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ON GLOBAL INVERSE THEOREMS OF SZÁSZ AND BASKAKOV OPERATORS

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The Szász and Baskakov approximation operators are given by

$$S_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} \equiv \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) P_{k,n}(x) \quad \text{and} \tag{1.1}$$

$$V_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} x^k (1+x)^{-n-k} \equiv \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) b_{k,n}(x) \tag{1.2}$$

respectively. For continuous functions on $[0, \infty)$ with exponential growth (i.e. $\|f\|_A \equiv \sup_x |f(x)e^{-Ax}| < M$) the modulus of continuity is defined by

$$w_2(f, \delta, A) \equiv \sup_{h \leqq \delta, 0 \leqq x < \infty} |f(x) - 2f(x+h) + f(x+2h)|e^{-Ax} \tag{1.3}$$

$$\equiv \sup_{h \leqq \delta, 0 \leqq x < \infty} |\Delta_h^2 f(x)| e^{-Ax},$$

where $f \in \mathrm{Lip}^*(\alpha, A)$ for some $0 < \alpha \leqq 2$ if $w_2(f, \delta, A) \leqq M\delta^\alpha$ for all $\delta < 1$. We shall find a necessary and sufficient condition on the rate of convergence of $A_n(f, x)$ (representing $S_n(f, x)$ or $V_n(f, x)$) to $f(x)$ for $f(x) \in \mathrm{Lip}^*(\alpha, A)$. In a recent paper of M. Becker [**1**] such conditions were found for functions of polynomial growth (where $(1 + |x|^N)^{-1}$ replaced e^{-Ax} in the above). M. Becker explained the difficulties in treating functions of exponential growth. For $S_n(f, x)$ he promised to treat $C = \bigcap_{\beta>0} C_\beta$ (the intersection of spaces treated here) that would not contain even the function e^{Ax} in a future paper. Concerning the Baskakov operators, Becker states: "For the Baskakov operator the situation is even more difficult as $V_n(1/w_\beta(t); x)$ (i.e. $V_n(e^{\beta t}; x)$) only exists for $x < (\exp(\beta/n) - 1)^{-1}$. Thus one has to restrict oneself to compact intervals, so that one may regard polynomial growth as a frame best suited for global (i.e. approximation on the whole $[0, \infty)$) results for the Baskakov operators". Our interest stems from the above and from the fact that point-wise convergence of $A_n(f, x)$ to $f(x)$ is known for functions of exponential growth (but no faster growth). In fact I received later a preprint of a paper by M. Becker, D. Kucharski and R. J. Nessel [**2**] also mentioned by the referee in which the authors find a global inverse theorem for Szász operators for $C = \bigcap_{\beta>0} C_\beta$ and $\bigcap_{\beta>\alpha} C_\beta$ but C_α is not treated there nor are the Baskakov operators treated for any C_α, for the same reason already quoted from [**1**].

Here as well as in [**1**] the method of the proof is the "elementary" one, i.e.,

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interpolation spaces are not used. The "elementary" method for proving inverse theorems was introduced by Berens and Lorentz [**3**] who succeeded in applying it only to the case $0 < \alpha < 1$. In a similar context the author and C. P. May [**5**] used it to prove a local inverse theorem for $0 < \alpha < 2$ and L_p functions. (The global case of that theorem is still open.) In numerous papers Becker and others in collaboration with him used that method to obtain global inverse theorems for $0 < \alpha < 2$ and continuous functions (of which one can mention [**1**] and [**2**] as well as a new proof of the inverse theorem of [**3**] using the elementary method for $0 < \alpha < 2$).

2. Some preliminary estimates. In this section some preliminary estimates will be gathered.

LEMMA 2.1. *For $S_n(f, x)$ defined by* (1.1) *we have*

$$(2.1)\qquad S_n(e^{Au}, x) \equiv \sum_{k=0}^{\infty} e^{-nx} e^{kA/n} \frac{(nx)^k}{k!} = \exp\left(\left(\exp\frac{A}{n} - 1\right)nx\right)$$

and

$$(2.2)\qquad S_n((u-x)^2 e^{Au}, x) = \{(x/n)e^{A/n} + x^2(1 - e^{A/n})^2\} \exp((\exp(A/n) - 1)nx).$$

If we assume in addition that $x \leqq n$, we have

$$(2.3)\qquad S_n(e^{Au}, x) \leqq \exp\{(A^2/2)e^A\} \cdot e^{Ax} \text{ for } x \leqq n \quad \text{and}$$

$$(2.4)\qquad S_n((u-x)^2 e^{Au}, x) \leqq (x/n)e^{Ax}(e^A + A^2 e^{2A}) \exp \times \{(A^2/2)e^A\} \equiv M(x/n)e^{Ax} \text{ for } x \leqq n.$$

Proof. While (2.1) and (2.2) are the result of straightforward computation, (2.3) and (2.4) follow (2.1) and (2.2) respectively, using the estimates

$$e^{A/n} - 1 \leqq \frac{A}{n} e^{A/n},\ e^{A/n} - 1 - \frac{A}{n} \leqq \frac{1}{2}\left(\frac{A}{n}\right)^2 e^{A/n} \quad \text{and} \quad x \leqq n \quad (\text{and} \quad n \geqq 1).$$

LEMMA 2.2. *For $V_n(f, x)$ defined by* (1.2) *we have*

$$(2.5)\qquad V_n(e^{Au}, x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} e^{kA/n} = (1 + x(1 - e^{A/n}))^{-n} \text{ for } x < (e^{A/n} - 1)^{-1} < \frac{n}{A}$$

and

$$(2.6)\qquad V_n((u-x)^2 e^{Au}, x) = (1 + x(1 - e^{A/n}))^{-n-2} \frac{x(1+x)}{n} \times \left[e^{A/n} + \frac{x(1+x)}{n}(n(e^{A/n} - 1))^2\right] \text{ for } x < (e^{A/n} - 1)^{-1} < \frac{n}{A}.$$

If we assume in addition that $x \leqq \eta\sqrt{n}$ *where* $\eta = \frac{1}{3}\min(A^{-2}, 1)$ *and* $n \geqq 2A$, *we have*

(2.7) $\quad V_n(e^{Au}, x) \leqq e \cdot e^{Ax}$ *for* $x \leqq \eta\sqrt{n}$ *and* $n \geqq 2A$ *and*

(2.8) $\quad V_n((u-x)^2 e^{Au}, x) \leqq M\dfrac{x(1+x)}{n} e^{Ax}$ *for* $x \leqq \eta\sqrt{n}$ *and* $n \geqq 2A$.

Proof. Equalities (2.5) and (2.6) follow straightforward computation using

$$\sum_{k=0}^{\infty}\binom{n+k-1}{k}Z^k = \left(\frac{1}{1-Z}\right)^n \quad \text{and} \quad Z = \frac{xe^{A/n}}{1+x}$$

for $|Z| < 1$, that is, for $x < (e^{A/n} - 1)^{-1} < n/A$. From (2.5) and (2.6) we will derive (2.7) and (2.8) respectively (using also $x \leq \eta\sqrt{n}$ and $n \geqq 2A$ to derive $[1 + x(1 - e^{A/n})]^{-2} \leqq M$). We first show

(2.9) $\quad 1 < (1 + x(1 - e^{A/n}))^{-n}e^{-Ax} < e$ for $x \leqq \eta\sqrt{n}$ and $n > 2A$.

It is enough to show that $0 < -Ax + n\sum_{k=1}^{\infty} k^{-1}x^k(e^{A/n} - 1)^k < 1$. Using $e^{A/n} - 1 > A/n$ the left inequality follows. One can write $x(e^{A/n} - 1) \leqq xAn^{-1}e^{A/n} \leqq \frac{1}{3}n^{-1/2}e^{1/2} < \frac{2}{3}$ which implies

$$\begin{aligned} -Ax + n\textstyle\sum_{k=1}^{\infty} k^{-1}x^k(e^{A/n} - 1)^k &< -Ax + nx(e^{A/n} - 1) \\ &\qquad + (3/2)nx^2(e^{A/n} - 1)^2 \\ \leqq -Ax + Ax + \frac{1}{2^2}\, nx\frac{A^2}{n^2}e^{A/n} + \frac{3}{2}nx^2(A/n)^2e^{2A/n} &< \frac{e^{1/2}}{2}\frac{1}{3}\frac{1}{\sqrt{n}} + \frac{e}{6} < 1, \end{aligned}$$

and completes the proof of (2.9). Using (2.9), we derive (2.7) and (2.8).

3. Estimates of $A_n(f, x)$ and $\dfrac{d^2}{dx^2}A_n(f, x)$.

Using Lemmas of Section 2, we shall estimate $A_n(f, x)$ for functions of exponential growth.

THEOREM 3.1. *For* $||f||_A \equiv \sup|f(x)|e^{-Ax} < \infty$ *we have*

(3.1) $\quad e^{-Ax}|S_n(f, x)| \leqq ||f||_A \cdot e^{A^2/2}$ *for* $x \leqq n$, *and*

(3.2) $\quad e^{-Ax}|V_n(f, x)| \leqq ||f||_A e$ *for* $x \leqq \eta\sqrt{n}$, $n \geqq 2A$ *and* $\eta = \frac{1}{3}(\min(A^{-2}, 1))$.

If in addition $||f''||_A < \infty$, *we have*

(3.3) $\quad e^{-Ax}|S_n(f, x) - f(x)| \leqq ||f''||_A \cdot (x/n)(M+1)/2$ *for* $x \leqq n$ *and*

(3.4) $\quad e^{-Ax}|V_n(f,x) - f(x)| \leq ||f''||_A \dfrac{x(1+x)}{n}\left(\dfrac{M+1}{2}\right)$ *for* $x \leqq \eta\sqrt{n}$, $n \geq 2A$

where M *depends only on* A.

Proof. Since $S_n(f, x)$ and $V_n(f, x)$ are positive functionals, we derive (3.1) and (3.2) from (2.3) and (2.7) respectively, using $|f(x)| \leqq ||f||_A e^{Ax}$. We could have expressions for all x instead of (3.1) and for $x < (e^{A/n} - 1)^{-1}$ instead of (3.2) using (2.1) and (2.5) instead of (2.3) and (2.7) respectively, but the added information would just complicate our expressions and not help in later investigations. Using Taylor's formula $f(u) - f(x) = (u - x)f'(x) + \frac{1}{2}(u - x)^2 f''(\xi)$ (ξ is between x and u) and $S_n((u - x), x) = V_n((u - x), x) = 0$, we have

$$|A_n(f, x) - f(x)| = |A_n(\tfrac{1}{2}(u - x)^2 f''(\xi))| \qquad (A_n(f, x) \text{ is } S_n(f, x) \text{ or } V_n(f, x)).$$

We write $|f''(\xi)| \leqq ||f''||_A e^{A\xi} \leqq ||f''||(e^{Au} + e^{Ax})$. Using positivity and linearity' we have

$$e^{-Ax}|A_n(f, x) - f(x)| \leqq ||f''||_A e^{-Ax} A_n(\tfrac{1}{2}(x - u)^2(e^{Au} + e^{Ax}), x) = \tfrac{1}{2}||f''||_A[e^{-Ax}A_n((x - u)^2 e^{Au}, x) + A_n((x - u)^2, x)].$$

Using the estimates (2.4) and (2.8), we complete the proof.

We shall now estimate $A_n''(f, x)$.

THEOREM 3.2. *For* $||f||_A = \sup_x |e^{-Ax}f(x)| < \infty$ *and* $\eta = \frac{1}{3}\min(A^{-2}, 1)$, *we have:*

(3.5) $e^{-Ax}|S_n''(f, x)| \leqq M(A)(n/x)||f||_A$ *for* $0 < x \leqq n$;

(3.6) $e^{-Ax}|S_n''(f, x)| \leqq M(A)n^2||f||_A$ *for* $0 < x \leqq n$;

(3.7) $e^{-Ax}|V_n''(f, x)| \leqq M(A)nx^{-1}(1 + x)^{-1}||f||_A$ *for* $0 < x \leqq \eta\sqrt{n}$, $2A \leqq n$; *and*

(3.8) $e^{-Ax}|V_n''(f, x)| \leqq M(A)n^2||f||_A$ *for* $x \leqq \eta\sqrt{n}$, $2A \leqq n$.

If in addition $||f''||_A < \infty$, *we have*

(3.9) $e^{-Ax}|S_n''(f, x)| \leqq M(A)||f''||_A$ *for* $x \leqq n$ *and*

(3.10) $e^{-Ax}|V_n''(f, x)| \leqq M(A)||f''||_A$ *for* $x < \eta\sqrt{n}$, $2A \leqq n$.

Proof. We observe that for $|f(x)| \leqq Me^{Ax}$ we have

$$S_n''(f(u), x) = (n/x)^2 S_n((u - x)^2 f(u), x) + n^{-1}(n/x)^2 S_n(uf(u), x) \quad \textit{for} \quad 0 < x \leqq n \tag{3.11}$$

and

$$V_n''(f(u), x) = \left(\frac{n}{x(x + 1)}\right)^2\left[V_n((u - x)^2 f(u), x) - \frac{1 + 2x}{n} \times V_n((u - x)f(u), x) - \frac{x(x + 1)}{n}V_n(f(u), x)\right] \text{for } 0 < x < (e^{A/n} - 1)^{-1}. \tag{3.12}$$

Similar computations were done in other cases (see [**1**], [**3**] and [**7**, p. 1231]). Since $|S_n(uf(u), x)| \leqq ||f||_A S_n(ue^{Au}, x) \leqq ||f||_A xe^{A/n}e^{Ax}e^{A^2/2}$, $x < n$

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and $|S_n((u-x)^2 f(u), x)| \leqq ||f||_A S_n((u-x)^2 e^{Au}, x)$, we obtain (3.5) combining (2.4) and (3.11). For $x \geqq 1/n$ the Cauchy-Schwartz inequality yields

$$|V_n((u-x)f(u), x)| \leqq V_n((u-x)^2|f(u)|, x)^{1/2} \cdot V_n(|f(u)|, x)^{1/2}$$

which together with (3.2), (3.12), (2.8) and an argument similar to the above implies (3.7).

We recall also [**8**, Theorem 1, p. 475] (see also [**6**])

$$(3.13) \quad S_n''(f, x) = n^2 \textstyle\sum_{k=0}^{\infty} \Delta_{1/n}{}^2 f(k/n) P_{k,n}(x) \quad \text{where}$$
$$\Delta_{1/n} f(k/n) = f((k+1)/n) - f(k/n).$$

Since $|\Delta_{1/n}{}^2 f(k/n)| \leqq 4||f||_A \cdot e^{kA/n}$, (3.1) implies (3.6). Using

$n^2 \Delta_{1/n}{}^2 f(k/n) = f''(\xi)$, $k/n < \xi < (k+2)/n$, we have

$$|n^2 \Delta_{1/n}{}^2 f(k/n)| = |f''(\xi)e^{-A\xi}|e^{A\xi} \leqq e^{2A/n}||f''||_A e^{kA/n} \leqq e^{2A}||f''||_A e^{kA/n}$$

which combined with (3.1) implies (3.9).

Similarly (using again [**8**, p. 475] or [**6**])

$$(3.14) \quad V_n''(f, x) = n(n+1) \textstyle\sum_{k=0}^{\infty} \Delta_{1/n}{}^2 f(k/n) b_{k,n+2}(x)$$

will, using (3.2) for V_{n+2}, imply (3.8) and (3.10).

4. The direct and inverse theorem. The main result is given by the following theorem.

THEOREM 4.1. *Let* $\sup |f(x)e^{-Ax} = ||f||_A < \infty$, *then for* $0 < \alpha < 2$ *the following are equivalent:*

$$(4.1) \quad e^{-Ax}|S_n(f, x) - f(x)| \leqq M_1(x/n)^{\alpha/2} \text{ for } x \leqq n;$$

$$(4.2) \quad e^{-Ax}|V_n(f, x) - f(x)| \leqq M_2(x(1+x)/n)^{\alpha/2} \text{ for } x \leqq \eta\sqrt{n}$$

where $\eta = \frac{1}{3}\min(A^{-2}, 1)$ *and* $n \geqq 2A$; *and*

$$(4.3) \quad f \in \text{Lip}^*(\alpha, A) \quad \text{that is} \quad e^{-Ax}|f(x) - 2f(x+h)$$
$$+ f(x+2h)|h^{-\alpha} \leqq M_3 \quad \text{for} \quad 0 \leqq x < \infty \quad \text{and} \quad 0 < h < 1;$$

where the M_i *do not depend on* n *or* x.

Remark 4.1. Since the rate at which $(x/n)^{\alpha/2}$ or $(x(1+x)/n)^{\alpha/2}$ tend to zero (in (4.1) and (4.2) respectively) is related to the smoothness, we naturally restrict ourselves to $x \leqq n$ or $x \leqq \eta\sqrt{n}$. We also observe that for $x > n$ or $x > \eta\sqrt{n}$ better estimates follow from $||f||_A < \infty$. We could replace (4.1) by

$$(4.4) \quad e^{-Ax}|S_n(f; x) - f(x)| \leqq M_3(x/n)^{\alpha/2} \exp[n(\exp(A/n) - 1)x - Ax]$$
$$0 \leqq x < \infty,$$

but this harsher condition would not yield any new information on the smoothness of f.

Proof. We recall (see [**1**], [**4**]) the function $f_\delta(x)$ given by

$$(4.5)\quad f_\delta(x) \equiv (2/\delta)^2 \int_0^{\delta/2}\int_0^{\delta/2} [2f(x+u+v) - f(x+2(u+v))]dudv.$$

$f_\delta(x)$ is a C^2 approximation of f. Obviously $f_\delta''(x) = \delta^{-2}[8\Delta_{\delta/2}{}^2 f(x) - \Delta_\delta{}^2 f(x)]$ which, together with the definition, yields

$$(4.6)\quad \|f - f_\delta\|_A \leqq w_2(f;\delta, A) \quad \text{and} \quad \|f_\delta''\|_A \leqq 9\delta^{-2}w_2(f,\delta,A).$$

To show (4.1) or (4.2) implies (4.3) we write (compare also [**1**])

$$\begin{aligned} e^{-Ax}&|f(x) - 2f(x+h) + f(x+2h)| \\ &\leqq e^{-Ax}\{|f(x) - A_n(f,x)| + 2|f(x+h) - A_n(f,x+h)| \\ &\quad + |f(x+2h) - A_n(f,x+2h)|\} + e^{-Ax}|\Delta_h{}^2 A_n(f,x)| = \\ &\qquad\qquad I_1 + I_2, \quad \text{and} \end{aligned}$$

$$\begin{aligned} I_2 &= e^{-Ax}|\Delta_h{}^2 A_n(f,x)| \\ &= e^{-Ax}(|\Delta_h{}^2 A_n(f - f_\delta, x)| + |\Delta_h{}^2 A_n(f_\delta, x)|) = J_1 + J_2. \end{aligned}$$

For the Szász operator we have for fixed x and h satisfying $x \leqq n - 1$ and $h \leqq \frac{1}{2}$ the following estimates:

$$I_1 \leqq M(x+2h)/n)^\alpha \leqq M(\max(3/n^2, (x+2h)/n))^{\alpha/2} \quad \text{using (4.1)};$$

$$J_2 = h^2|S_n''(f_\delta,\xi)| \leqq M(h^2/\delta^2)w_2(f,\delta,A) \qquad \text{combining (4.6) and (3.9)};$$

$$J_1 = h^2|S_n''(f - f_\delta,\xi)| \leqq Mh^2(n/\xi)w_2(f,\delta,A) \qquad \text{combining (3.5) and (4.6)};$$

$$J_1 = h^2|S_n''(f - f_\delta,\xi)| \leqq Mh^2n^2w_2(f,\delta,A) \qquad \text{combining (3.6) and (4.6)};$$

and

$$J_1 = e^{-Ax}|\Delta_h{}^2(S_n(f - f_\delta, x))| \leqq Mw_2(f,\delta,A) \qquad \text{combining (3.1) and (4.6)}.$$

Therefore $J_1 \leqq Mw_2(f,\delta,A)\min(1, h^2n^2, h^2n/\xi)$, but for $\xi \geqq h$, $3\xi > x + 2h$ or $h^2n/\xi < h^2 3n/(x+2h)$, and for $\xi < h$, $h^2n/\xi > hn > \min(1, h^2n^2)$ and so is $h^2 3n/(x+2h) > 3h^2n/3h > hn$, and therefore

$$\begin{aligned} J_1 \leqq Mw_2(f,\delta,A)\min(1, h^2n^2, h^2 3n/(x+2h)) &\leqq 3Mh^2w_2(f,\delta,A) \\ &\times \min(n^2/3, n/(x+2h)). \end{aligned}$$

Let $\delta_{n,x}{}^2 = \max(3/n^2, (x + 2h/n))$, then $\delta_{n+1,x} > \frac{3}{4}\delta_{n,x}$ for $n \geqq 5$, and also for every $\delta < \frac{1}{4}$ and every x, n can be chosen such that $\frac{3}{4}\delta_{n,x} < \delta \leqq \delta_{n,x}$, since for $n \geqq 6$, there is n such that $\frac{1}{4} < \delta_{n,x} < \frac{1}{3}$ which, using $\delta_{n+1,x} > \frac{3}{4}\delta_{n,x}$, implies the above. Therefore

$$e^{-Ax}|f(x) - 2f(x+h) + f(x+2h)|$$
$$\leqq M(\delta_{n,x}{}^{\alpha} + h^2 w_2(f, \delta, A)(\delta^{-2} + \delta_{n,x}{}^{-2}))$$
$$\leqq M_1(\delta^{\alpha} + (h^2/\delta^2)w_2(f, \delta, A)).$$

Therefore for $\delta < \frac{1}{4}$, $w_2(f, h, A) \leqq M_1(\delta^{\alpha} + (h^2/\delta^2)w_2(f, \delta, A))$ which, using H. Berens and G. G. Lorentz's technique [**3**, p. 695–696], completes the proof for Szász operators. Similarly for Baskakov operators we have the following estimates:

$$I_1 \leqq M((x+2h)(1+x+2h)/n)^{\alpha/2}$$
$$\leqq M(\max(1/n^2, (x+2h)(1+x+2h)/n))^{\alpha/2}$$
$$\text{for } x \leqq \eta\sqrt{n} \text{ using (4.2);}$$

$$J_2 \leqq M(h^2/\delta^2)w_2(f, \delta, A) \text{ combining (3.10) and (4.6); and}$$

$$J_2 \leqq Mw_2(f, \delta, A) \min(1, n(n+1)h^2, h^2(n/\xi(1+\xi)))$$
$$\leqq M_1 h^2 w_2(f, \delta, A) \min(n^2, n/(x+2h)(1+x+2h))$$

combining (4.6) with (3.7), (3.8) and (3.10) and following considerations similar to those used for Szász operators. We set

$$\delta_{n,x}{}^2 = \max(1/n^2, (x+2h)(1+x+2h)/n),$$

and since $x + 2h < \eta\sqrt{n}$ for $\eta = \frac{1}{3}\min(A^{-2}, 1)$, we have

$$(x+2h)(1+x+2h)/n < \eta^2 = \tfrac{1}{9}\min(A^{-4}, 1).$$

We can show for $\delta < \min(1/4A, 1/4)$ and every x that $n > 2A$ can be chosen such that $\delta_{n+1,x} < \delta < \delta_{n,x}$. Therefore

$$e^{-Ax}|f(x) - 2f(x+2h) + f(x)| \leqq M[\delta^{\alpha} + (h^2/\delta^2)w_2(f, \delta, A)]$$

which, using the technique of [**3**, p. 696] again, concludes the proof.

To prove the direct result we write

$$e^{-Ax}|A_n(f, x) - f(x)| = e^{-Ax}|A_n(f_\delta - f, x)| + e^{-Ax}|f_\delta(x) - f(x)|$$
$$+ e^{-Ax}|A_n(f_\delta, x) - f_\delta(x)| = I_1 + I_2 + I_3.$$

One can estimate the I_i by the following:

$$I_1 \leqq K_1 w_2(f, \delta, A)\begin{cases}\text{for } x \leqq n \text{ where } A_n = S_n \text{ using (4.6) and (3.1);}\\ \text{for } x \leqq \eta\sqrt{n} \text{ where } A_n = V_n \text{ using (4.6) and (3.2);}\end{cases}$$

$$I_2 \leqq w_2(f, \delta, A) \quad \text{using (4.6);}$$

$$I_3 \leqq K_2\delta^{-2}w_2(f, \delta, A)(x/n) \quad \text{for } x \leqq n \text{ where } A_n \equiv S_n$$
$$\text{using (4.6) and (3.3);}$$

$$I_3 \leqq K_2\delta^{-2}w_2(f, \delta, A)(x(1+x)/n) \quad \text{for } x < \eta\sqrt{n} \text{ where } A_n \equiv V_n$$
$$\text{using (4.6) and (3.4).}$$

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Substituting $\delta^2 = x/n$ and $\delta^2 = x(1+x)/n$ for $A_n = S_n$ and $A_n = V_n$ respectively, we have $w_2(f, \delta, A) \leqq M\delta^\alpha$ completing the proof.

5. On Saturation. The global saturation result for Szász or Baskakov operators can be easily deduced from its local counterpart.

THEOREM 5.1. *For* $\|f\|_A < \infty$, $S_n(f, x)$ *and* $V_n(f, x)$ *defined by* (1.1) *and* (1.2) *respectively, the following are equivalent:*

(A) $e^{-Ax}|S_n(f, x) - f(x)| \leqq Mx/n$ *for* $x \leqq n$, $n = 1, 2, \ldots$;

(B) $e^{-Ax}|V_n(f, x) - f(x)| \leqq Mx(1+x)/n$ *for* $x < \eta\sqrt{n}$, $\eta \leqq \frac{1}{3}\min(A^{-2}, 1)$ *and* $n \geqq 2A$; *and*

(C) $f'(x)$ *is locally absolutely continuous and* $|e^{-Ax}f''(x)| \leqq M$, *for all* x.

Proof. We have already shown (C) $\Rightarrow$ (A) and (C) $\Rightarrow$ (B) (Theorem 3.1). To show (A) $\Rightarrow$ (C) ((B) $\Rightarrow$ (C) follows similarly) we use the corresponding local result [7] first on $[k - \frac{1}{2}, k + \frac{3}{2}]$. $|S_n(f, x) - f(x)| \leqq Mx/ne^{Ak}e^{3A/2}$ implies $f'' \in L_\infty[k, k+1]$ and

$$\|f''\|_{L_\infty[k,k+1]} \leqq \lim_n \sup \|(n/x)[S_n(f, x) - f(x)]\|_{C[k-1/2,k+3/2]}$$
$\leqq Me^{3A/2}e^{Ak}$ since f'' is achieved in [7] as the weak* limit of

$$(n/x)[S_n(f, x) - f(x)].$$

Similarly, using the same theorem for $[1/2^{\nu+1}, 1/2^{\nu-2}]$, $\nu \geqq 1$,

$$\|(n/x)(S_n(f, x) - f(x))\|_{C[2^{-\nu-1},2^{-\nu+2}]} \leqq Me^{2A}$$

implies $f'' \in L_\infty[2^{-\nu}, 2^{-\nu+1}]$ and

$$\|f''\|_{L_\infty[2^{-\nu},2^{-\nu+1}]} \leqq \lim_n \sup \|(n/x)(S_n(f,x) - f(x))\|_{C[2^{-\nu-1},2^{-\nu+2}]}$$
$\leqq Me^{2A}$.

Remark. This very simple technique will yield the converse part of the global saturation for positive exponential-type operators.

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