REAL ANALYTIC FUNCTIONS ON PRODUCT SPACES AND SEPARATE ANALYTICITY

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Let f be a function on the product space $V \times W$, where V and W are analytic manifolds, both either real or complex. The function f is said to be analytic (or bi-analytic) on $V \times W$ if it is analytic in the analytic structure induced on $V \times W$ by the corresponding structures on V and W. The function f is said to be separately analytic on $V \times W$ if, for each x in V, the function $f(x,\cdot)$ is analytic on W while, for each y in W, the function $f(\cdot,y)$ is analytic on V. In the case of complex analytic manifolds, the classical theorem of Hartogs (**3**, chapter VII) states that the two notions of analyticity and separate analyticity are equivalent. For real analytic manifolds, it is known that such an equivalence does not hold, even if one adds the additional hypothesis that f is infinitely differentiable on $V \times W$.

It is the purpose of the present paper to establish a positive criterion for bi-analyticity in terms of the analytic properties of f in its separate variables. There are several equivalent forms in which we may state this criterion. The most direct of these is the following:

(A) *Consider V and W as imbedded in $\tilde{V}$ and $\tilde{W}$, their complexifications which are complex analytic manifolds.* Let K_1 and K_2 be compact subsets of V and W, respectively. Then there exists a neighbourhood U_1 of K_1 in $\tilde{V}$ and a neighbourhood U_2 of K_2 in $\tilde{W}$ and a constant M (depending on K_1 and K_2) such that for each x in K_1, the function $f(x,\cdot)$ on K_2 may be extended to a complex analytic function $f(x, z)$ on U_2 for which $|f(x, z)| \leqslant M$, $z \in U_2$, while similarly for each y in K_2, the function $f(\cdot,y)$ on K_1 may be extended to a complex analytic function $f(z, y)$ for z in U_1 for which $|f(z, y)| \leqslant M$, $z \in U_1$.*

Our study of this problem arose from a question raised by de Barros-Neto in connection with his investigation (**1**) of the structure of distribution kernels (in the sense of Schwartz) which are analytically very regular; the application of our result (Theorem 1 below) to such kernels is carried out in a joint paper by Barros and the author which appears immediately after the present paper in the same issue of this journal.

The criterion (A) is of local character, as follows easily from the elementary properties of holomorphic functions. We therefore may give another equivalent form if we assume that V and W each lie in a single co-ordinate patch, and therefore without loss of generality, that V and W are E^n and $E^{n'}$, respectively.

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*As defined, for example, by Bruhat and Whitney, Comm. Helv., *33* (1959), 132–60.

(B) There exists a constant C_0 such that for all x in V,

$$\left|\left(\frac{\partial}{\partial y_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial y_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial y_n}\right)^{\alpha_n} f(x, y)\right| \leqslant C_0^{(\Sigma\alpha_j)}\cdot\left(\sum\alpha_j\right)!$$

and for all y in W,

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_2}\right)^{\alpha_n} f(x, y)\right| \leqslant C_0^{(\Sigma\alpha_j)}\cdot\left(\sum\alpha_j\right)!$$

THEOREM 1. *Let $f(x, y)$ be a Borel measurable function on the product $V \times W$ of two real analytical manifolds V and W such that f is separately analytic on $V \times W$ and satisfies condition* (A) *above. Then f is analytic on $V \times W$.*

Proof of Theorem 1. Since analyticity is a local property, we may assume without loss of generality that V and W are open subsets of Euclidean spaces. Introducing dummy variables into the space of lower dimension, we may suppose that both V and W lie in E^n for some integer n, and that they both contain the origin O. We need only prove that f is analytic at (O, O).

We introduce the usual notation for partial derivatives, setting

$$D_j = i^{-1}\frac{\partial}{\partial x_j}$$

for $1 \leqslant j \leqslant n$,

$$D^\alpha = \prod_{j=1}^{n} D_j^{\alpha_j}$$

for any n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers,

$$\alpha = \sum_{j=1}^{n}\alpha_j, \alpha! = \prod_{j=1}^{n}(\alpha_j)!$$

If h is a function of two variables x and y in E^n, we indicate derivatives with respect to the x-variables by D_x^α and derivatives with respect to the y-variables by D_y^β.

If condition (A) holds, it follows immediately from the Cauchy integral formula for polycylinders that on some neighbourhood of (O, O) we have inequalities similar to those of condition (B), that is,

(C) *There exists a constant c_0 such that for all α and β,*

$$|D_x^\alpha f(x; y)| \leqslant c_0^{|\alpha|}\alpha! \leqslant C_0^{|\alpha|}|\alpha|!$$
$$|D_y^\beta f(x, y)| \leqslant c_0^{|\beta|}\beta! \leqslant C_0^{|\beta|}|\beta|!$$

Since f is Borel measurable on $V \times W$ and uniformly bounded on the neighbourhood N for which the bounds of (C) hold, it follows easily that its distribution derivatives $D_x^\alpha f$ and $D_y^\beta f$ coincide with the derivatives of f taken as a function of x with y held fixed and of y with x held fixed. It follows then from the bounds of condition (C) that on a fixed neighbourhood of

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(O, O), all the derivatives $D_x^\alpha f$ and $D_y^\beta f$ are bounded functions. In particular, we know that $\{(\Delta_x)^r + (\Delta_y)^r\}f$ is bounded for every positive integer r, $(\Delta = -\sum_j D_j^2)$. It follows from the standard regularity theorems for solutions of elliptic partial differential equations that $D_x^\alpha D_y^\beta f$ lies in $L_{\text{loc}}^2(N)$ for every α and β, and hence by the Sobolev Imbedding Theorem (Schwartz **(4)**, vol. 2) f is infinitely differentiable in N.

We now remark that to prove that f is analytic at (O, O), it suffices to show the following:

(C′) *There exists a constant c_1 such that for all α and β,*

$$|(D_x^\alpha D_y^\beta f)(O, O)| \leqslant c_1^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!$$

Indeed, suppose that (C′) holds. Then we may form the power series

$$f_1(x, y) = \sum_{\alpha,\beta} (\alpha!)^{-1}(\beta!)^{-1}(D_x^\alpha D_y^\beta f)(O, O)x^\alpha y^\beta,$$

where

$$x^\alpha = \prod_{j=1}^n x^{\alpha_j}, \; y^\beta = \prod_{j=1}^n y^{\beta_j}.$$

If the inequalities (C′) hold, the power series for f_1 may be majorized by the series

$$\sum_{\alpha,\beta} (\alpha!)^{-1}(\beta!)^{-1}c_1{}^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!|x|^\alpha|y|^\beta,$$

where

$$|x|^\alpha = \prod_j |x_j|^{\alpha_j}.$$

The majorizing series is the expansion of the function

$$\left(1-\sum_j c_1|x|_j+c_1|y|_j\right)^{-1}$$

which converges for x and y sufficiently small. It follows that $f_1(x, y)$ is an analytic function of (x, y) on a suitably small neighbourhood of the origin (O, O). On the other hand, for every α and β,

$$(D_x^\alpha D_y^\beta f)(O, O) = (D_x^\alpha D_y^\beta f_1)(O, O).$$

The functions $f(\cdot, y)$ and $f_1(\cdot, y)$ are both analytic on a neighbourhood of O and have the same x-derivatives at O. Hence $f(x, O) = f_1(x, O)$ for $|x| \leqslant d_0$. Similarly, $D_y^\beta f(x, O) = (D_y^\beta f_1)(x, O)$ for $|x| \leqslant d_0$. Finally, for x held fixed, $f(x,\cdot)$ and $f_1(x,\cdot)$ are both analytic in a neighbourhood of the origin and have the same derivatives at $y = O$. Hence $f(x, y) = f_1(x, y)$. for all (x, y) near (O, O). Thus in order to show that f is analytic near (O, O) it clearly suffices to prove the inequalities (C′) for some value of the constant c_1.

To carry through the latter proof, we may assume after making a linear change of variables (which will only affect the constant c_0 in condition (C)) that the inequalities of (C) hold on the cube

$$R = \{|x_j| \leqslant 1, \quad |y_j| \leqslant 1, \quad 1 \leqslant j \leqslant n\}.$$

We define the function of a single real variable

$$\zeta_0(s) = \begin{cases} 0, & |s| \geqslant 1 \\ 1, & |s| \leqslant \frac{1}{2} \\ 2 - 2s, & \frac{1}{2} \leqslant |s| \leqslant 1. \end{cases}$$

The function ζ_0 is clearly continuous and piecewise continuously differentiable. Using ζ_0, we define the auxiliary function $\zeta(x, y)$ on $E^n \times E^n$ by

$$\zeta(x, y) = \prod_{j=1}^{n} \zeta_0(x_j) \cdot \prod_{j=1}^{n} \zeta_0(y_j).$$

For each positive integer r, we let

$$\zeta_r(x, y) = (\zeta(x, y))^{r+2}.$$

For each r, the function $\zeta_r(x, y)$ is $(r + 1)$-times continuously differentiable on $E^n \times E^n$ and has its support contained in R.

As another piece of auxiliary equipment, we consider for each sufficiently large integer m, the partial differential operator with constant coefficients on $E^n \times E^n$ defined by

$$A_{2m} = (-\Delta_x)^m + (-\Delta_y)^m + 1,$$

We construct an elementary solution for A_{2m} on $E^n \times E^n$ by a Fourier transform. Let

$$\langle x, \xi \rangle = \sum_j x_j \xi_j$$

for x and ξ in E^n. Then we define the function $e_{2m}(x, y)$ on $E^n \times E^n$ for $2m > 2n$ by

$$(1) \quad e_{2m}(x, y) = \int_{E^n} \int_{E^n} \exp(i\langle x, \xi \rangle + i\langle y, \xi' \rangle)(|\xi|^{2m} + |\xi'|^{2m} + 1)^{-1} d\xi\, d\xi'.$$

The integral converges uniformly on $E^n \times E^n$. We verify immediately by taking Fourier transforms (in the sense of tempered distributions of Schwartz **(4)**) that for every function v in $C^{2m}(E^n \times E^n)$ with support in R, we have

$$(2) \quad v(x, y) = \int_R e_{2m}(x - x_1, y - y_1)(A_{2m} v)(x_1, y_1) dx_1 dy_1; \ ((x, y) \in R)).$$

Let α and β be two indices of differentiation with $|\alpha| + |\beta| < 2m - 2n$. We verify by inspection that we can differentiate the integral defining e_{2m} in equation (1), $(|\alpha| + |\beta|)$-times obtaining

$$(3) \quad (D_x^\alpha D_y^\beta e_{2m})(x, y) = \int_{E^n} \int_{E^n} \xi^\alpha (\xi')^\beta \exp(i\langle x, \xi \rangle + i\langle y, \xi' \rangle)(|\xi|^{2m} + |\xi'|^{2m} + 1)^{-1} d\xi\, d\xi'.$$

It follows from (3) that $(D_x^{\alpha} D_y^{\beta} e_{2m})$ is continuous and bounded for $|\alpha| + |\beta| < 2m - 2n$, with

$$(4)\quad |D_x^{\alpha} D_y^{\beta} e_{2m}(x, y)| \leqslant \int_{E^n} \int_{E^n} |\xi|^{|\alpha|} |\xi'|^{|\beta|} (|\xi|^{2m} + |\xi'|^{2m} + 1)^{-1} d\xi d\xi' = K(m, n)$$

where, by the inequality

$$(5)\qquad |\xi|^{|\alpha|} |\xi'|^{|\beta|} \leqslant \frac{\alpha}{\alpha + \beta} |\xi|^{|\alpha| + |\beta|} + \frac{\beta}{\alpha + \beta} |\xi'|^{|\alpha| + |\beta|},$$

we know that

$$(6)\quad K(m, n) \leqslant \int_{E^n} \int_{E^n} (|\xi|^{2m-2n-1} + |\xi'|^{2m-2n-1} + 1)(|\xi|^{2m} + |\xi'|^{2m} + 1)^{-1} d\xi\, d\xi',$$

and the integral on the right side of the inequality (6) is bounded by

$$(7)\qquad 5 \int_{E^n} \int_{E^n} (|\xi|^{2n+1} + |\xi'|^{2n+1} + 1)^{-1} d\xi\, d\xi',$$

which is independent of m. Thus we have for all (x, y) in $E^n \times E^n$,

$$(8)\qquad |D_x^{\alpha} D_y^{\beta} e_{2m}(x, y)| \leqslant K(n)$$

for $|\alpha| + |\beta| < 2m - 2n$.

Differentiating the equation (2) under the integral sign $|\alpha| + |\beta|$ times, we obtain for $|\alpha| + |\beta| < 2m - 2n$,

$$(9)\quad D_x^{\alpha} D_y^{\beta} v(x, y) = \int_R (D_x^{\alpha} D_y^{\beta} e_{2m})(x - x_1, y - y_1) A_{2m} v(x_1, y_1)\, dx_1\, dy_1,$$

which yields, since the measure of R is exactly 1, the inequality

$$(10)\qquad |D_x^{\alpha} D_y^{\beta} v(x, y)| \leqslant K(n) \sup_{(x_1, y_1) \epsilon R} |A_{2m} v(x_1, y_1)|$$

for (x, y) in R and $|\alpha| + |\beta| < 2m - 2n$.

We apply the inequality (10) to the given function $f(x, y)$ by setting

$$v(x, y) = f(x, y) \zeta_{2m}(x, y).$$

We put $(x, y) = (O, O)$ on the left-hand side of the inequality (10) and obtain, since ζ_{2m} is identically equal to 1 on a neighbourhood of (O, O), that

$$(11)\qquad |(D_x^{\alpha} D_y^{\beta} f)(O, O)| \leqslant K(n) \sup_{(x, y) \epsilon R} |A_{2m}(\zeta_{2m}(x, y) \cdot f(x, y))|.$$

Let us examine the term on the right of the last inequality. It follows from the definition of A_{2m} that

$$A_{2m}(\zeta_{2m} \mathrm{f}) = \{(-\Delta_x)^m + (-\Delta_y)^m + 1\}(\zeta_{2m} f) = \zeta_{2m} \{(-\Delta_x)^m + (-\Delta_y)^m + 1\} f + R_1,$$

where the remainder term R_1 is of the form

$$(12)\qquad R_1 = \sum_{\substack{\alpha, \beta \\ |\alpha| + |\beta| = 2m,\, |\alpha| > 0}} C_{\alpha\beta} D_x^{\alpha}(\zeta_{2m}) D_x^{\beta} f + \sum_{\substack{\alpha, \beta \\ |\alpha| + |\beta| = 2m,\, |\alpha| > 0}} C_{\alpha\beta} D_y^{\alpha}(\zeta_{2m}) D_y^{\beta} f.$$

The derivatives of ζ_{2m} are considered only on the set where ζ_{2m} is equal neither to 0 nor to 1. On this set ζ_{2m} is a polynomial of degree $(2m + 2)$ in each variable. It follows from an easy application of a well-known theorem of Bernstein that for all (x, y) in R, $|\alpha| > 0$,

$$(13) \qquad |(D_x^\alpha \zeta_{2m})(x, y)| \leqslant (2m + 2)^{|\alpha|} 2^{|\alpha|},$$

and

$$(14) \qquad |(D_y^\alpha \zeta_{2m})(x, y)| \leqslant (2m + 2)^{|\alpha|} 2^{|\alpha|}.$$

We now estimate R_1. The expansion of $(- \Delta_x)^m$ in elementary differential operators has n^m terms with coefficient 1. Each of these terms applied to a product $w_1 w_2$ can be written as the sum of 2^{2m} terms of the form $D^\alpha w_1 D^\beta w_2$ with coefficient 1. Hence the sum of the coefficeints

$$\sum_{\alpha,\beta} c_{\alpha,\beta}$$

in equation (12) is less than $2^{2m} n^m$. By the condition (C) and inequalities (13) and (14), it follows therefore that (assuming $c_0 > 1$)

$$(15) \qquad |R_1| \leqslant 2(2n)^{2m} C_0^{2m} (2m + 2)^{2m} 2^{2m}.$$

A similar, but sharper, estimate holds for the term $\zeta_{2m} A_{2m} f$. Combining these estimates, we obtain finally for $|\alpha| + |\beta| < 2m - 2n$

$$(16) \qquad |(D_x^\alpha D_y^\beta f)(O, O)| \leqslant k_0 (4nC_0)^{2m} (2m)^{2m} \left(1 + \frac{2}{2m}\right)^{2m}.$$

By Stirling's formula, however, we know that

$$(2m)! \sim (4\pi m)^{\frac{1}{2}} (2m)^{2m} e^{-2m}.$$

Hence there exists an absolute constant k such that

$$(2m)^{2m} \leqslant k e^{2m} (2m)!$$

We obtain,

$$(17) \quad |(D_x^\alpha D_y^\beta f)(O, O)| \leqslant k\, k_0 e^2 (4nC_0 e)^{2m} (2m)! \leqslant (4nC_0 e)^{2m} \frac{(2m)!}{(2m - n - 1)!}$$

$$(2m - n - 1!) c_1^{2m-2n-1} (2m - n - 1)!$$

if $|\alpha| + |\beta| = 2m - 2n - 1$, and similarly,

$$(18) \qquad |(D_x^\alpha D_y^\beta f)(O, O)| \leqslant c_1^{2m-n-2} (2m - n - 2)!$$

if $|\alpha| + |\beta| = 2m - 2n - 2$.

In particular, for any given α and β, we may choose m such that either $2m = |\alpha| + |\beta| + 2n + 1$, or $2m = |\alpha| + |\beta| + 2n + 2$. Since the inequalities (17) and (18) are therefore equivalent to the inequalities of condition (C′) for various α and β, the proof of Theorem 1 is therefore complete.

THEOREM 2. *Let f be an infinitely differentiable, separately analytic function on $V \times W$, where V and W are real analytic manifolds. Then there exists an everywhere dense open subset G of $V \times W$ such that f is analytic on G.*

Proof of Theorem 2. It clearly suffices to suppose that V and W are both cubes with centre at the origin in E^n and to show that f is analytic in some open subset of $V \times W = R$.

For each positive integer M, let

$$S_M = \{(x, y): (x, y) \in R, |D_x^\alpha f(x, y)| \leqslant M^{|\alpha|}\alpha! \quad \text{for all } \alpha\},$$

$$S'_M = \{(x, y): (x, y) \in R, |D_y^\alpha f(x, y)| \leqslant M^{|\alpha|}\alpha! \quad \text{for all } \alpha.\}$$

By the separate analyticity of the function f, each point (x, y) of R belongs to at least one of the sets S_M and at least one of the sets S_{M_1}'. By the continuity of the derivatives of f, each S_M is the intersection of closed sets and hence closed. Similarly each S_{M_1}' is closed, and so is

$$S_M \cap S'_{M_1}.$$

R is the union of the sets

$$S_M \cap S'_{M_1},$$

and by the Baire category theorem, one of these closed sets must have an interior. Let R' be a disk in

$$S_M \cap S'_{M_1}.$$

Then if M_2 is the larger of M and M_1, we have for all points of R' and all α,

$$|D_x^\alpha f(x, y)| \leqslant M_2^{|\alpha|}\alpha!$$

$$|D_y^\alpha f(x, y)| \leqslant M_2^{|\alpha|}\alpha!$$

It follows by the proof of Theorem 1 that f is then analytic at each interior point of R', and the proof of Theorem 2 is complete.

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