to $\lambda_{\tilde{\omega}}^\varepsilon$,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon \cdot \nabla_\varepsilon u_\varepsilon \, dx &\leq \int_{\Omega \times X} A(x, \omega) \cdot U_0(x, \omega) \, d\mu(\omega) \, dx \\ &\quad + 2P(X) \int_{\Sigma} u_0(x')^2 \, dx' - \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon^2 \, d\lambda_{\tilde{\omega}}^\varepsilon(x) \\ &\leq \int_{\Omega \times X} A(x, \omega) \cdot U_0(x, \omega) \, d\mu(\omega) \, dx. \end{aligned} \tag{4.36}$$

Now, if we pass to the limit superior in inequality (4.33) whenever $\varepsilon \rightarrow 0$ making use of (4.34)–(4.36), we get as a result

$$\int_{\Omega \times X} [A(x, \omega) - a(\omega, \eta(x, \omega))] \cdot (U_0(x, \omega) - \eta(x, \omega)) \, d\mu(\omega) \, dx \geq 0 \tag{4.37}$$

for every function $\eta(x, \omega) \in \mathcal{R}^N$. But, since $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ and $\mathcal{C}^1(X)$ is dense in $L^2(X, P)$, we have also the density of \mathcal{R} in $L^2(\Omega \times X)$. So, there exists a sequence of functions $(U_k)_k$ in \mathcal{R}^N such that $U_k \rightarrow U_0$ strongly in $[L^2(\Omega \times X)]^N$ as $k \rightarrow \infty$. If we take $\eta(x, \omega) = U_k(x, \omega) - t\psi(x, \omega)$ in (4.37), where $\psi(x, \omega) = \phi(x) \cdot b(\omega)$, $\phi \in [\mathcal{C}_0^\infty(\Omega)]^N$, $b \in \mathcal{C}^1(X)$ and $t > 0$, and, by the continuity of the function a with respect to its variable ξ (by hypothesis (H_3)), firstly we pass to the limit as $k \rightarrow +\infty$ in (4.37), then we divide the whole by t and we pass to the limit as $t \rightarrow 0$. This procedure leads to

$$\int_{\Omega \times X} [A(x, \omega) - a(\omega, U_0(x, \omega))] \cdot \phi(x) \cdot b(\omega) \, d\mu(\omega) \, dx \geq 0$$

for every $\phi \in [\mathcal{C}_0^\infty(\Omega)]^N$ and $b \in \mathcal{C}^1(X)$, which provides that $A(x, \omega) = a(\omega, U_0(x, \omega))$ almost everywhere in $\Omega \times X$ and so proves (4.29).

4.5. Step 5. Homogenized and auxiliary equations

To finish the proof of Theorem 3.1, it remains to verify the equations satisfied by u_0 and u_1 . At first, we end the proof of (3.7) by claiming that

$$v_0(x, \omega) = 0, \quad \text{a.e. } x \in \Omega \text{ and } \omega \in X. \tag{4.38}$$

Indeed, condition (H_2) implies that

$$a_N(\omega, \nabla u_0(x') + u_1(x, \omega), 0) = 0 \tag{4.39}$$

and (4.19), (4.23) and (4.29) mean that

$$a_N(\omega, \nabla u_0(x') + u_1(x, \omega), v_0(x, \omega)) = 0. \tag{4.40}$$

So, taking $\xi_1 = (\nabla u_0(x') + u_1(x, \omega), v_0(x, \omega))$ and $\xi_2 = (\nabla u_0(x') + u_1(x, \omega), 0)$ in condition (H_4) and making use of (4.39) and (4.40), it follows that

$$0 \geq C\rho(\omega)(1 + |\nabla u_0(x') + u_1(x, \omega)| + |v_0(x, \omega)|)^{2-\beta} \cdot |v_0(x, \omega)|^\beta,$$

which gives (4.38). Now, by (4.20), (4.23) and (4.29) we see immediately that the function u_1 solves equation (3.10) and, by (3.9), we have

$$\begin{aligned} B(\nabla_{x'} u_0(x')) &= \int_{]-1,1[\times X} a'(\omega, U_0(x, \omega)) d\mu(\omega) dx_N \\ &= \int_{]-1,1[\times X} A'(x, \omega) d\mu(\omega) dx_N. \end{aligned}$$

Now, integrating (4.26) by parts and by (3.11), it follows directly that u_0 is the variational solution of the homogenized system (3.8) and the proof of Theorem 3.1 is then accomplished.

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