CHAPTER 8.

Cosmic Voids

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Jaanilaupäev - St. John's Eve,
the most important day and evening in the Estonian calendar.

The Midsummer Night's sun setting over the Baltic Sea,
and casting its last rays over traditional Estonian farms.

Photo courtesy (top): Steven Rieder.
Photo (bottom): Rien van de Weijgaert

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CHAPTER 8A.

Cosmic Voids
Structure, Dynamics and Cosmology

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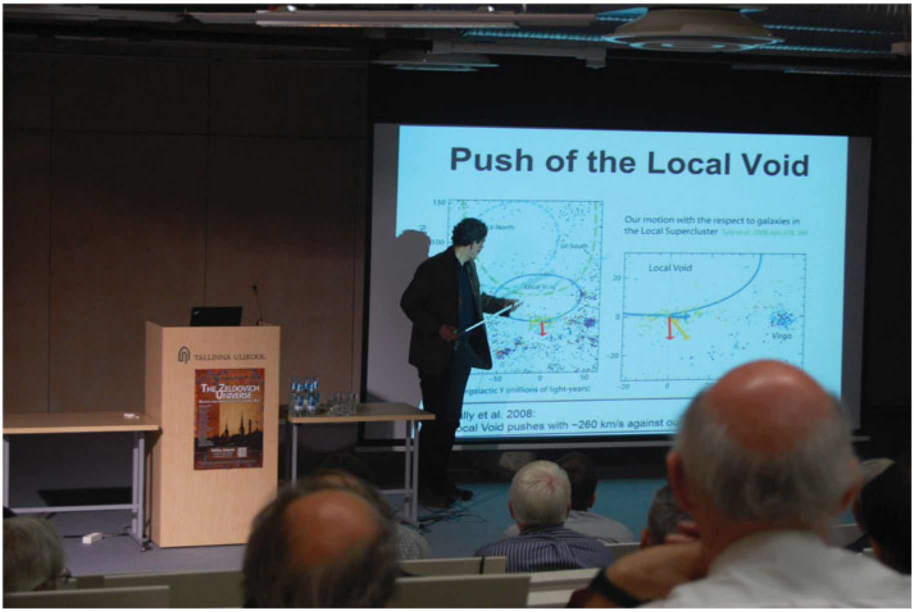

Much ado about Nothing:
Rien van de Weygaert reviewing cosmic voids.

On the stairs, behind the panels,
participants were to be found all over the conference hall.

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The Zeldovich Universe:
Genesis and Growth of the Cosmic Web
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Voids and the Cosmic Web: cosmic depression & spatial complexity

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Abstract. Voids form a prominent aspect of the Megaparsec distribution of galaxies and matter. Not only do they represent a key constituent of the Cosmic Web, they also are one of the cleanest probes and measures of global cosmological parameters. The shape and evolution of voids are highly sensitive to the nature of dark energy, while their substructure and galaxy population provides a direct key to the nature of dark matter. Also, the pristine environment of void interiors is an important testing ground for our understanding of environmental influences on galaxy formation and evolution. In this paper, we review the key aspects of the structure and dynamics of voids, with a particular focus on the hierarchical evolution of the void population. We demonstrate how the rich structural pattern of the Cosmic Web is related to the complex evolution and buildup of voids.

Keywords. Cosmology, large-scale structure, voids, dark energy, modified gravity

1. Introduction

Voids form a prominent aspect of the Megaparsec distribution of galaxies and matter (Chincarini & Rood 1975; Gregory & Thompson 1978; Einasto, Joeveer & Saar 1980; Kirshner *et al.* 1981, 1987; de Lapparent, Geller & Huchra 1986; Colless *et al.* 2003; Tegmark *et al.* 2004; Guzzo *et al.* 2013, 2014). They are enormous regions with sizes in the range of $20 - 50h^{-1}$ Mpc that are practically devoid of any galaxy, usually roundish in shape and occupying the major share of space in the Universe (see fig. 1 and van de Weygaert & Platen (2011) for a recent review). Forming an essential and prominent aspect of the *Cosmic Web* (Bond, Kofman & Pogosyan 1996), they are instrumental in the spatial organization of the Cosmic Web (Icke 1984; Sahni, Sathyaprakash & Shandarin 1994; Sheth & van de Weygaert 2004; Aragon-Calvo *et al.* 2010; Einasto *et al.* 2011). Surrounded by elongated filaments, sheetlike walls and dense compact clusters, they weave the salient weblike pattern of galaxies and matter pervading the observable Universe.

Several recent studies came to the realization that voids not only represent a key constituent of the cosmic mass distribution, but that they are also one of the cleanest probes and measures of the global cosmology. Particularly interesting is the realization that their structure, morphology and dynamics reflects the nature of dark energy, dark matter and that of the possibly non-Gaussian nature of the primordial perturbation field. Another major aspect of voids is that their pristine environment represents an ideal and pure setting for the study of galaxy formation and the influence of cosmic environment on the evolution of galaxies. In addition, voids play a prominent role in the reionization process of the universe, forming the principal regions along which the ionizing radiation produced by the first stars in the Universe propages.

In a void-based description of the evolution of the cosmic matter distribution, voids mark the transition scale at which density perturbations have decoupled from the Hubble

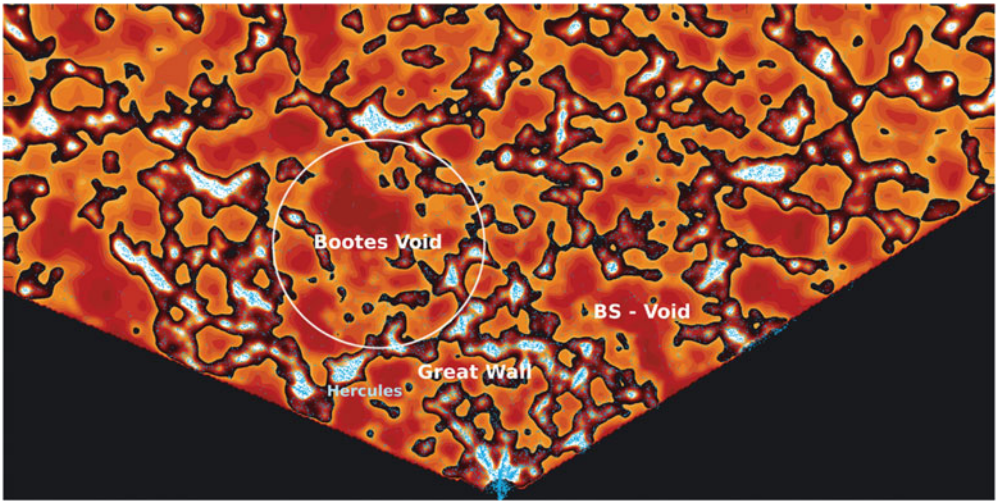

Figure 1. SDSS density map and galaxies in the a region of the SDSS galaxy redshift survey region containing the canonical Boötes void. The DTFE computed galaxy density map, Gaussian smoothed on a scale of $R_f = 1h^{-1}Mpc$, is represented by the color scale map. The galaxies in the SDSS survey are superimposed as dark dots. The underdense voids are clearly outlined as the lighter region outside the high-density weblike filamentary and wall-like features. We have indicated the location of the Hercules supercluster, the CfA Great Wall, and the Boötes void.

flow and contracted into recognizable structural features. At any cosmic epoch the voids that dominate the spatial matter distribution are a manifestation of the cosmic structure formation process reaching a non-linear stage of evolution. On the basis of theoretical models of void formation one might infer that voids may act as the key organizing element for arranging matter concentrations into an all-pervasive cosmic network (Icke 1984; Regős & Geller 1991; van de Weygaert 1991; Sheth & van de Weygaert 2004; Aragon-Calvo *et al.* 2010; Aragon-Calvo & Szalay 2013). As voids expand, matter is squeezed in between them, and sheets and filaments form the void boundaries. This view is supported by numerical studies and computer simulations of the gravitational evolution of voids in more complex and realistic configurations (Martel & Wasserman 1990; Regős & Geller 1991; Dubinski *et al.* 1993; van de Weygaert & van Kampen 1993; Goldberg & Vogeley 2004; Colberg *et al.* 2005; Padilla, Ceccarelli & Lambas 2005; Aragon-Calvo *et al.* 2010; Aragon-Calvo & Szalay 2013; Sutter *et al.* 2014b; Wojtak *et al.* 2016). A marked example of the evolution of a typical large and deep void in a ΛCDM scenarios is given by the time sequence of six frames in fig. 2.2.

The relatively simple structure and dynamical evolution of voids remains strongly influenced by the evolving large scale environment, through dominant tidal influences and direct contact between neighbouring voids. It is this aspect that has been recognized in a surge of recent studies for its considerable potential for measuring the value of cosmological parameters. The evolution of their structure and shape appear to be a direct manifestation of the nature of dark energy, while their dynamics reflects the nature of dark matter or that of the possibly nonstandard nature of gravity.

2. Depressions in the Universe

Voids are an outstanding aspect of the weblike cosmic mass distribution (de Lapparent, Geller & Huchra 1986; Colless *et al.* 2003; Huchra *et al.* 2012; Pan *et al.* 2012; Sutter

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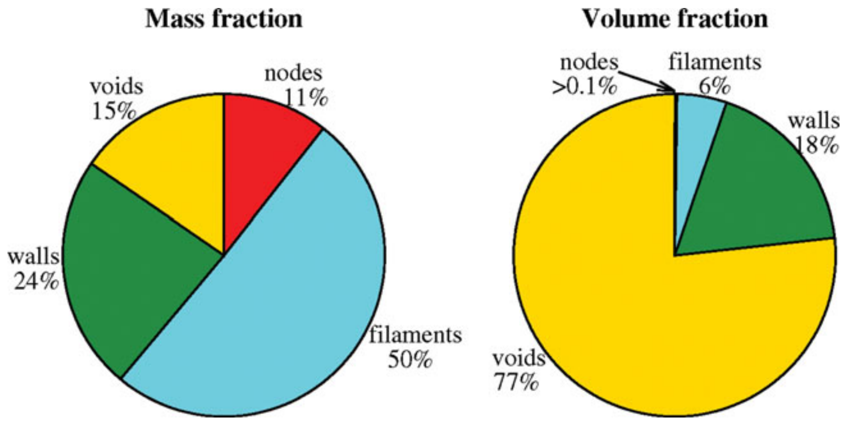

Figure 2. The mass and volume fractions occupied by cosmic web environments detected by the NEXUS+ method. From Cautun *et al.* 2014

et al. 2012). They have been known as a feature of galaxy surveys since the first surveys were compiled (Chincarini & Rood 1975; Gregory & Thompson 1978; Einasto, Joeveer & Saar 1980; Zeldovich *et al.* 1982). Following the discovery by Kirshner *et al.* (1981) and Kirshner *et al.* (1987) of the most dramatic specimen, the Boötes void, a hint of their central position within a weblike arrangement came with the first CfA redshift slice (de Lapparent, Geller & Huchra 1986). This view has been dramatically endorsed and expanded by the redshift maps of the 2dFGRS and SDSS surveys (Colless *et al.* 2003; Tegmark *et al.* 2004). They have established voids as an integral component of the Cosmic Web. The 2dFGRS maps and SDSS maps (see e.g. fig. 1) are telling illustrations of the ubiquity and prominence of voids in the cosmic galaxy distribution. Recent studies have also revealed the prominence of voids in the distant and early U niverse. A beautiful example is that of the deep probe of the VIPERS galaxy redshift survey, which meticulously outlined the Cosmic Web up to redshifts $z \approx 0.5$ (Guzzo *et al.* 2013, 2014). Perhaps even more impressive is the identification of the considerably more intricate void population in the dark matter distribution, as revealed by its reconstruction in the 2MRS survey volume (Kitaura 2012; Hess *et al.* 2013) and the SDSS volume (Leclercq *et al.* 2015; Leclercq 2015).

A relatively crude analysis of voids in computer simulations provides an impression of the status of voids in the Megaparsec universe. They clearly occupy a major share of the volume of the Universe. This has been confirmed in a recent systematic analysis by Cautun *et al.* (2014), which included an inventory of morphological components of the cosmic web with respect in the dark matter distribution in the LCDM Millennium simulation (see fig. 2). Around 77% of the cosmic volume should be identified as a void region. Nonetheless, it represents less than 15% of the mass content of the Universe. This implies that the average density in voids is $\approx 20\%$ of the average cosmic density, which is indeed reasonably close to what is expected on simple theoretical grounds (see below).

2.1. *Voids and the Cosmos*

With voids being recognized as prominent aspects of the Megaparsec galaxy and matter distribution, we should expect them to be a rich source of information on a range of cosmological questions. We may identify at least four:

• They are a prominent aspect of the Megaparsec Universe, instrumental in the spatial organization of the Cosmic Web (Icke & van de Weygaert 1987; van de Weygaert & van

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Kampen 1993; Sheth & van de Weygaert 2004; Aragon-Calvo & Szalay 2013). Their effective repulsive influence over their surroundings has been recognized in surveys of the Local Universe (Courtois *et al.* 2012; Tully *et al.* 2014).

• Voids contain a considerable amount of information on the underlying cosmological scenario and on global cosmological parameters. They are one of the cleanest cosmological probes and measures of dark energy, dark matter and tests with respect to possible modifications of gravity as described by General Relativity. This realization is based on the fact that it is relatively straightforward to relate their dynamics to the underlying cosmology, because they represent a relatively modest density perturbation. Their structure and shape, as well as mutual alignment, are direct reflections of dark energy (Park & Lee 2007; Lee & Park 2009; Platen, van de Weygaert & Jones 2008; Lavaux & Wandelt 2010, 2012; Bos *et al.* 2012; Pisani *et al.* 2015). Notable cosmological imprints are also found in the outflow velocities and accompanying redshift distortions (Dekel & Rees 1994; Martel & Wasserman 1990; Ryden & Melott 1996). In particular interesting is the realization that the dynamics and infrastructure of voids are manifestations of the nature of dark matter or of modified gravity (Li 2011; Peebles & Nusser 2010; Clampitt *et al.* 2013; Cai *et al.* 2015; Cautun *et al.* 2016). The cosmological ramifications of the reality of a supersized voids akin to the identified ones by Rudnick, Brown & Williams (2007), Granett, Neyrinck & Szapudi (2009) and Szapudi *et al.* (2015) would obviously be far-reaching.

• The pristine low-density environment of voids represents an ideal and pure setting for the study of galaxy formation and the influence of cosmic environment on the formation of galaxies (e.g. Kreckel *et al.* 2011, 2012). Voids are in particular interesing following the observation by Peebles that the dearth of low luminosity objects in voids is hard to understand within the ΛCDM cosmology (Peebles 2001).

• Voids are prominent in the key reionization transition in the early universe, key targets of LOFAR and SKA (e.g. Furlanetto *et al.* 2006; Morales & Wyithe 2010).

2.2. *Voids in the Local Universe*

The most detailed view of the structure and galaxy population of voids is offered by voids in the local Universe. A particularly important source of information is the Karachentsev LV catalog of galaxies in the Local Volume, a volume-limited sample of galaxies within a radius of 11 Mpc around the Milky Way (Karachentsev *et al.* 2004). It provides a meticulously detailed view of the Local Void that appears to dominate a major fraction of space in our immediate cosmic neighbourhood. The desolate emptiness of this vast volume is most strikingly borne out by the adhesion reconstruction of the Local Void by Hidding *et al.* (2016) and Hidding (2016).

Moving to a slightly larger volume of the surrounding universe, we obtain a more representative impression of the prominence and role of voids in the overall large scale mass distribution. The 2MRS survey (Huchra *et al.* 2012) provides a uniquely complete census of the galaxy distribution out to distances of 100-150 Mpc. It entails the entire cosmic environment out to that distance, and has enabled remarkably precise and detailed reconstructions of the underlying matter distribution in our cosmic neighbourhood.

Figure 2.2 provides a remarkably detailed reconstruction of the cosmic web in the 2MRS volume. It shows the (surface) density of the weblike structures in the Local Universe. These are the result of adhesion simulations by Hidding *et al.* (2016), based on the the constrained Bayesian KIGEN reconstruction by Kitaura (2012) of the initial conditions in the local volume traced by the 2MRS redshift survey. For a given Gaussian primordial field, the adhesion formalism allows the accurate reconstruction of the rich pattern of weblike features that emerge in the same region as a result of gravitational evolution.

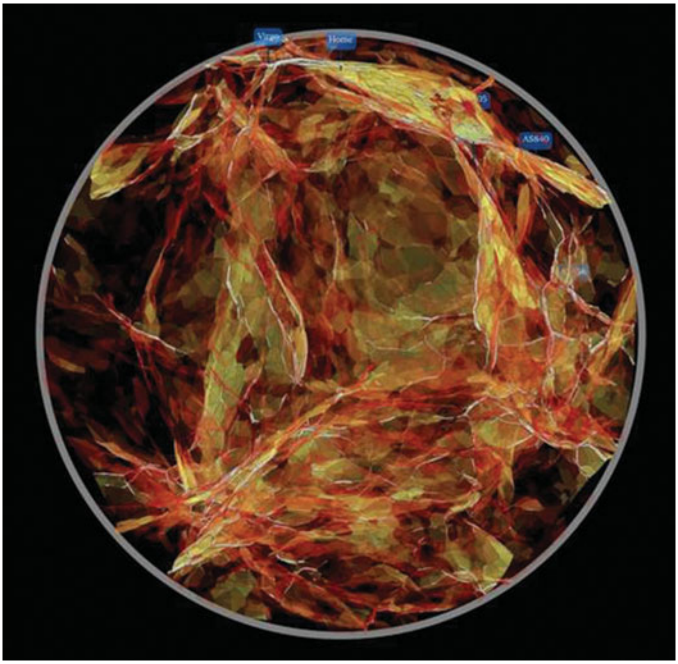

Figure 3. The Local Void. 3D rendering of the adhesion reconstruction of the Local Void. The reconstruction is based on the Bayesian KIGEN reconstruction by Kitaura *et al.* (2012) of the initial conditions in the local volume traced by the 2MRS redshift survey. The Local Supercluster, including the Virgo cluster and the Local Group, are located at the top of the image. Particularly striking is the precipitous emptiness of the Local Void. Image courtesy J. Hidding (see Hidding *et al.* 2016).

The adhesion formalism was applied to 25 constrained realizations of the 2MRS based primordial density field (Hidding *et al.* 2012, 2016). The mean of these realizations gives a reasonably accurate representation of the significant filamentary and wall-like features in the Local Universe. Most outstanding is the clear outline of the void population in the local Universe. The reconstruction also includes the velocity flow in the same cosmic region. It reveals the prominent nature of the outflow from the underdense voids, clearly forming a key aspect of the dynamics of the Megaparsec scale universe.

The Local Universe structure in figure 2.2 presents a telling image of a void dominated large scale Universe. Many of the voids in the adhesion reconstruction can be identified with the void nomenclature proposed by Fairall (Fairall 1998), who mainly identified these voids by eye from the 6dFGRS survey. It is interesting to see that the socalled Tully void appears to be a richly structured underdense region, containing at least the Microscopium Void, the Local Void and the "Trans Tully Void".

3. Formation and Evolution of Voids

Voids emerge out of the density troughs in the primordial Gaussian field of density fluctuations. Early theoretical models of void formation concentrated on the evolution of isolated voids (Hoffman & Shaham 1982; Icke 1984; Bertschinger 1985; Blumenthal *et al.* 1992). Nearly without exception they were limited to spherically symmetric configurations (but see Icke 1984). Neither of these simplifications appears to be close to what happens in reality. Nonetheless, the spherical model of isolated void evolution

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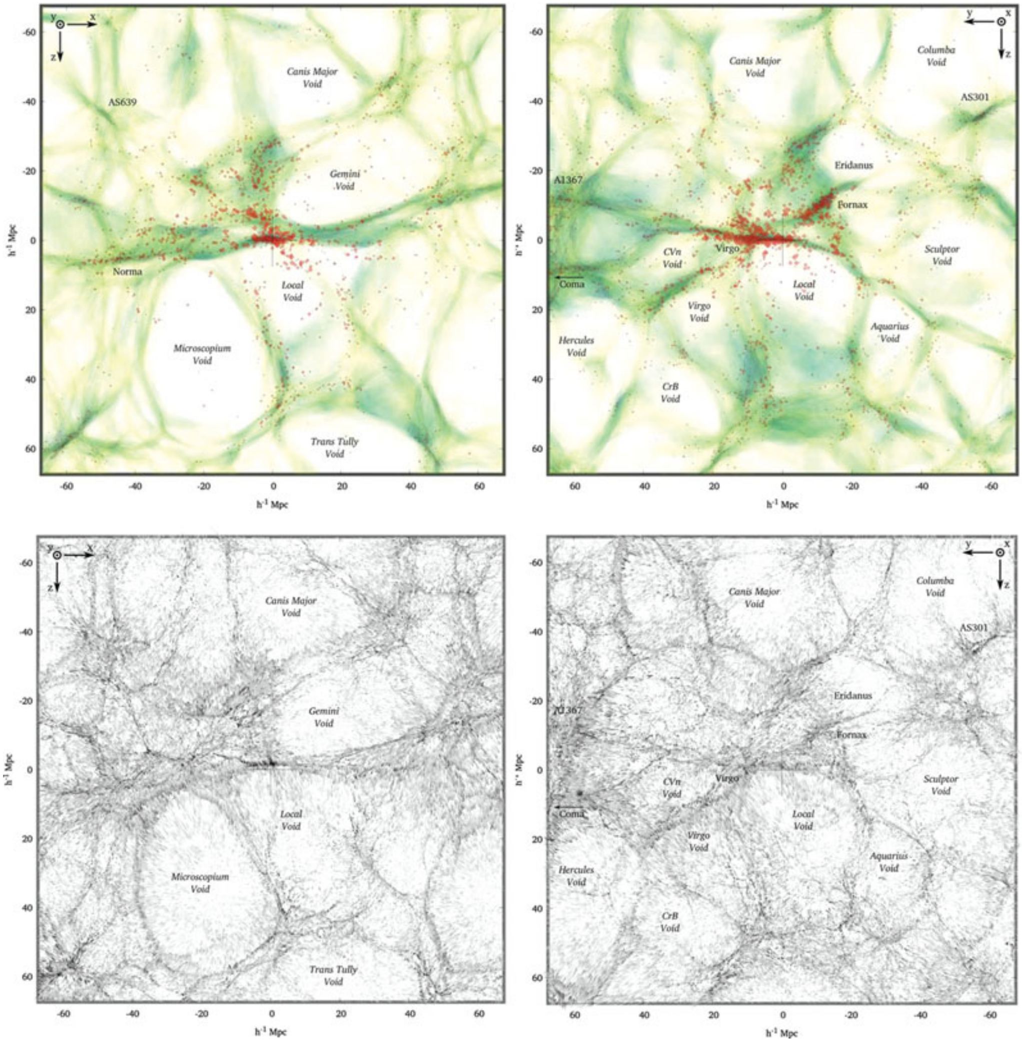

Figure 4. Voids in the Local Universe. The reconstruction of the weblike structure in the local Universe, sampled by the 2MRS survey, has been obtained on the basis of the adhesion formalism applied to a set of 25 constrained Bayesian KIGEN realizations of the primordial density and velocity field in the Local Universe. Top frames: (surface) density field along two perpendicular slices perpendicular to the plane of the Local Supercluster. Note that the density field concerns the dark matter distribution. The red dots are the 2MRS galaxies in the same volume. Bottom frame: the corresponding implied velocity flow in the same slices. Image courtesy J. Hidding (see Hidding *et al.* 2016).

appears to provide us with valuable insights in major physical void characteristics and, as important, with sometimes surprisingly accurate quantitative reference benchmarks (see below).

In the next section (sect.4.2), we will see that a major and dominant aspect of voids is the fact that they are the opposite of isolated objects. To understand the dynamical evolution of voids we need to appreciate two major aspects in which the dynamics of voids differs fundamentally from that off dark halos. Because voids expand and increase in size, they will naturally run up against their expanding peers. Their spatial distribution and organization will be substantially influenced by the way in which the mutually competing

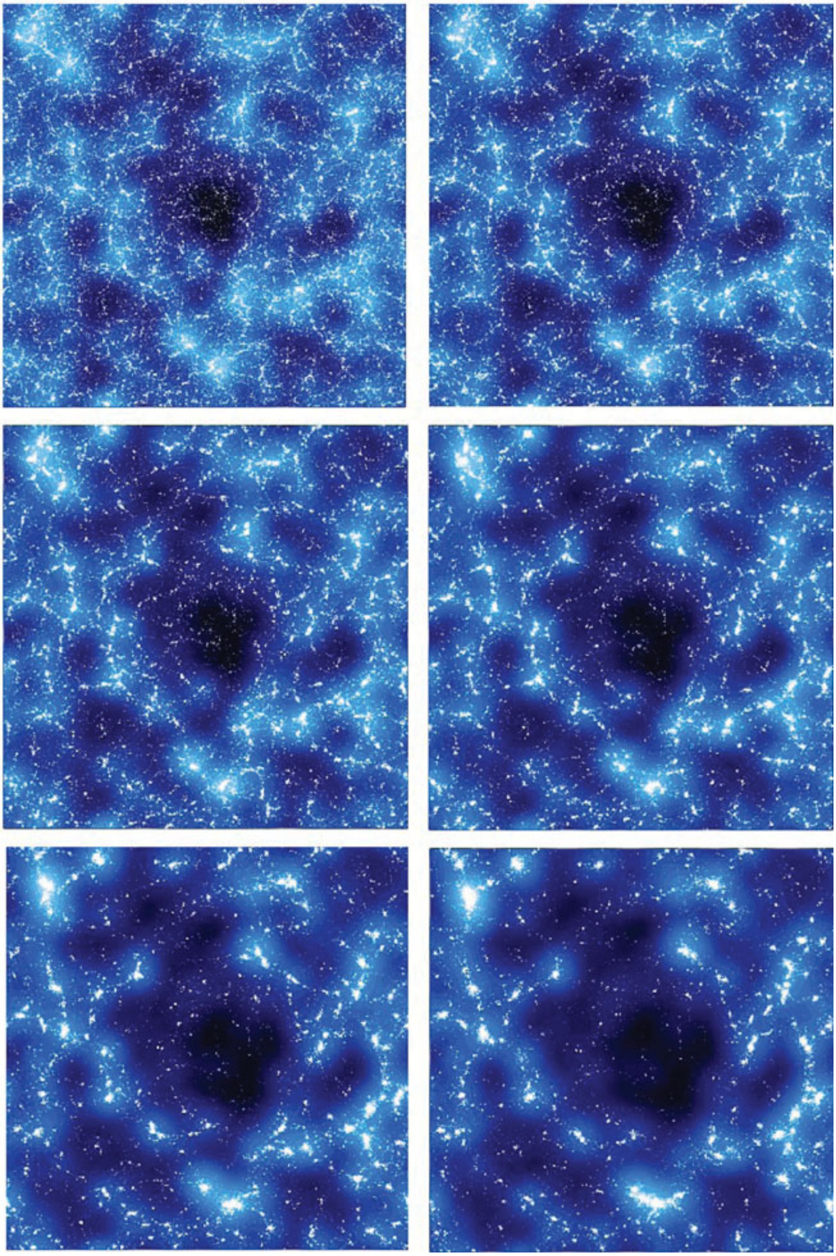

Figure 5. Simulation of evolving void (LCDM scenario). A void in a $n = 0$ power-law power spectrum model. The slice is $50h^{-1}$Mpc wide and $10h^{-1}$Mpc thick. Shown are the particles and smoothed density field (smoothed on a scale of $4h^{-1}$Mpc) at six different timesteps: a=0.05, 0.15, 0.35, 0.55, 0.75 and 1.0. Image courtesy of Erwin Platen

voids distribute their share of space. Equally important is the fact that voids will always be limited to a rather modest density deficit: they cannot become more empty than empty. An immediate consequence of this is that the dynamical influence of the external mass inhomogeneities retains its dominant role in the evolution of a void. Voids will never and cannot decouple from their surroundings!

3.1. *Expansion, Evacuation, Dilution*

The essence of void evolution stems from the fact that they are underdensities in the mass distribution. As a result, they represent regions of weaker gravity. It translates in an effective repulsive peculiar gravitational influence. Most of the principal characteristics of void evolution can be recognized in the illustration of a typical evolving void in a ΛCDM Universe in fig. 2.2 (also see van de Weygaert & van Kampen 1993). The sequence of 6 timestepe of the simulation reveals at least three of the principal aspects of void evolution. Because of the effective repulsive and thus outward peculiar gravitational accleration, initially underdense regions expand faster than the Hubble flow: they expand with respect to the background Universe. Also we see that mass is streaming out of the underdensity. As a result of this evacuation, the density within voids continuously decrease. A clear census of the continuously decreasing mass content in voids can be found in the study of cosmic web evolution by Cautun *et al.* (2014). Isolated voids would asympotically evolve towards an underdensity $\delta = -1$, pure emptiness.

Also observable in figure 2.2 is how the mass distribution inside the void appears to become increasingly uniform. At the same time we see an accumulation of mass around its boundary. This is a direct consequence of the differential outward peculiar gravitational acceleration in voids. Because the density within underdense regions gradually increases outward, we see a decrease of the corresponding peculiar (outward) gravitational acceleration. It means that matter at the centre of voids moves outward faster than matter at the boundary regions. As a result, matter tends to accumulate in - filamentary or planar - ridges surrounding the void, while the interior evolves into a uniform low-density region resembling a low-density homogeneous FRW universe (Goldberg & Vogeley 2004).

Another key feature of void evolution is the diminishing prominence of substructure in its interior. Figure 2.2 clearly shows how the internal structure gradually disappears as mass moves out of the void. This is a direct manifestation of the complex hierarchical evolution of voids (Sheth & van de Weygaert 2004). A direct implication of this is that massive features and objects cannot form in voids, which e.g. manifests itself in a strong shift of the mass spectrum of dark halos - and hence galaxies - towards small masses (Cautun *et al.* 2014). Goldberg & Vogeley (2004) showed how this can be rather accurately modelled by means of a modification of the spectrum of density fluctuations towards one more appropriate for a low-density FRW universe.

Note that while by definition voids correspond to density perturbations of at most unity, $|\delta_v| \leqslant 1$, mature voids in the nonlinear matter distribution do represent *highly nonlinear* features. This may be best understood within the context of Lagrangian perturbation theory (Sahni & Shandarin 1996). Overdense fluctuations may be described as a converging series of higher order perturbations. The equivalent perturbation series is less well behaved for voids: successive higher order terms of both density deficit and corresponding velocity divergence alternate between negative and positive.

3.2. *Spherical Voids*

While in reality voids will never be isolated, nor spherical, we may obtain a basic understanding of the quantitative aspects of void evolution from the simple model of an evolving isolated spherical void. Figure 6 illustrates the evolution of spherical isolated voids. The void in the lefthand is a typical pure and uncompensated "bucket" void†. We notice the principal characteristics of void formation that we discussed in the previous section. The expansion of the void is evident as we see the boundary edge, and the ridge

† This is more commonly known as *tophat*, but given the configuration, *bucket* seems a more appropriate description.

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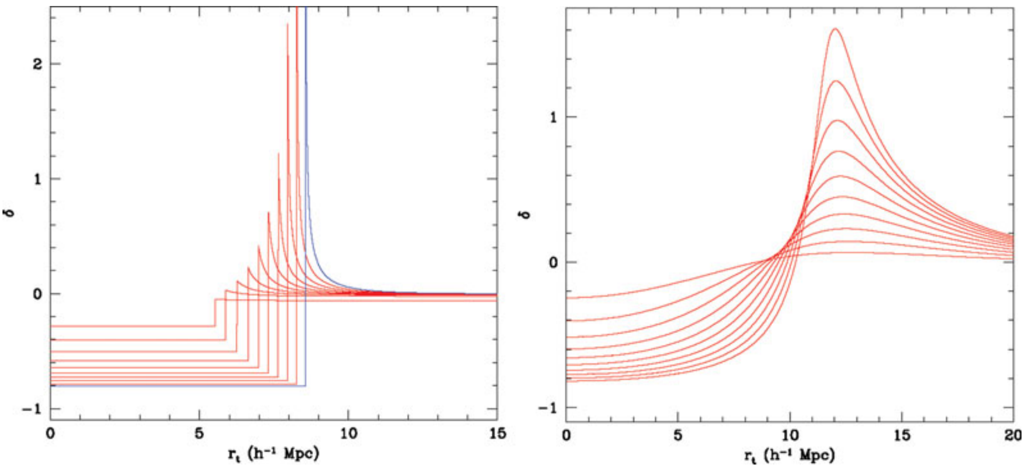

Figure 6. Spherical model for the evolution of voids. Left: a pure (uncompensated) "bucket" void evolving up to the epoch of shell-crossing. Initial (linearly extrapolated) density deficit was $\Delta_{lin,0} = -10.0$, the initial (comoving) radius $\widetilde{R}_{i,0} = 5.0h^{-1}$Mpc. Righthand: a void with an angular averaged SCDM profile. Initial density deficit and characteristic radius are same as for the tophat void (left). The tendency of this void to evolve into a tophat configuration by the time of shell crossing is clear. Shell-crossing, and the formation of a ridge, happens only if the initial profile is sufficiently steep.

at its boundary, move outward. Meanwhile, matter moves out of the void, leading to an increasingly empty void.

The spherical void model also clarifies an additional major aspect, the formation of a ridge around the expanding void. This is the result of the differential outward expansion of the mass in and around the void. While the uniform underdensity profile implies a uniform expansion within the void, near the boundary the inner layers move outward faster than the more outward layers. This is the result of the latter feeling a more moderate interior underdensity. In all, this ultimately leads to the interior mass shells taking over the initially exterior shells. This leads to a *fundamental* evolutionary timescale for void evolution, that of *shellcrossing.*

Bertschinger (1985) demonstrated that once voids have passed the stage of shellcrossing they enter a phase of self-similar expansion. Subsequently, their expansion will slow down with respect to the earlier linear expansion. This impelled Blumenthal *et al.* (1992) to identify voids in the present-day galaxy distribution with voids that have just reached the stage of shell-crossing. It happens when a primordial density depression attains a linearly extrapolated underdensity $\delta_v = f_v = -2.81$ (strictly speaking for an EdS universe). That happens when a perfectly spherical "bucket" void will have expanded by a factor of 1.72 at shellcrossing, and therefore have evolved into an underdensity of $\sim 20\%$ of the global cosmological density, ie. $\delta_{v,nl} = -0.8$. In other words, the voids that we see nowadays in the galaxy distribution do probably correspond to regions whose density is $\sim 20\%$ of the mean cosmic density (note that it may be different for underdensity in the galaxy distribution).

Interestingly, for a wide range of initial radial profiles, voids will evolve into a bucket shaped void profile. The righthand frame of fig. 6 shows the evolution of a spherical void that develops out of an initial underensity that is an angular averaged density profile of an underdensity in a CDM Gaussian random field (see Bardeen *et al.* 1986; van de Weygaert & Bertschinger 1996). Due to the differential expansion of the interior mass shells, we get an acccumulation of mass near the exterior and boundary of the void, meanwhile

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evening out the density distribution in the interior. We also recognized this behaviour in the more complex circumstances of the LCDM void in fig. 2.2. To a considerable extent this is determined by the steepness of the density profile of the protovoid depression (Palmer & Voglis 1983). In nearly all conceivable situations the void therefore appears to assume a *bucket shape*, with a uniform interior density depression and a steep outer boundary (fig. 6, righthand frame). The development of a *bucket* shaped density profile may therefore be considered a generic property of cosmic voids.

Recently, there have been a range of studies on the issue of void density profiles, and the question whether they display universal behaviour (see e.g. Hamaus *et al.* 20144; Cautun *et al.* 2016). In a range of studies, Hamaus *et al.* (e.g Hamaus *et al.* 20144) concluded that spherically averaged density profiles of voids do indeed imply a universal density profile, that could be parameterized by 2 factors. Interestingly, these density profiles have a less prominent bucket shaped interior profile than those seen for the spherical voids in fig. 6. This may be understood from the fact that voids in general are not spherical (see sec. 4.2.2), so that spherical averaging will lead to the mixing of different layers in the void's interior. The recent study by Cautun *et al.* (2016) confirms this: when taking into account the shape of voids, a remarkably strong bucket void density profile appears to surface.

The corresponding void expansion velocity profile confirms the above. A uniform density distribution within a void's interior directly translates into a superHubble velocity outflow, ie. an outflow in which the expansion velocity scales linear with the distance to the void's center. In the next section we will discuss this within the context of the dynamics of voids.

4. Void Dynamics

Soon after their discovery, various studies pointed out their essential role in the organization of the cosmic matter distribution (Icke 1984; Regős & Geller 1991), based on the realization that they have a substantial dynamical influence on their surroundings. Voids have an effective repulsive influence over their surroundings (see below). Even though they represent relatively limited density deficits - voids will never exceed $|\delta| > 1$ - the fact that they occupy a major share of cosmic volume translates into a dominant factor in the dynamical interplay of forces on Megaparsec scales. The flow in and around the void is dominated by the outflow of matter from the void, culminating into the void's own expansion near the outer edge. This has indeed been recognized in various galaxy surveys and surveys of galaxy peculiar motions in our Local Universe. A telling illustration is the map of velocity flows implied by the PSCz galaxy redshift survey (see fig. 7). The gravitational impact of the Sculptor Void on our immediate cosmic vicinity is directly visible. The influence of voids on the peculiar velocities of galaxies have been even recognized at an individual level (Tully *et al.* 2008).

To fully to understand the dynamical influences on the evolution of voids, and as well to appreciate the dynamical impact of voids on their environment, we may obtain substantial insight from the idealized configurations of a spherically isolated void, and that of the force field on a homogeneous ellipsoidal void. The spherical model allows us to understand the principal aspects of the flow field in the interior of voids. The ellipsoidal model allows us to evaluate the importance of external forces on the dynamical evolution of voids.

In terms of their dynamics, the key observation is that external mass inhomogeneities retain a dominant role in the evolution of void. Key to this realization is the fact that voids are always be limited to a rather modest density deficit of not more than $\delta = -1$.

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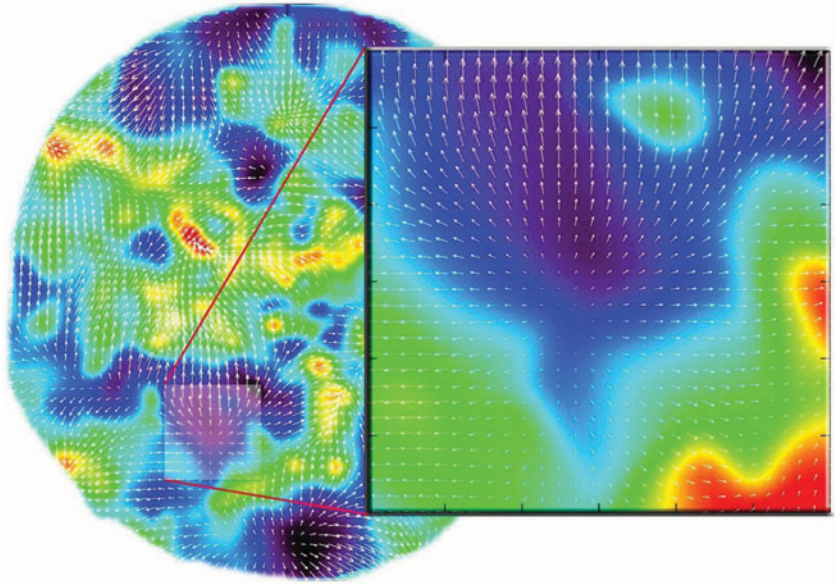

Figure 7. Gravitational impact of the Sculptor Void. The righthand frame shows the inferred velocity field in and around the Sculptor void near the Local Supercluster. The colour map represents the density values, with dark blue at $\delta \sim -0.75$ and cyan near $\delta \sim 0.0$. The vectors show the implied velocity flow around the void, with a distinct nearly spherically symmetric outflow. It is a zoom-in onto the indicated region in the density and velocity map in the Local Universe (lefthand) determined on the basis of the PSCz galaxy redshift survey. From: Romano-Díaz & van de Weygaert 2007.

This will strongly limit their own local gravitational action. It also means that the implicit tidal influences induces by the large scale environment of voids represent a major factor in their dynamical evolution. Their shape, mutual alignment and overall expansion are aspects that are strongly dependent on these external tidal influences.

It also means that global cosmological influences retain a strong impact on the void's evolution. While highly nonlinear overdensities internally hardly notice the presence of the repulsive force induced by the presence of dark energy, in the desolate interiors of voids dark energy plays an even considerably more prominent role than on global cosmological scales (Park & Lee 2007; Lee & Park 2009; Lavaux & Wandelt 2010, 2012). Along the same line, it is in the diluted density field of a void that the imprint of possible modifications of the force of gravity will be most noticeable (Li 2011; Clampitt *et al.* 2013). While this direct dynamical influence of dark energy, and gravity modifications, in voids may be one factor of importance, there is an equally and possibly even more sizeable secondary influence. The dominant role of external tidal influences on the void's dynamics depends on the amplitude of the inhomogeneities in the large scale environment of voids. The growth rate of mass fluctuations is a sensitive function of factors such as the nature of dark energy, and the nature of gravity on large scales, as these influence the dynamical timescales involved in the formation of structure.

To fully appreciate the forces at work in voids, we first looks at their own principal imprint, that of the void's expansion and the corresponding outflow velocities. Subsequently, we will investigate the external influences on the basis of the ellipsoidal model.

4.1. *Superhubble Bubbles*

Because voids are emptier than the rest of the universe, they have an outward directed velocity flow: mass is flowing out of the void. Evidently, the resulting void velocity field profile is intimately coupled to that of its interior density field profile.

In the situation of a mature, evolved void, the velocity field of a void resembles that of a Hubble flow, in which the outflow velocity increases linearly with distance to the void center. In other words, voids are *super-Hubble bubbles* (Icke 1984)! The linear velocity increase is a manifestation of its general expansion. The near constant velocity divergence within the void conforms to the *super-Hubble flow* expected for the near uniform interior density distribution that voids attain at more advanced stages (see fig. 6).

It is straightforward to appeciate this from the *continuity equation*. For a uniform density field, it tells us that the velocity divergence in the void will be uniform, corresponding to a Hubble-like outflow. Because voids are emptier than the rest of the universe they will expand faster than the rest of the universe, with a net velocity divergence equal to

$$\theta \quad = \quad \frac{\nabla \cdot \mathbf{v}}{H} = 3(\alpha - 1)\,, \qquad \alpha = H_{\rm void}/H\,, \tag{4.1}$$

where α is defined to be the ratio of the super-Hubble expansion rate of the void and the Hubble expansion of the universe. Inspection of the velocity flow in the Sculptor void (see fig. 7) shows that this is indeed a good description of the observed reality.

van de Weygaert & van Kampen (1993) confirmed that the velocity outflow field in viable cosmological scenarios does indeed resemble that of a superHubble expanding bubble (see fig. 8b). They also established that the superHubble expansion rate is directly proportional to the nonlinear void density $\Delta(t)$ (see fig. 8a),

$$H_{\rm void}/H \;=\; -\frac{1}{3} f(\Omega)\,\Delta(t)\,. \tag{4.2}$$

This relation, known within the context of a linearly evolving spherical density perturbation, in the case of fully evolved voids appears to be valid on the basis of the *nonlinear* void density deficit. As van de Weygaert & van Kampen (1993) illustrated, voids should therefore be considered as distinctly nonlinear objects. Several recent studies (e.g. Hamaus *et al.* 20144) have confirmed this finding for voids in a range of high resolution cosmological simulations.

4.2. *Tidal Forces - Ellipsoidal Voids*

Of decisive importance for understanding the dynamics and evolution of voids is the realization that voids will never and cannot decouple from their surroundings. One aspect of this is that voids expand and increase in size, they will naturally run up against their expanding peers. Their spatial distribution and organization will be substantially influenced by the way in which the mutually competing voids distribute their share of space.

The second strong environmental influence on the evolution of voids is that of the tidal influences induced by the large scale environment of voids. Their shape, mutual alignment and overall expansion are aspects that are strongly dependent on these external tidal influences.

4.2.1. *Homogeneous Ellipsoidal Model*

Arguably one of the most direct and transparent means of developing our insight into the tidally affected dynamics of voids is to look at the evolution of homogeneous underdense ellipsoids. This idea was forwarded by Icke (1984). In particular the dominant

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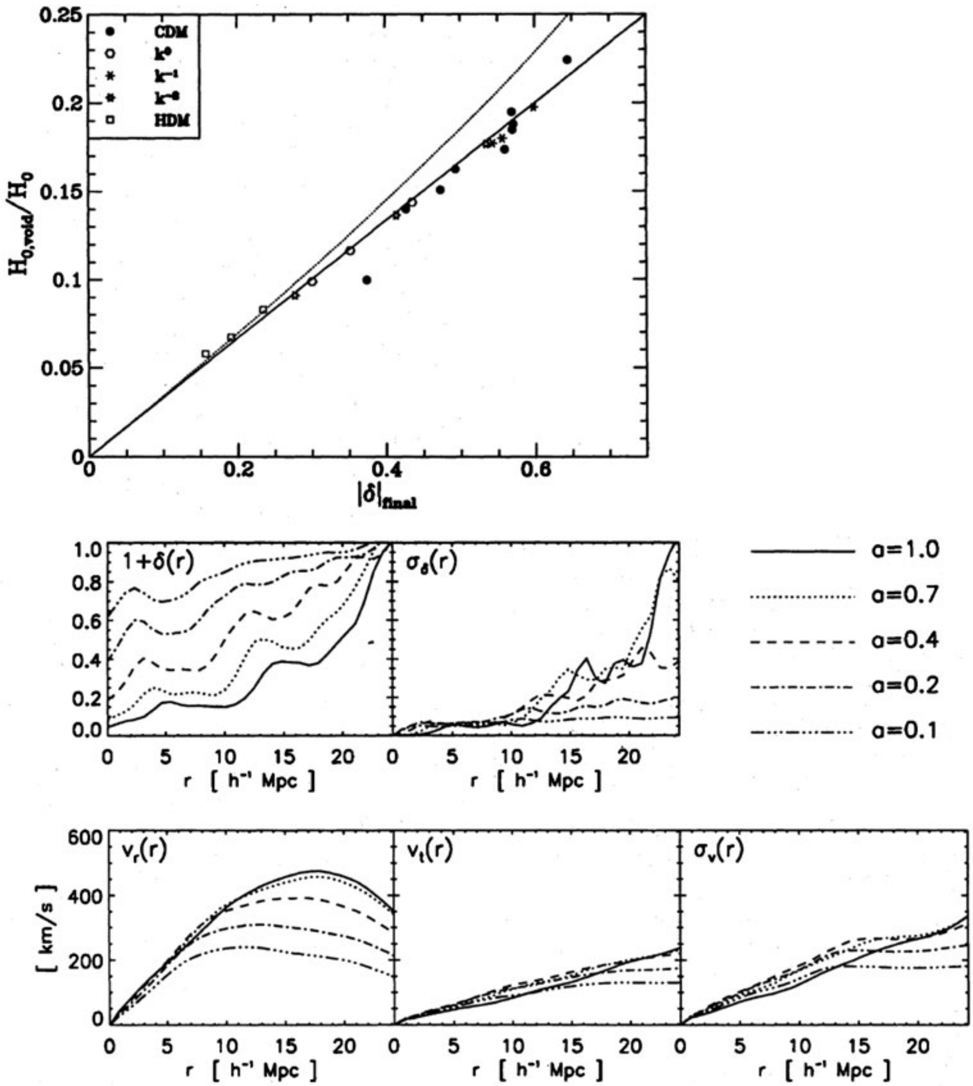

Figure 8. Superhubble Void expansion. Bottom rows: the density profiles and velocity profiles of an evolving void in a constrained simulation of a void in a CDM cosmology. Clearly visible is the evolution of the void towards a bucket shaped density profile, and the corresponding development of a linear superHubble flow field in the interior of the void. Top: the superHubble void expansion rate, in units of the global Hubble parameter $H(t)$ for a set of simulated voids in various cosmologies, as a function of the underdensity $\Delta(t)$ of a void (here called δ_{final}. From: van de Weygaert & van Kampen 1993.

role of external tidal forces on the evolution of voids can be understood by assessing the evolution of homogeneous underdense ellipsoids, within the context of *Homogeneous Ellipsoidal Model* (Icke 1973; White & Silk 1979; Eisenstein & Loeb 1995; Bond & Myers 1996; Desjacques 2008).

It is interesting to realize that the ellipsoidal model, even while idealized, is a rather good approximation for the main aspects of a void's evolution. In many respects the homogeneous model is far more suitable as an approximation for underdense regions than it is for overdense ones: voids expand and evolve towards an increasingly uniform interior

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density field over a vast range of their volume (see fig. 6): while they expand their interior gets drained of matter and develops a flat "bucket-shaped" density profile. Overdense regions contract into more compact and hence steeper density peaks, so that the area in which the ellipsoidal model represents a reasonable approximation will continuously shrink. Evidently, the approximation is restricted to the interior and fails at the void's outer fringes. The latter is a result of the accumulation of material near the void's edge, and the encounter with neighbouring features of the cosmic web and surrounding voids.

The homogeneous ellipsoidal model assumes an object to be a region with a triaxially symmetric ellipsoidal geometry and a homogeneous interior density, embedded within a uniform background density $\rho_{\rm u}$. Consider the simple situation of the external tidal shear directed along the principal axes of the ellipsoid. The gravitational acceleration along the principal axes of an ellipsoid with over/underdensity δ can be evaluated from the expression for the corresponding scale factors $\mathcal{R}_{\rm i}$,

$$\frac{d^2\mathcal{R}_{\rm m}}{dt^2} \;=\; -4\pi G\rho_{\rm u}(t)\left[\frac{1+\delta}{3}+\frac{1}{2}\,(\alpha_m-\frac{2}{3})\,\delta\right]\mathcal{R}_{\rm m}\;-\;\tau_{\rm m}\,\mathcal{R}_{\rm m}\;+\;\Lambda R_m\,, \quad (4.3)$$

where we have also taken into account the influence of the cosmological constant Λ. The factors $\alpha_{\rm m}(t)$ are the ellipsoidal coefficients specified by the integral equation,

$$\alpha_{\rm m}(t)\;=\;\mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)\int_0^{\infty}\frac{{\rm d}\lambda}{\left(\mathcal{R}_{\rm m}^2(t)+\lambda\right)\prod_{n=1}^{3}\left(\mathcal{R}_{\rm n}^2(t)+\lambda\right)^{1/2}}\,. \quad (4.4)$$

The influence of the external (large-scale) tidal shear tensor $T_{\rm mn}^{(ext)}$ enters via the eigenvalue τ_m. From eqn. 4.3, it is straightforward to appreciate that as δ grows strongly nonlinear, the relative influence of the large-scale (near-)linear tidal field will decline. However, the density deficit δ of voids will never exceed unity, $|\delta|<1$, so that the importance of the factor τ_m remains relatively large.

The impact of the external tidal forces can be so strong they not only effect an anisotropic expansion of the void, but even may manage to make an initially spherically expanding void to collapse. The latter can be seen in fig. 9, where we compare the evolution of the two initially spherical and equivalent void regions. The isolated one assumes the regular evolution of a spherical isolated void, while its peer develops into an increasingly anisotropic configuration. At some point the external tides go as far to squeeze the void into contraction and ultimately towards collapse. This is the situation that we find in the hierarchical buildup of voids, to be described in the next section 5. As illustrated in the accompanying illustration of the hierarchically evolving void population, small voids near the boundary of large void bubbles get squeezed into collapse as a result of the tidal impact of their overdense surroundings.

4.2.2. *Void Shapes and Tides*

Icke (1984) pointed out that any (isolated) aspherical underdensity will become more spherical as it expands. The effective gravitational acceleration is stronger along the short axis than along the longer axes. For overdensities this results in a stronger inward acceleration and infall, producing increasingly flattened and elongated features. By contrast, for voids this translates into a larger *outward* acceleration along the shortest axis so that asphericities will tend to diminish. For the interior of voids this tendency has been confirmed by N-body simulations (van de Weygaert & van Kampen 1993).

In reality, voids will never reach sphericity. Even though voids tend to be less flattened or elongated than the halos in the dark matter distribution, they are nevertheless quite nonspherical: Platen, van de Weygaert & Jones (2008) find that they are slightly prolate

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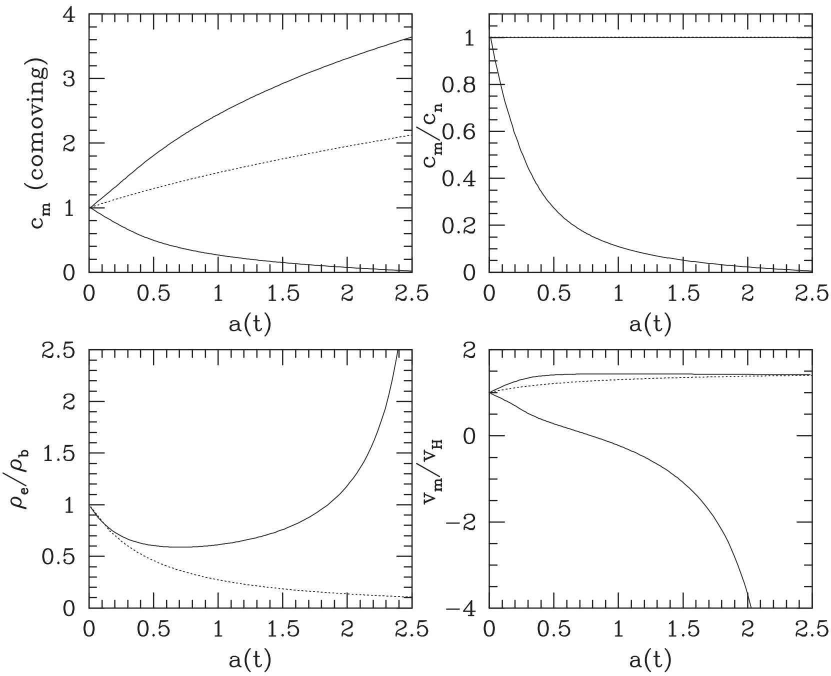

Figure 9. Anisotropic Void Expansion and Collapse. Evolution of the same initially spherical void. Dashed line: without external influences Solid line: under the influence of an axisymmetric and linearly evolving external tidal force field ($E_{mm,0} = (-E, -E, 2E)$). *Topleft:* (comoving) void axis, $c'_m = c_m/a(t)$. *Topright:* axis ratios c_2/c_1 and c_3/c_1. *Bottom Left:* Internal density ρ_e of the void, in units of the cosmic density $\rho_u(t)$. *Bottom Right:* The velocity v_m along the axes of the voids, in units of the Hubble velocity v_H. Note the collapse along axis 1 and 2.

with axis ratios of the order of $c : b : a \approx 0.5 : 0.7 : 1$. This agrees with the statement by Shandarin *et al.* (2006) and Park & Lee (2007) that in realistic cosmological circumstances voids will be nonspherical. This is also quite apparent in images of simulations, such as those in figure 5.

The flattening is a result of large scale dynamical and environmental factors, amongst which we can identify at least two important factors (Platen, van de Weygaert & Jones 2008). Even while their internal dynamics pushes them to a more spherical shape they will never be able to reach perfect sphericity before encountering surrounding structures such as overdense filaments or planar walls. Even more important is the fact that, for voids, the external tidal influences remain important (see discussion above). These external tidal forces are responsible for a significant anisotropic effect in the development of the voids. In extreme cases they may even induce fullscale collapse and demolition of the void.

4.2.3. *Void Shape Measurement: the WVF Watershed Void Finder*

Figure 4.2.4 illustrates a typical example of the shape distribution of voids in a ΛCDM cosmology. The identification of voids by means of the watershed transform, which delineates the region of a void independent of its scale and shape, a direct objective measurement of the volume, shape and orientation of the void population is obtained. This may be directly inferred from the comparison between the bottom panels with the dark matter distribution in the top panel of fig. 4.2.4. Introduced and proposed by Platen, van de Weygaert & Jones (2007) (also see contribution by Jones & van de Weygaert in this

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volume), the Watershed Void Finder (WVF) identifies voids via a watershed transform applied to the DTFE (Delaunay Tessellation Field Estimator) density field reconstruction (Schaap & van de Weygaert 2000; van de Weygaert & Schaap 2009). The latter exploits the scale and shape sensitivity of Voronoi and Delaunay tessellations to retain the intricate multiscale and weblike nature of the mass distribution probed by a discrete particle or galaxy distribution. The idea of the use of the watershed transform for the objective identification of voids, given the close relation to the topology of the weblike Megaparsec matter distribution (see Aragon-Calvo *et al.* 2010), has in the meantime been recognized as a true watershed with respect to setting a standard definition for what should be considered as a void (Colberg *et al.* 2008; Neyrinck 2008; Sutter *et al.* 2015; Nadathur & Hotchskiss 2015).

4.2.4. *Void Alignments*

Large scale tidal influences not only manifest themselves in the shaping of individual voids. They are also responsible for a distinct alignment of substructures along a preferred direction, while they are also instrumental in their mutual arrangement and organization. Locally, the orientation of a void turns out to be strongly aligned with the tidal force field generated by structures on scales up to at least $20-30h^{-1}$Mpc. This goes along with a similar mutual alignment amongst voids themselves. They have strongly correlated orientations over distances $> 30h^{-1}$Mpc (Platen, van de Weygaert & Jones 2008), a scale considerably exceeding the typical void size. It forms a strong confirmation of the large scale tidal force field as the dominant agent for the evolution and spatial organization of the Megaparsec Universe, as emphasized by the Cosmic Web theory (Bond, Kofman & Pogosyan 1996; van de Weygaert & Bond 2008).

4.2.5. *Void Shapes and Dark Energy*

A third interesting influence on the shape of vois is the possible impact of dark energy on the dynamics and evolution of voids. This influence is a result of its direct repulsive effect on the force field of the void, as well as indirectly via the external tidal force field induced by the surrounding large scale inhomogeneities.

Following the earlier suggestions by Park & Lee (2007); Lee & Park (2009), studies by Wandelt and collaborators (Lavaux & Wandelt 2010; Biswas *et al.* 2010) showed that void shapes may be used as precision probes of dark energy. Biswas *et al.* (2010) even quoted the possibility of improving the figure of the Dark Energy Task Force figure of merit by an order of hundred for future experiments like Euclid. An elaborate study by Bos *et al.* (2012) of void shape evolution in simulations within a range of different dark energy cosmologies confirmed the high level of sensitivity of the shapes of dark matter voids to the underlying dark energy (see fig. 4.2.4). However, it is less straightforward to reach similar conclusions of the shapes of voids in the observed galaxy distribution (Bos *et al.* 2012).

Lavaux & Wandelt (2012) forwarded the suggestion of using the Alcock-Paczysnki test on the stacking of voids in a sufficiently large cosmic volume. Assuming that the orientation of voids is random, the shapes of properly scaled voids would average out to purely spherical. This comes along with the significant advantage of substantially increasing the signal-to-noise of the resulting void stack. Given the observation of the voids in a galaxy redshift survey, as a result of the cosmic expansion the resulting void stack in redshift space will be stretched differently in the radial direction than in the transverse direction. The ratio between the transverse and radial stretching of an intrinsically spherical feature yields direct information on the angular diameter distance and Hubble parameter at a redshift z. Since the suggestion by Lavaux & Wandelt (2012), several groups realized

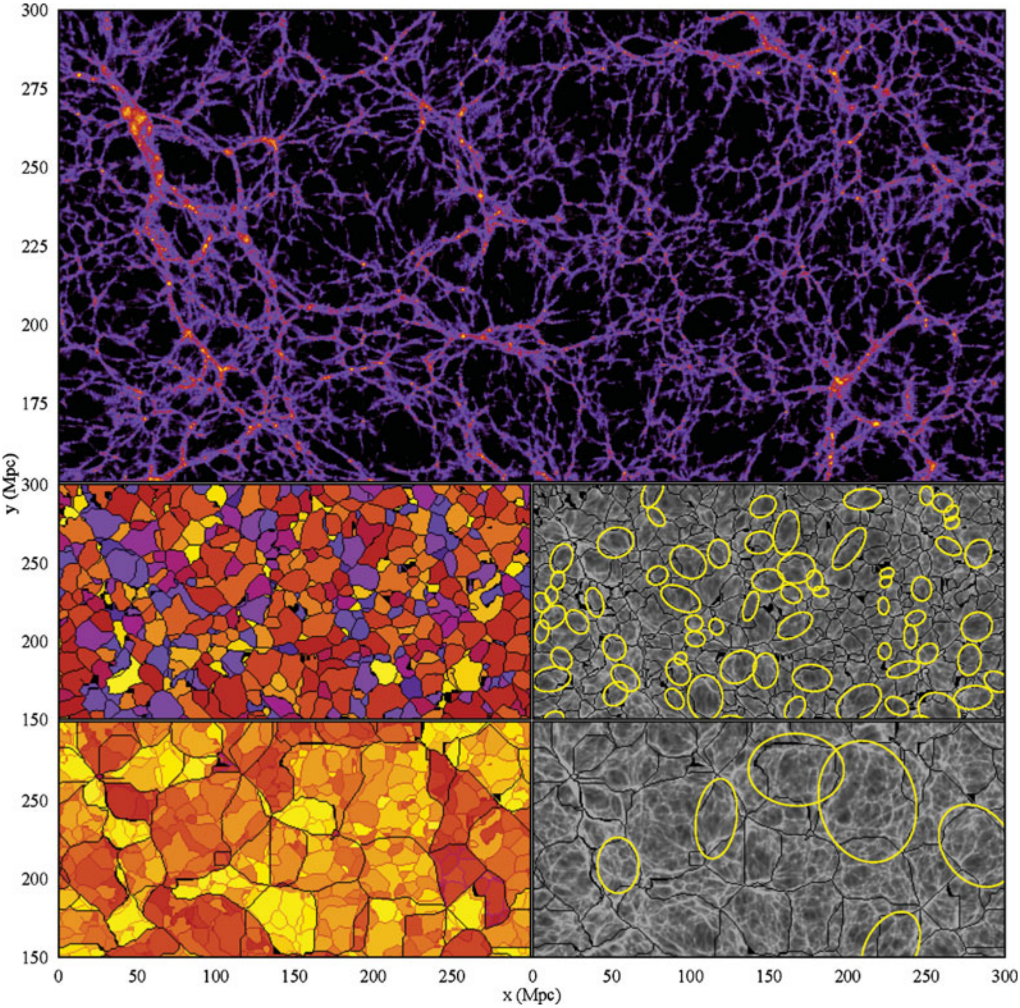

Figure 10. Shapes of voids in a ΛCDM simulation. Top: a density field slice of thickness $0.25h^{-1}$Mpc, and size 300 by 150 h^{-1}Mpc. *Bottom left two*: the corresponding distribution of voids. The voids have been identified with the Watershed Void Finder (Platen *et al.* 2007), for the top panel a Gaussian filter with radius $R_f = 1.5h^{-1}$Mpc, for the bottom panel $R_f = 6.0h^{-1}$Mpc (the $1.5h^{-1}$Mpc ones are transparently inset). *Bottom right two*: the shapes of the watershed identified voids have been determined via the inertia tensor of the corresponding region. A random selection of the ellipsoid fits (yellow) is overlaid on the density field (now in grayscale), again at two radii 1.5 and $6.0h^{-1}$Mpc. From Bos *et al.* 2012.

the large cosmological potential of the application of the Alcock-Paczysnki test to voids in large surveys. It has also resulted in a range of interesting studies (e.g. Sutter *et al.* 2012b, 2014; Pisani *et al.* 2014; Pisani 2014), and is generating substantial interest for exploiting voids in upcoming large galaxy surveys (e.g. Nadathur 2016).

Of key importance for a succesful application of the Alcock-Paczysnki test to samples of voids is to correct for some systematic effects. One concerns the correction for the peculiar velocity outflow of the void itself (see Pisani 2014, for a clear discussion). Another effect that should be accounted for is the alignment of voids on large scales, as discussed in the previous subsection 4.2.4 (also see fig. 4.2.4). Clearly, if the region from which the void sample is extracted is not much larger than the scale on which tidally induced alignment

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are expected, the stacked voids may not define an intrinsically spherical object and hence influence the outcome of the determined cosmological parameters.

4.3. *Dynamical Influence of Voids*

Various studies have found strong indications for the imprint of voids in the peculiar velocity flows of galaxies in the Local Universe. Bothun *et al.* (1992) made the first claim of seeing pushing influence of voids when assessing the stronger velocity flows of galaxies along a filament in the first CfA slice. Stronger evidence came from the extensive and systematic POTENT analysis of Mark III peculiar galaxy velocities (Willick *et al.* 1997) in the Local Universe (Dekel, Bertschinger & Faber 1990; Bertschinger *et al.* 1990). POTENT found that for a fully selfconsistent reconstruction of the dynamics in the Local Universe, it was inescapable to include the dynamical influence of voids (Dekel 1994).

With the arrival of new and considerably improved data samples the dynamical influence of voids in the Local Universe has been investigated and understood in greater detail. The reconstruction of the density and velocity field in our local cosmos on the basis of the 2MASS redshift survey has indeed resulted in a very interesting and complete view of the dynamics on Megaparsec scales. This conclusion agree with that reached on the basis of an analysis of the peculiar velocity of the Local Group by Tully *et al.* (2008). Their claim is that the Local Void is responsible for a considerable repulsive influence, accounting for ~ 259 km s^{-1} of the ~ 631 km s^{-1} Local Group motion with respect to the CMB (also see fig. 2.2).

The substantial dynamical role of voids in the large scale Universe has been most convincingly demonstrated in the recent advances in the study of cosmic flows enabled by the completion of the Cosmicflows2 and Cosmicflows3 surveys (Tully *et al.* 2008; Courtois *et al.* 2012). The implied velocity-based reconstructions of the Local Universe, culminating in the suggestion of the hypercluster Laniakea as local dynamical entity (Tully *et al.* 2014), clearly reveal the substantial repulsive influence of voids in the Local Universe.

5. Void Sociology & Void Hierarchy: Bubbles in Soapsuds

Computer simulations of the gravitational evolution of voids in realistic cosmological environments do show a considerably more complex situation than that described by idealized spherical or ellipsoidal models (Martel & Wasserman 1990; Regős & Geller 1991; Dubinski *et al.* 1993; van de Weygaert & van Kampen 1993; Goldberg & Vogeley 2004; Colberg *et al.* 2005; Padilla, Ceccarelli & Lambas 2005; Ceccarelli *et al.* 2006; Bos *et al.* 2012; Aragon-Calvo & Szalay 2013; Sutter *et al.* 2014b; Wojtak *et al.* 2016). In recent years the huge increase in computational resources has enabled N-body simulations to resolve in detail the intricate substructure of voids within the context of hierarchical cosmological structure formation scenarios (Mathis & White 2002; Gottlöber *et al.* 2003; Goldberg & Vogeley 2004; Colberg *et al.* 2005; Padilla, Ceccarelli & Lambas 2005; Ceccarelli *et al.* 2006; Bos *et al.* 2012; Aragon-Calvo & Szalay 2013; Sutter *et al.* 2014b; Wojtak *et al.* 2016). They confirm the theoretical expectation of voids having a rich substructure as a result of their hierarchical buildup (see e.g. fig. 5).

Sheth & van de Weygaert (2004) treated the emergence and evolution of voids within the context of *hierarchical* gravitational scenarios. It leads to a considerably richer view of the evolution of voids. The role of substructure within their interior and the interaction with their surroundings turn out to be essential aspects of the *hierarchical* evolution of the void population in the Universe. An important guideline are the heuristic void

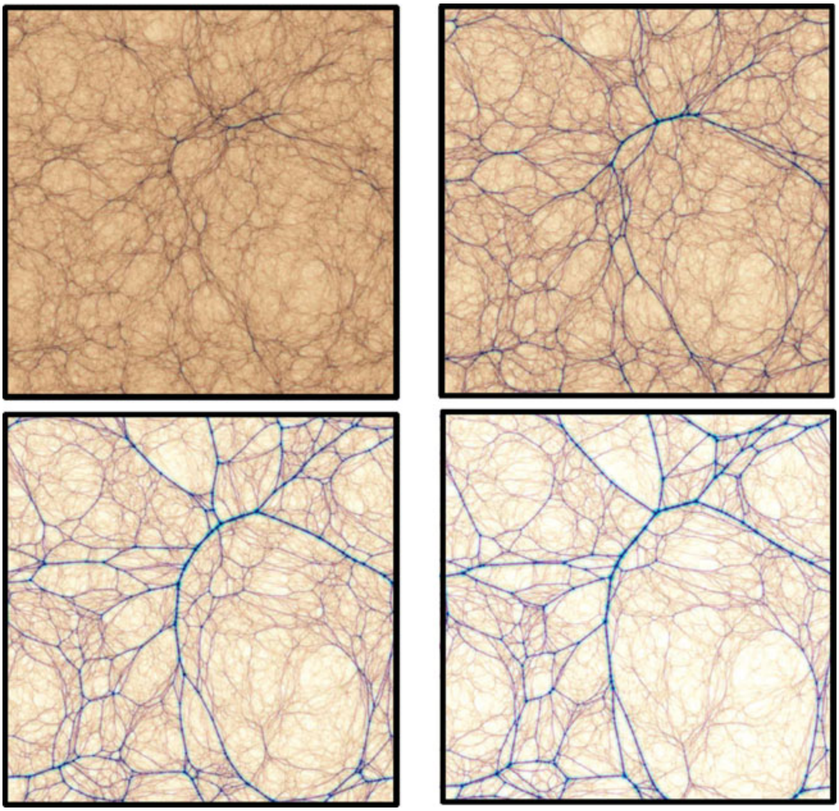

Figure 11. The hierarchical evolution of a void. The evolving mass distribution in and around a large void in an LCDM cosmology. The evolution runs from top lefthand to the bottom righthand frame. The structure depicted shows the result of an adhesion calculation of the evolving weblike network of walls, filaments and nodes in the cosmic volume. It shows in detail and with considerable contrast the evolution of the void population. From Hidding *et al.* 2016.

model simulations by Dubinski *et al.* (1993), and the theoretical void study by Sahni, Sathyaprakash & Shandarin (1994) within the context of a Lagrangian adhesion model description.

In some sense voids have a considerably more complex evolutionary path than overdense halos. Their evolution is dictate by two processes: their *merging* into ever larger voids as well as the *collapse* and disappearance of small ones embedded in overdense regions (see fig. 12). As argued by Sheth & van de Weygaert (2004), the implied hierarchical development of voids, akin to the evolution of overdense halos, may be described by an *excursion set* formulation (Press & Schechter 1974; Bond *et al.* 1991; Sheth 1998). To take account of the more complex evolutionary history, the evolution of voids needs to be described by a *two-barrier* excursion set formalism.

The resulting hierarchical buildup of the void population resembles that of a gradually diluting soapsud. At early time, emerging from a primordial Gaussian random field, the first bubbles to appear are small matured voids. The ones coexisting within the realm of a larger underdense region, will gradually merge into a larger void. This is the typical fate of a void in a *void-in-void* configuration. Meanwhile, small voids that find themselves in overdense regions or near the boundary of larger underdensities get squeezed out of existence, the fate for a void in a *void-in-cloud* configuration. What remains is a sud of larger void bubbles. This sequence proceeds continuously. Given the almost volume-filling

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nature of voids, this void hierarchical process conjures up the impression of a weblike pattern evolving as the result of continuously merging bubbles, with filaments, walls and cluster nodes at the interstices of the network (see Icke & van de Weygaert 1987; Aragon-Calvo 2014, for a geometric model along these lines).

5.0.1. *Void Merging*

First, consider a small region which was less dense than the critical void density value. It may be that this region is embedded in a significantly larger underdense region which is also less dense than the critical density. Many small primordial density troughs may exist within the larger void region. Once small density depressions located within a larger embedding underdensity have emerged as true voids at some earlier epoch, their expansion tends to slow down.

When the adjacent subvoids meet up, the matter in between is squeezed in thin walls and filaments. The peculiar velocities perpendicular to the void walls are mostly suppressed so that the flow of matter is mostly confined to tangential motions. The subsequent merging of the subvoids is marked by the gradual fading of these structures while matter evacuates along the walls and filaments towards the enclosing boundary of the emerging void (Dubinski *et al.* 1993) (also see top row fig. 12). The timescale on which the internal substructure of the encompassing void is erased is approximately the same as that on which it reaches maturity.

The final result is the merging and absorption of the subvoids in the larger void emerging from the embedding underdensity. Hence, as far as the void population is concerned only the large void counts, while the smaller subvoids should be discarded as such. Only a faint and gradually fading imprint of their original outline remains as a reminder of the initial internal substructure.

5.0.2. *Void Collapse*

A *second* void process is responsible for the radical dissimilarity between void and halo populations. If a small scale minimum is embedded in a sufficiently high large scale density maximum, then the collapse of the larger surrounding region will eventually squeeze the underdense region it surrounds: the small-scale void will vanish when the region around it has collapsed completely. Alternatively, though usually coupled, they may collapse as a result of the tidal force field in which they find themselves. If the void within the contracting overdensity has been squeezed to vanishingly small size it should no longer be counted as a void (see fig. 12, bottom row).

When inspecting the evolving matter distribution in and around voids, such as in the high resolution configuration of fig. 5, we find that most of the collapsing or squeezed voids are small voids to be found in and near the boundary regions of large underdense void regions. The small voids in these regions are strained by the high density boundary regions, mostly filaments and clusters, and in many situations squeezed out of existence. On the other hand, subvoids in the interior of a larger void tend to merge with surrounding peers. A detailed assessment of the evolution of these regions reveals that the void collapse process is an important aspect of the evolution and buildup of the cosmic web. Interestingly, recent observational studies have indeed identified the existence of collapsing void regions by studying the redshift structure around voids in the SDSS galaxy redshift survey (Paz 2013).

The collapse of small voids is an important aspect of the symmetry breaking between underdensities and overdensities. In the primorial Universe, Gaussian primordial conditions involve a perfect symmetry between under- and overdense. Any inspection of a galaxy redshift map or an N-body simulation shows that there is a marked difference

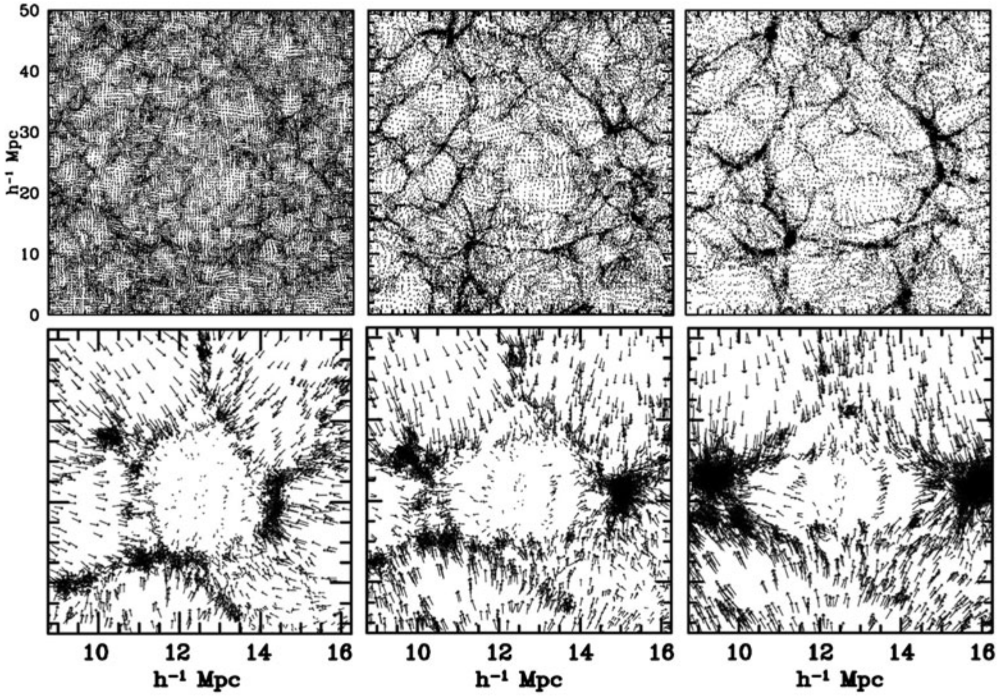

Figure 12. The two modes of void evolution: void merging (top row) and void collapse (bottom row). Top: three timesteps of evolving void structure in a 128^3 particle N-body simulation of structure formation in an SCDM model ($a_{\rm exp} = 0.1, 0.3, 0.5$). The sequence shows the gradual development of a large void of diameter $\approx 25h^{-1}$Mpc as the complex pattern of smaller voids and structures which had emerged within it at an earlier time, merge with one another. It illustrates the *void-in-void* process of the evolving void hierarchy. Bottom: a choice of three collapsing voids in a constrained N-body simulation, each embedded within an environment of different tidal shear strength. The arrows indicate the velocity vectors, showing the infall of outer regions onto the void region. As a result the voids will be crushed as the surrounding matter rains down on them.

between matter clumps and voids. While the number density of halos is dominated by small objects, void collapse is responsible for the lack of small voids.

5.1. *Void Excursions*

Marked by the two processes of merging and collapse, the complex hierarchical buildup of the void population may be modelled by a two-barrier excursion set formalism (Sheth & van de Weygaert 2004). The barriers refer to the critical (linear) density thresholds involved with the *merging* and *collapse* of voids. Whenever the linearly extrapolated (primordial) $\delta_L(\vec{r}, t|R)$ on a scale R,

$$\delta_L(r, t|R) = \frac{D(t)}{D(t_i)}\,\delta_L(r, t_i|R)\,, \tag{5.1}$$

exceeds a density threshold f_c it will collapse. For an Einstein-de Sitter $\Omega_m = 1$ Universe the critical value has the well-known value $f_c \simeq 1.686$ (Gunn & Gott 1972). A void will form when an underdensity reaches the critical density threshold of *shell-crossing*, corresponding to a value of $f_v \simeq -2.81$ for spherical voids in an Einstein-de Sitter Universe. The linear density growth factor $D(t)$, normalized to unity at the present epoch, follows from the integral (Heath 1977; Peebles 1980; Hamilton 2001; Lahav &

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Suto 2004),

$$D(t) = D(t, \Omega_{m,0}, \Omega_{\Lambda,0}) = \frac{5\,\Omega_{m,0} H_0^2}{2}\, H(a) \int_0^a \frac{da'}{a'^3 H^3(a')} \,. \tag{5.2}$$

Emerging from a primordial Gaussian random field, many small voids may coexist within one larger void. Small voids from an early epoch will merge with one another to form a larger void at a later epoch. The excursion set formalism takes account of this *void-in-void* configuration by discarding these small voids from the void count. To account for the impact of voids disappearing when embedded in collapsing regions, the two-barrier formalism also deals with the *void-in-cloud* problem.

By contrast, the evolution of overdensities is governed only by the *cloud-in-cloud* process: the *cloud-in-void* process is much less important, because clouds which condense in a large scale void are not torn apart as their parent void expands around them. This asymmetry between how the surrounding environment affects halo and void formation is incorporated into the *excursion set approach* by using one barrier to model halo formation and a second barrier to model void formation. Only the first barrier matters for halo formation, but both barriers play a role in determining the expected abundance of voids.

5.2. *Void Population Statistics.*

The analytical evaluation of the two-barrier random walk problem in the extended Press-Schechter approach leads directly to a prediction of the distribution function $n_v(M)$ for voids on a mass scale M (or corresponding void size R†). The resulting void spectrum‡. is peaked, with a sharp cutoff at both small and large values of the peak mass $M_{v,*}$ (fig. 5.2, bottom lefthand frame),

$$n_v(M)\,\mathrm{d}M \approx \sqrt{\frac{2}{\pi}}\,\frac{\rho_u}{M^2}\,\nu_v(M)\,\exp\left(-\frac{\nu_v(M)^2}{2}\right)\left|\frac{\mathrm{d}\ln\sigma(M)}{\mathrm{d}\ln M}\right| \exp\left\{-\frac{|f_{\rm v}|}{f_{\rm c}}\,\frac{\mathcal{D}^2}{4\nu_v^2} - 2\frac{\mathcal{D}^4}{\nu_v^4}\right\}, \tag{5.3}$$

where $\sqrt{\nu_v(M)}$ is the fractional relative underdensity,

$$\nu_v(M) \equiv \frac{|f_{\rm v}|}{\sigma(M)}\,, \tag{5.4}$$

in which with the dependence on the mass scale M entering via the rms density fluctuation on that scale, $\sigma(M)$. The quantity $\mathcal{D}$ is the "*void-and-cloud parameter*", $\mathcal{D} \equiv |f_{\rm v}|/(f_{\rm c} + |f_{\rm v}|)$. It parameterizes the impact of halo evolution on the evolving population of voids: the likelihood of smaller voids being crushed through the *void-in-cloud* process decreases as the relative value of the collapse barrier $f_{\rm c}$ with respect to the void barrier $f_{\rm v}$ becomes larger.

Assessment of the relation above shows that the population of large voids is insensitive to the *void-in-cloud* process. The large mass cutoff of the void spectrum is similar to the ones for clusters and reflects the Gaussian nature of the fluctuation field from which the objects have condensed. The *characteristic void size* increases with time: the gradual

† The conversion of the void mass scale to equivalent void radius R is done by assuming the simplest approximation, that of the spherical tophat model. According to this model a void has expanded by a factor of 1.7 by the time it has mature, so that $V_v = (M/\rho_u) * 1.7^3$.

‡ Note that for near-empty voids the mass scale M is nearly equal to the corresponding void mass deficit.

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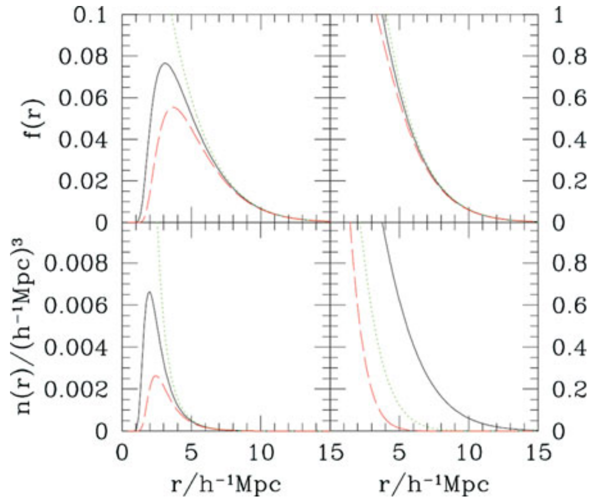

Figure 13. Distribution of void radii predicted by the two-barrier extended PS formalism, in an Einstein de-Sitter model with $P(k) \propto k^{-1.5}$, normalized to $\sigma_8 = 0.9$ at $z = 0$. Top left panel shows the mass fraction in voids of radius r. Bottom left panel shows the number density of voids of radius r. Note that the void-size distribution is well peaked about a characteristic size provided one accounts for the void-in-cloud process. Top right panel shows the cumulative distribution of the void volume fraction. Dashed and solid curves in the top panels and bottom left panel show the two natural choices for the importance of the void-in-cloud process discussed in the text: $\delta_c = 1.06$ and 1.686, with $\delta_v = -2.81$. Dotted curve shows the result of ignoring the *void-in-cloud* process entirely. Clearly, the number of small voids decreases as the ratio of $\delta_c/|\delta_v|$ decreases. Bottom right panel shows the evolution of the cumulative void volume fraction distribution. The three curves in this panel are for $\delta_c = 1.686(1+z)$, where $z = 0$ (solid), 0.5 (dotted) and 1 (dashed). From Sheth & van de Weygaert 2004.

merging of voids into ever larger ones is embodied in a self-similar shift of the peak of the void spectrum.

When evaluating the corresponding fraction of contained in voids on mass scale M, $f(M) = M\,n_v(M)/\rho_u$, we find that this also peaks at the characteristic void scale. It implies a mass fraction in voids of approximately thirty percent of the mass in the Universe, with most of the void mass to be found in voids of this characteristic mass.

While the two-barrier excursion set formalism offers an attractive theoretical explanation for the distinct asymmetry between clumps and voids and for the peaked void size distribution, realistic cosmological simulations are needed to identify where the disappearing small-scale voids are to be found in a genuine evolving cosmic matter distribution.

An important aspect of the implied void population is that it is approximately *space-filling*. It underlines the adagio that the large scale distribution of matter may be compared to a *soapsud of expanding bubbles*. This follows from evaluation of the cumulative integral

$$\mathcal{F}_V(M) \equiv \int_M^{\infty} (1.7)^3\, \frac{M'\, n_v(M')}{\rho_u}\, dM' \, . \tag{5.5}$$

where the factor 1.7 is an estimate of the excess expansion of the void based upon the spherical model for void evolution (see footnote). The top righthand panel of fig. 5.2 shows the resulting (current) cumulative void volume distribution: for a finite value of void radius R the whole of space indeed appears to be occupied by voids, while there is the gradual shift of the cumulative volume distribution towards larger voids. In other

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words, the correct image appears to be that of a gradually unfolding bubbly universe in which the average size of the voids grows as small voids merge into ever larger ones.

While the SvdWmodel has proven to provide a good model for the evolving void population, it should be considered as a basic framework. Substantial finetuning to realistic circumstances, devoid of spherical symmetry and other implicit simplifications, will likely improve the model. One major issue concerns the value of the typical void formation threshold or density barrier. The simple choice for the shellcrossing of a hypothetical spherical configuration certainly deserves scrutiny, and considerable improvements to this have been forwarded (Paranjape *et al.* 2012; Jennings *et al.* 2013; Pontzen *et al.* 2016).

Also, to connect the SvdW formalism to the voids in the observed galaxy distribution involves a range of factors. Galaxies are diluted and biased tracers of the underlying dark matter distribution. While the dark matter voids define a rich and complex structural pattern (see e.g. Leclercq *et al.* 2015; Leclercq 2015), voids in the galaxy distribution will only represent a limited aspect of the void populations. Furlanetto & Piran (2006) elaborated on the SvdW formalism to describe what it would imply for voids in the galaxy distribution.

The advantage of having an analytical formalism for describing the void population has recently been recognized in the context of casting constraints on the nature of dark energy. By evaluating the implied void population in a range of dark energy scenarios, Pisani *et al.* (2015) demonstrated that in upcoming surveys such as Euclid it will be possible to infer competitive estimates of the equation of state of dark energy, as well as on its time derivative. The use of the analytical description of the two-barrier excursion set model provides highly versatile and flexible framework for readily identifying the most sensitive factors and situations. The same philosophy was followed in the elaboration by Li & Efstathiou (2012), who applied the excursion set formalism towards a general description of the evolving void population in modified gravity scenarios.

5.3. *Void Substructure*

An important issue within the hierarchically proceeding evolution of voids and the Cosmic Web is the fate of its substructure. In voids the diluted and diminished infrastructure remains visible, at ever decreasing density contrast, as cinders of the earlier phases of the *void hierarchy* in which the substructure stood out more prominently.

N-body simulations show that voids do retain a rich infrastructure. Examples such as the fig. 5, and images of large cosmological simulations such as the Millennium and the Illustris simulation (Springel *et al.* 2005) show that while void substructure does fade, it does not disappear. We may find structures ranging from filamentary and sheetlike structures to a population of low mass dark matter halos and galaxies (see e.g. Rieder *et al.* 2013). Although challenging, these may also be seen in the observational reality. The galaxies that populate the voids do currently attract quite some attention (see next section). Also, the SDSS galaxy survey has uncovered a substantial level of substructure within voids, such as a the VGS31 filament by Beygu *et al.* (2013).

The gradual fading of the initially rich substructure as a void expands, and its matter Rcontent evacuates its interior, is clearly visible in the development of the void region in fig. 2.2. This may be seen as the slowing of structure growth in the low-density environment of voids. The resulting observation is one in which the density distribution in a void region undergoes a manifest and continuous transformation towards an ever larger dominating scale, while it mostly retains the topological features of the initial density field. In a sense, it reflects of the buildup of the Cosmic Web itself, its basic pattern already recognizable in the primordial matter distribution (Bond, Kofman & Pogosyan 1996; van de Weygaert & Bond 2008).

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