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Uniqueness of invariant means on certain introverted spaces

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Let S be a topological semigroup with separately continuous multiplication and H a uniformly closed invariant subspace of $LUC(S)$ (the space of left uniformly continuous bounded functions on S) that contains the constants. It is shown that if H is left introverted and H admits a tight two-sided invariant mean m, then for each $h \in H$, $m(h)$ is the unique constant function in the norm closed convex hull of the left orbit of h; consequently, H has a unique left invariant mean. (In fact, it is enough for H to admit a tight right invariant mean and a left invariant mean.) For certain S, a similar result is obtained when H is a left compact-open introverted subspace of $LCC(S)$ (the space of left compact-open continuous functions on S).

Introduction

The following theorem, which is a generalization of the uniqueness of Haar measure on a compact topological group, is apparently well-known (see for example, [3, p. 10] for the case when S has an identity and $H = W(S)$).

THEOREM. *Let S be a topological semigroup with separately continuous multiplication and H a uniformly closed invariant subspace of $W(S)$ (the space of weakly almost periodic functions on S). If H admits a two-sided invariant mean m, then for each $h \in H$, $m(h)$ is the*

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unique constant function in the norm closed convex hull of the right orbit of h *and the unique constant function in the norm closed convex hull of the left orbit of* h ; *in particular,* H *has a unique two-sided, left or right invariant mean.*

A quick proof, part of the folklore, of this theorem is as follows. Let $h \in H$ and let K_h be the norm (weakly) closed convex hull of the right orbit $R(h)$ of h. Then K_h is weakly compact (Krein-Šmulian) and $m(h)$ is in the pointwise closed convex hull of $R(h)$ (see for example, [9, Theorem 1]). Since the pointwise topology is weaker than the weak topology, the pointwise and weak topologies coincide on K_h and therefore $m(h)$ is in the norm closed convex hull of $R(h)$. The argument for $L(h)$ is similar since $L(h)$ is necessarily weakly relatively compact (for example, [3]). The uniqueness assertions are then obvious (see the proof of Theorem 1 below).

In this paper we obtain two theorems of the above type, one for tight invariant means on introverted subspaces of LUC(S) and one for compact-open continuous invariant means on compact-open introverted subspaces of LCC(S) (the compact-open analogue of LUC(S)). The motivation for the proofs of these theorems was an alternative argument of the author (in the spirit of [3, pp. 10, 11]) for the theorem stated above under the additional assumption that H is introverted. The basic tool employed was a form of Day's fixed point theorem due to Argabright (see [17], [2]). For the proofs of our main theorems we employ two fixed point results of the latter type whose proofs are implicitly contained in the treatment of Section 4 in [10].

1. Definitions and notations

Throughout this paper a *topological semigroup* is a semigroup equipped with a Hausdorff topology for which multiplication is separately continuous. If S is a topological semigroup, then $\mathbb{R}^S$ $(m(S))$ denotes the space of all (bounded) real-valued functions on S and $C(S)$ $(BC(S))$ the space of all (bounded) continuous real-valued functions on S. If $f \in \mathbb{R}^S$ and $s \in S$, then f_s, f^s denote the functions defined on S by

$f_s(a) = f(sa)$ and $f^s(a) = f(as)$ for all $a \in S$. If $f \in \mathbb{R}^S$, then $R(f)$ denotes the *right orbit* $\{f^s : s \in S\}$ of f and $L(f)$ the *left orbit* $\{f_s : s \in S\}$ of f. A linear subspace H of $\mathbb{R}^S$ is *invariant* if $h_s \in H$ and $h^s \in H$ for all $h \in H$ and all $s \in S$. An invariant subspace H of $m(S)$ is *left (right) introverted* if for every $\psi \in H^*$ (the Banach dual of H) and every $h \in H$, $\psi(h_s)$ $(\psi(h^s))$ considered as a function of s lies in H. Similarly, an invariant subspace H of $\mathbb{R}^S$ is *left (right) compact-open introverted* if the latter condition holds where H^* is the dual space of H with H equipped with the compact-open topology (the topology of uniform convergence on compacta).

The space of *left uniformly continuous* functions on S, denoted $\mathrm{LUC}(S)$, is the set of f in $BC(S)$ such that the map $s \to f_s$ is continuous where $BC(S)$ has the sup norm. The space of *left compact-open continuous* functions on S, denoted $\mathrm{LCC}(S)$, is the set of f in $C(S)$ such that the map $s \to f_s$ is continuous where $C(S)$ has the compact-open topology. In a similar manner, one defines the space $\mathrm{RUC}(S)$ of *right uniformly continuous* functions on S and the space $\mathrm{RCC}(S)$ of *right compact-open continuous* functions on S.

Let H be an invariant subspace of $\mathbb{R}^S$ that contains the constant functions. A linear functional m on H is said to be a *mean* on H if $m(h) \geq 0$ for all $h \in H$ with $h \geq 0$ and $m(1) = 1$. m is a *left (right) invariant mean* on H if m is a mean on H and $m(h_s) = m(h)$ $(m(h^s) = m(h))$ for all $h \in H$ and all $s \in S$. m is *two-sided* invariant if m is both left and right invariant.

If K is a convex subset of a real locally convex Hausdorff space E, then $E(K)$ denotes the set of restrictions of the real-valued continuous affine functions on E to K; that is, $E(K) = E^*|K + \mathbb{R}$ and $A(K)$ denotes the space of all real-valued continuous affine functions on K. $BE(K)$ and $BA(K)$ denote respectively the bounded functions in $E(K)$ and $A(K)$. If S is a semigroup and K is a convex subset of E with the relative topology, then an *(restricted) affine action* of S on K denoted (S, K), is a map from $S \times K$ into K, $(s, x) \to s\cdot x$, such that

(1) $s_1\cdot(s_2\cdot x) = (s_1s_2)\cdot x$ for all $s_1, s_2 \in S$ and all $x \in K$;

(2) for each $s \in S$, the map $x \to s\cdot x$ is a (restriction of a) continuous affine map of K (E) into K (E) .

If (S, K) is an affine action of S on K and $x \in K$, then $Tx : A(K) \to \mathbb{R}^S$ is the map defined by $Txf(s) = f(s\cdot x)$ for all $f \in A(K)$ and all $s \in S$.

2. Two fixed point results

The two lemmas below were stated in [10]. Lemma 1 is a slight generalization of a result of Khurana [13, Theorem 2.1]. We include a proof for the sake of completeness (see the proof of Theorem 2.2 in [13]).

DEFINITION. Let X be a Hausdorff topological space and H a linear space of bounded continuous real-valued functions on X that contains the constant functions. A mean μ on H (μ is linear, $\mu(h) \geq 0$ for $h \geq 0$ and $\mu(1) = 1$) is said to be *tight* if for each uniformly bounded net $\{h_\alpha\}$ in H with $h_\alpha \to 0$ uniformly on compact subsets of X, we have $\mu(h_\alpha) \to 0$. A mean μ on H is *σ-additive* if for each sequence $\{h_n\} \subset H$ with $h_n \downarrow 0$, we have $\mu(h_n) \to 0$.

LEMMA 1. *Let K be a complete bounded convex subset of a locally convex Hausdorff space E. If μ is a tight mean on $E(K)$ (respectively $BA(K)$), then there exists a unique x_0 in K such that $\mu(f) = f(x_0)$ for all f in $E(K)$ (respectively $BA(K)$).*

Proof. Let $\tilde{E}$ be the completion of E and equip $\tilde{E}^*$ with the weak topology $\sigma(\tilde{E}^*, \tilde{E})$. Define the linear functional $\hat{\mu} : \tilde{E}^* \to \mathbb{R}$ by $\hat{\mu}(g) = \mu(g|K)$ for all $g \in \tilde{E}^*$. We need only show that $\hat{\mu}$ is $\sigma(E^*, \tilde{E})$-continuous. For then $\hat{\mu}$ must be the evaluation functional at some $x_0 \in \tilde{E}$ and since K is closed convex in $\tilde{E}$ we must have $x_0 \in K$. x_0 is unique since $E(K)$ separates points. (If μ is a tight mean on $BA(K)$, then μ is the restriction of a finitely additive positive measure of total mass one on K (Hahn-Banach) and therefore by [14, Theorem 1] $\mu(f) = f(x_0)$ for f in $BA(K)$ as well.)

In order to show that $\hat{\mu}$ is $\sigma(\tilde{E}^*, \tilde{E})$-continuous it suffices by [12, p. 156] to verify that $\hat{\mu}$ is continuous on every equicontinuous subset G of $\tilde{E}^*$. Let $\{g_\alpha\}$ be a net in G and $g \in G$ such that $g_\alpha \to g$ pointwise on $\tilde{E}$. Since K is bounded and G is equicontinuous, $\{g_\alpha\}$ is uniformly bounded on K. Since G is equicontinuous and $g_\alpha \to g$ pointwise on $\tilde{E}$, $g_\alpha \to g$ uniformly on totally bounded sets [12, p. 76]. Since μ is tight, $\mu(g_\alpha|K) \to \mu(g|K)$.

A slight modification of the proof of Lemma 1 yields

LEMMA 2. *Let K be a complete convex subset of a locally convex Hausdorff space E. If μ is a mean on $E(K)$ which is continuous when $E(K)$ has the compact-open topology, then there exists a unique x_0 in K such that $\mu(f) = f(x_0)$ for all f in $E(K)$.*

The following definitions are adapted from Mitchell [17] and Argabright [2].

Let S be a topological semigroup and H a subset of $C(S)$. Let (S, K) be a (restricted) affine action of S on K where K is a convex subset with relative topology of a locally convex space E. The action (S, K) is an *E-representation of S, H on K by (restricted) continuous affine maps* if there is an x in K such that the map $s \to s\cdot x$ is continuous and $Tx(A(K)) \subset H$ $(Tx(E(K)) \subset H)$. The action (S, K) is a *bounded E-representation of S, H on K by (restricted) continuous affine maps* if there is an x in K such that $s \to s\cdot x$ is continuous and $Tx(BA(K)) \subset H$ $(Tx(BE(K)) \subset H)$. Let $\mathcal{K}$ be a class of convex subsets of locally convex Hausdorff spaces. The pair S, H has the *common fixed point property on the sets in $\mathcal{K}$ with respect to E-representations by (restricted) affine maps* if for each K in $\mathcal{K}$ and E-representation (S, K) of S, H by (restricted) continuous affine maps, K has a common fixed point for the action of S. The common fixed point property with respect to bounded E-representations is defined similarly.

REMARK. Note that if K is compact and $x \in K$ with $Tx(E(K)) \subset H$, then necessarily $s \to s \cdot x$ is continuous. Consequently, the above definition of an E-representation is consistent with that given by Argabright in [2].

The following two propositions are implicit in the proofs of Proposition 4.7 and "(1) implies (2)" of Proposition 4.13 in [10]. The technique of the proof appears throughout the literature (see, for example, [17], [2]). We include the proof since it is so short.

PROPOSITION 1. *Let S be a topological semigroup and H an invariant subspace of $BC(S)$ that contains the constant functions. If H admits a tight left invariant mean, then S, H has the common fixed point property on complete bounded convex sets with respect to E-representations by restricted affine maps and on complete bounded convex sets with respect to bounded E-representation by affine maps.*

Proof. Let (S, K) be an E-representation of S, H by restricted continuous affine maps on the complete bounded convex set K. Let $x \in K$ be such that $s \to s \cdot x$ is continuous and $Tx\big(E(K)\big) \subset H$. If $Tx^* : H^* \to E(K)^*$ is the adjoint of Tx and m is a tight left invariant mean on H, then Tx^*m is a tight mean on $E(K)$ which is invariant under the action of S on K. Consequently, by Lemma 1, there is an x_0 in K such that $f\big(s \cdot x_0\big) = f\big(x_0\big)$ for all $f \in E(K)$ and all $s \in S$. Since $E(K)$ separates points, $s \cdot x_0 = x_0$ for all $s \in S$. The argument is similar for bounded E-representations by continuous affine maps.

Applying in the above proof Lemma 2 in place of Lemma 1, we obtain

PROPOSITION 2. *Let S be a topological semigroup and H an invariant subspace of $C(S)$ that contains the constant functions. If H admits a compact-open continuous left invariant mean, then S, H has the common fixed point property on complete convex sets with respect to E-representations by restricted affine maps.*

REMARK. We note that if in Proposition 2, S is realcompact and for every $f \in C(S)$ with $f \geq 0$ there exists $h \in H$ such that $f \leq h$, then every mean on H is compact-open continuous. For every mean on H has an extension to a mean on $C(S)$ [18, p. 82] and every mean on $C(S)$ is necessarily compact-open continuous [6, Theorem 5.3].

3. The main theorems

We now apply Propositions 1 and 2 to obtain our main results. The proof of the following theorem is an adaptation of the proof of Lemma 5.1 in [4] to our present setting (*cf.* also [3, pp. 10, 11]).

THEOREM 1. *Let S be a topological semigroup and H a uniformly closed invariant subspace of* $\mathrm{LUC}(S)$ (*respectively* $\mathrm{RUC}(S)$) *that contains the constant functions and is left (respectively right) introverted. If H admits a tight two-sided invariant mean m, then for each $h \in H$, $m(h)$ is the unique constant function in the norm closed convex hull of $L(h)$* (*respectively* $R(h)$)*; in particular, H has a unique left (respectively right) invariant mean.*

Proof. We assume H is right introverted and $H \subset \mathrm{RUC}(S)$. The proof for H left introverted with $H \subset \mathrm{LUC}(S)$ follows then by considering the multiplication $s_1 \circ s_2 = s_2 s_1$ on S. If for every h in H the norm closed convex hull K_h of $R(h)$ in H contains the constant function $m(h)$, then necessarily H has a unique right invariant mean. For if ν is a right invariant mean on H, then by linearity and continuity of ν we must have $\nu(m(h)) = \nu(h)$; that is, $m(h) = \nu(h)$. We show, in fact, that $m(h)$ is the unique function g in K_h such that $g = g^s$ for all $s \in S$.

Fix $h \in H$ and let (S, K_h) denote the left action of S on K_h defined by $s \cdot f = f^s$ for all $s \in S$ and all $f \in K_h$. Then (S, K_h) is a restricted affine action of S on K_h (for fixed $s \in S$, the map $f \to f^s$ on H is linear and norm continuous). Now let $f \in K_h$ and consider the map $Tf : E(K_h) \to m(S)$. Since $f \in \mathrm{RUC}(S)$, the map $s \to f^s$ is continuous and since H is right introverted with $1 \in H$, we have $Tf(E(K_h)) \subset H$. Thus, (S, K_h) is an E-representation of S, H by restricted continuous affine maps on K_h - a complete norm bounded convex subset of H. Since H admits a tight left invariant mean, the action (S, K_h) has a common fixed point by Proposition 1; that is, there is a

$g \in K_h$ such that $g = g^s$ for all $s \in S$.

Since K_h is the norm closed convex hull of $R(h)$, for each $\varepsilon > 0$, there exists a convex combination $\sum_{i=1}^{n} \lambda_i h^{s_i}$ such that $\left\| g - \sum_{i=1}^{n} \lambda_i h^{s_i} \right\| \le \varepsilon$. If we fix $x \in S$ and $\varepsilon > 0$, then $\left| g(x) - \sum_{i=1}^{n} \lambda_i h^{s_i}(xs) \right| \le \varepsilon$ for all $s \in S$; that is, $\left| g(x) - \sum_{i=1}^{n} \lambda_i (h_x)^{s_i} \right| \le \varepsilon$. If we apply m to the last inequality, we have $\left| g(x) - \sum_{i=1}^{n} \lambda_i m(h) \right| \le \varepsilon$; that is, $|g(x)-m(h)| \le \varepsilon$. Consequently, $m(h) = g(x)$ for all $x \in S$.

REMARKS. The above proof shows that in place of requiring $H \subset \mathrm{LUC}(S)$ $(\mathrm{RUC}(S))$ it is sufficient to assume that the norm closed convex hull of $L(h)$ $(R(h))$ meets $\mathrm{LUC}(S)$ $(\mathrm{RUC}(S))$ for every h in H. Also, the theorem is valid if H admits a tight right (left) invariant mean and a left (right) invariant mean where m is taken to be a two-sided invariant mean (necessarily unique) on H. (Since H is either left or right introverted, if H has a left invariant mean and a right invariant mean, then H has a two-sided invariant mean.) Under the additional assumption that S has a right (left) identity it follows from the first part of the proof (see Theorem 2) that if H admits a tight right (left) invariant mean, then the norm closed convex hull of $L(h)$ $(R(h))$ contains a constant function for each h in H.

It is of interest to note here a result of Granirer and Lau. In [9] it is shown that if $\mathrm{LUC}(S)$ has a left invariant mean m, then for each $h \in H$, $m(h)$ is in the compact-open closed convex hull of $R(h)$. Consequently, if $\mathrm{LUC}(S)$ admits a compact-open continuous right invariant mean and a left invariant mean, then $\mathrm{LUC}(S)$ has a unique compact-open continuous right invariant mean.

COROLLARY. *Let S be a semigroup. If $m(S)$ admits a σ-additive right (respectively left) invariant mean and a left (respectively right)*

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invariant mean, then $m(S)$ *has a unique left (respectively right) invariant mean.*

Proof. If we equip S with the discrete topology, then $LUC(S) = RUC(S) = m(S)$. If S is countable, then every σ-additive mean on $m(S)$ is tight (for example, [11, p. 40]) and therefore Theorem 1 applies (see the above remarks). For general S a result of Granirer can be used. Namely, if $m(S)$ has a σ-additive left invariant mean, then S contains a finite group which is a left ideal [7, Theorem 4.2]. Thus, every affine action of S on a convex subset of a vector space has a common fixed point. The proof of Theorem 1 then remains valid for S.

REMARK. If in the corollary S has left (right) cancellation, then S is a finite group [8, Corollary 2.1].

THEOREM 2. *Let* S *be a topological semigroup with right (respectively left) identity* e *such that* $C(S)$ *is complete in the compact-open topology (in particular,* S *a completely regular* k*-space). Let* H *be a compact-open closed invariant subspace of* $LCC(S)$ *(respectively* $RCC(S)$*) that contains the constant functions and is left (respectively right) compact-open introverted. If* H *admits a compact-open continuous two-sided invariant mean* m*, then for each* $h \in H$, $m(h)$ *is the unique constant function in the compact-open closed convex hull of* $L(h)$ *(respectively* $R(h)$*); in particular,* H *has a unique compact-open continuous left (respectively right) invariant mean.*

Proof. By applying Lemma 2 in place of Lemma 1, it follows exactly as in the proof of Theorem 1 that for each $h \in H$, there is a $g \in K_h$ (the compact-open closed convex hull of $R(h)$) with $g = g^s$ for all $s \in S$. Consequently, $g(e) = g(e \cdot s) = g(s)$ for all $s \in S$. It follows then $m(h) = g$ and m is the unique compact-open continuous right invariant mean on H.

REMARKS. There exist spaces S for which $C(S)$ is compact-open complete but S is not a k-space [19, p. 363]. Again, it is enough to assume that the compact-open closed convex hull of $L(h)$ $(R(h))$ meets $LCC(S)$ $(RCC(S))$ for every h in H. Of course, the argument shows that if H admits a compact-open continuous right (left) invariant mean, then the compact-open closed convex hull of $L(h)$ $(R(h))$ contains a constant function for each h in H.

COROLLARY. *Let S be a realcompact topological semigroup with jointly continuous product, right (respectively left) identity e and $C(S)$ complete in the compact-open topology. Let H be a compact-open closed invariant subspace of $C(S)$ such that H contains the constants, is left (respectively right) compact-open introverted and satisfies: if $f \in C(S)$ with $f \geq 0$, then there exists $h \in H$ with $f \leq h$. If H admits a two-sided invariant mean m, then for each $h \in H$, $m(h)$ is the unique constant function in the compact-open closed convex hull of $L(h)$ (respectively $R(h)$); in particular H has a unique left (respectively right) invariant mean.*

Proof. Every mean on H is necessarily compact-open continuous (see the remark in Section 2). Since multiplication on S is jointly continuous, $LCC(S) = RCC(S) = C(S)$ (for example, [15, Lemma 4.2]).

REMARKS. In the above corollary H can be taken to be $C(S)$ since $LCC(S) = RCC(S) = C(S)$ implies that $C(S)$ is both left and right compact-open introverted. For the case $H = C(S)$ Argabright in [1, Theorem 2.4] showed without an identity or completeness restriction that if $C(S)$ admits a two-sided invariant mean, then $C(S)$ has a unique left or right invariant mean.

We also note that the corollary is applicable to discrete semigroups S of non-measurable cardinal. For they are realcompact in the discrete topology [5, p. 163] and certainly $\mathbb{R}^S$ is compact-open complete.

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