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Christopher Brav and Tobias Dyckerhoff

Abstract

We introduce relative noncommutative Calabi–Yau structures defined on functors of differential graded categories. Examples arise in various contexts such as topology, algebraic geometry, and representation theory. Our main result is a composition law for Calabi–Yau cospans generalizing the classical composition of cobordisms of oriented manifolds. As an application, we construct Calabi–Yau structures on topological Fukaya categories of framed punctured Riemann surfaces.

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1. Introduction

A fundamental insight in algebraic topology is that the geometric notion of orientation can be captured in purely algebraic terms: An orientation of a given compact n-dimensional manifold X corresponds to a *fundamental class*

$$[X] \in H_n(X;\mathbb{Z}) \tag{1.1}$$

characterized by the requirement that, for every point $x \in X$, it maps to a generator of the local homology group $H_n(X, X\backslash\{x\};\mathbb{Z}) \cong \mathbb{Z}$. Further, evaluation on a chosen fundamental class $[X]$ yields a map

$$\mathrm{tr} : H^*(X;\mathbb{Z}) \longrightarrow \mathbb{Z}[-n] \tag{1.2}$$

such that the bilinear pairing $(\alpha, \beta) \mapsto \mathrm{tr}(\alpha \cup \beta)$ on $H^*(X;\mathbb{Z})$ is nondegenerate.

A key observation from noncommutative geometry is that this algebraic perspective on orientations has the following noncommutative variant which we formulate for a differential graded k-algebra A where k is a field: an *n-dimensional fundamental class* for A is a Hochschild homology class

$$[A] \in \mathrm{HH}_{-n}(A) \tag{1.3}$$

with the property that the induced map of A-bimodules

$$A^! \longrightarrow A[-n]$$

is a quasi-isomorphism. Here, using the notation $A^e = A \otimes_k A^{\mathrm{op}}$, the module

$$A^! = R\underline{\mathrm{Hom}}_{A^e}(A, A^e)$$

is called the *inverse dualizing bimodule*. The map $A^! \to A[-n]$ is then obtained by acting on the first factor of the fundamental class $[A] \in \mathrm{HH}(A) = A \otimes^L_{A^e} A$. Under the assumption that A is perfect as an A-bimodule, it can further be shown that, for every A-module M whose underlying complex of vector spaces is perfect, the fundamental class $[A]$ induces a k-linear map

$$\mathrm{tr}_M : \mathrm{Ext}^*_A(M, M) \longrightarrow k[-n] \tag{1.4}$$

such that the bilinear pairing $(f, g) \mapsto \mathrm{tr}(f \circ g)$ on $\mathrm{Ext}^*_A(M, M)$ is nondegenerate.

To provide some intuition for these noncommutative fundamental classes, we demonstrate how to interpret classical fundamental classes in noncommutative terms. Let X be a connected n-dimensional manifold and let $A = C_*(\Omega X)$ be the differential graded algebra of chains on the space of Moore loops based at a chosen point in X. Then, by [Jon87], we have an isomorphism $\mathrm{HH}_{-*}(A) \simeq H_*(LX)$ where LX denotes the free loop space of X. A fundamental class $[X]$ in the classical sense (1.1) induces a noncommutative fundamental class in the sense of (1.3) via pushforward along the inclusion X into LX as the space of constant loops. Further, the trace map (1.2) can be recovered from (1.4) by setting M to be the A-module k. More generally, A-modules M whose underlying complex is perfect can be interpreted as local systems of perfect complexes on X, so that k corresponds to the trivial rank 1 local system.

The noncommutative perspective has the advantage that it is widely applicable and covers variations of the notion of orientation that arise in other contexts. For example, when X is an algebraic variety, it is natural to ask whether the fundamental class can be represented by an algebraic volume form. This algebraic variant of orientation which, in light of the celebrated differential geometric results of Calabi and Yau, is often referred to as a Calabi–Yau structure, is captured noncommutatively by setting A to be the differential graded algebra of derived endomorphisms of a generator of the derived category of coherent sheaves on X. Similarly, noncommutative fundamental classes describe interesting versions of orientations in symplectic geometry, representation theory, and topological field theory.

When studying noncommutative fundamental classes in the various above contexts, it becomes apparent that, in essentially all examples, they admit a rather subtle additional circular symmetry which manifests itself in the form of a lift of the fundamental class from the Hochschild class (1.3) to a class in negative cyclic homology. In particular, this additional symmetry is necessary to interpret fundamental classes in terms of oriented topological field theories (see, e.g., [Lur09b]). Further, to avoid the necessity to choose generators, it is convenient to generalize from differential graded (dg) algebras to dg categories, thus arriving at the following refinement of (1.3).

Definition 1.5. Let $\mathcal{A}$ be a smooth k-linear dg category (i.e., $\mathcal{A}$ is Morita equivalent to a dg algebra which is perfect as a bimodule). An n-dimensional *left Calabi–Yau structure* on $\mathcal{A}$ consists of a cycle

$$\widetilde{[\mathcal{A}]} : k[n] \longrightarrow \mathrm{CC}_\bullet(\mathcal{A})^{S^1} \tag{1.6}$$

in the negative cyclic complex such that the induced morphism of $\mathcal{A}$-bimodules

$$\mathcal{A}^! \longrightarrow \mathcal{A}[-n] \tag{1.7}$$

is a quasi-isomorphism where $\mathcal{A}^!$ denotes the derived $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}$-linear dual of $\mathcal{A}$.

The smoothness hypothesis implies that $\mathcal{A}^!$ is the Morita-theoretic *left* adjoint of the evaluation functor

$$\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathrm{Ch}(k), (x, y) \mapsto \mathcal{A}(x, y)$$

which explains our choice of terminology. The idea to formulate Calabi–Yau structures in terms of the quasi-isomorphism (1.7) has appeared in [Gin06]. The additional S^1-equivariance data (1.6) has been proposed by Kontsevich and Vlassopoulos in [KV13]. The result that, for a compact oriented manifold X, the dg algebra $C_*(\Omega X)$ carries a canonical left Calabi–Yau structure is stated in [Lur09b] and proved in [CG15]. The natural refinement of (1.4) to the current context is captured by a different notion of Calabi–Yau structure, introduced in [KS06].

Definition 1.8. Let $\mathcal{A}$ be a (locally) proper k-linear dg category (i.e., all morphism complexes of $\mathcal{A}$ are perfect). An n-dimensional *right Calabi–Yau structure* on $\mathcal{A}$ consists of a cocycle

$$\widetilde{\sigma} : \mathrm{CC}_\bullet(\mathcal{A})_{S^1} \longrightarrow k[-n] \tag{1.9}$$

on the cyclic complex so that the induced morphism of $\mathcal{A}$-bimodules

$$\mathcal{A}[n] \longrightarrow \mathcal{A}^* \tag{1.10}$$

is a quasi-isomorphism where $\mathcal{A}^*$ denotes the k-linear dual of $\mathcal{A}$.

The properness hypothesis implies that $\mathcal{A}^*$ is the Morita-theoretic *right* adjoint of the evaluation functor $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathrm{Ch}(k)$ explaining our terminology. In this terminology, the fact that a chosen fundamental class (1.3) induces a nondegenerate trace pairing (1.4) is now expressed in the statement that a left Calabi–Yau structure on a smooth dg category $\mathcal{A}$ induces a right Calabi–Yau structure on its Morita dual $\mathcal{A}^\vee$, i.e., the derived dg category of functors from $\mathcal{A}$ to Perf_k (pseudo-perfect $\mathcal{A}$-modules in the terminology of [TV07]).

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For example, to relate to the initial example of an n-dimensional oriented manifold X, we introduce a dg category $\mathcal{L}(X)$, called the linearization of X, which can be informally described as follows: the objects of $\mathcal{L}(X)$ are the points of X, and the mapping complex between two points is the chain complex of the space of paths between the points. The Morita dual of $\mathcal{L}(X)$ can then be identified with the full dg subcategory of $\mathrm{Perf}_{\mathcal{L}(X)}$ consisting of ∞-local systems of complexes of vector spaces on X with perfect fibers. From the point of view of noncommutative geometry, as already strongly advocated in [Kon09], the dg category $\mathcal{L}(X)$ is much better behaved than its Morita dual $\mathcal{L}(X)^\vee$: the fact that X is homotopy equivalent to a finite CW complex implies that $\mathcal{L}(X)$ is *of finite type* in the sense of [TV07]. Most importantly for us, as shown in *loc. cit*, its derived moduli stack of pseudo-perfect modules is locally geometric and of finite presentation so that it has a perfect cotangent complex. This is in stark contrast to the Morita dual $\mathcal{L}(X)^\vee$ which is (locally) proper but almost never of finite type so that a reasonable moduli stack in the sense of [TV07] does not exist. It is for these reasons that, in the context of most examples appearing in this work, we are led to consider left Calabi–Yau structures as more fundamental than right Calabi–Yau structures.

Now assume that X is an oriented compact manifold but with possibly nonempty boundary ∂X. We obtain a corresponding dg functor

$$\mathcal{L}(\partial X) \longrightarrow \mathcal{L}(X)$$

of linearizations. To capture the orientation of $(X, \partial X)$ in terms of this functor, we propose the following relative version of Definition 1.5.

DEFINITION 1.11. An n-dimensional *relative left Calabi–Yau structure* on a functor $f : \mathcal{A} \to \mathcal{B}$ of smooth k-linear dg categories consists of a cycle

$$\widetilde{[\mathcal{B}, \mathcal{A}]} : k[n] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A})^{S^1} \tag{1.12}$$

in the relative negative cyclic complex, defined as the cone of the induced map $f_* : \mathrm{CC}_\bullet(\mathcal{A})^{S^1} \to \mathrm{CC}_\bullet(\mathcal{B})^{S^1}$ on absolute negative cyclic complexes, such that all vertical morphisms in the induced diagram of $\mathcal{B}$-bimodules

$$\begin{array}{ccccc} \mathcal{B}^! & \xrightarrow{\;c^!\;} & (\mathcal{B} \otimes^L_{\mathcal{A}} \mathcal{B})^! & \longrightarrow & \mathrm{cone}(c^!) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{cone}(c)[-n] & \longrightarrow & (\mathcal{B} \otimes^L_{\mathcal{A}} \mathcal{B})[-n+1] & \xrightarrow{\;c[-n+1]\;} & \mathcal{B}[-n+1] \end{array} \tag{1.13}$$

are quasi-isomorphisms where the morphism c represents the counit of the derived Morita-adjunction

$$Lf_! : \mathcal{D}(\mathrm{Mod}_{\mathcal{A}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{B}}) : Rf^*. \tag{1.14}$$

A relative left Calabi–Yau structure on the zero functor $0 \to \mathcal{B}$ can be identified with an absolute left Calabi–Yau structure on $\mathcal{B}$. Diagram (1.13) admits a particularly nice interpretation if we assume the dg functor f to be spherical in the sense of [AL17] (this is satisfied in many of our examples): in this case, the $\mathcal{B}$-bimodule $\mathrm{cone}(c)$ represents an autoequivalence of $\mathrm{Mod}_{\mathcal{B}}$ known as a *spherical twist* of the adjunction. Therefore, in this context, (1.13) amounts to an identification of the inverse dualizing bimodule $\mathcal{B}^!$ with a shifted spherical twist of the adjunction (1.14). There are also relative variants of right Calabi–Yau structures (these have already been

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introduced in [Toë14, 5.3]) and, generalizing the absolute case, the two notions are related via Morita duality.

A basic operation in cobordism theory is to glue two manifolds along a common boundary component to produce a new manifold: given manifolds X and X' with boundary decompositions $Z \amalg Z' = \partial X$ and $Z' \amalg Z'' = \partial X'$, we have a commutative diagram

$$\begin{array}{ccccc} & & & & Z \\ & & & & \downarrow \\ & & Z' & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ Z'' & \longrightarrow & X' & \longrightarrow & X \amalg_{Z'} X' \end{array}$$

where the square is a homotopy pushout square of spaces. Linearizing by applying $\mathcal{L}$ yields the commutative diagram

$$\begin{array}{ccccc} & & & & \mathcal{L}(Z) \\ & & & & \downarrow \\ & & \mathcal{L}(Z') & \longrightarrow & \mathcal{L}(X) \\ & & \downarrow & & \downarrow \\ \mathcal{L}(Z'') & \longrightarrow & \mathcal{L}(X') & \longrightarrow & \mathcal{L}(X \amalg_{Z'} X') \end{array}$$

where the square is a homotopy pushout of dg categories. The fact that composition of cobordisms is compatible with orientations admits the following noncommutative generalization which is the main result of this work.

Theorem 1.1. *Let*

$$\mathcal{A} \amalg \mathcal{A}' \longrightarrow \mathcal{B}$$

and

$$\mathcal{A}' \amalg \mathcal{A}'' \longrightarrow \mathcal{B}'$$

be functors of smooth dg categories equipped with relative left Calabi–Yau structures which are compatible on $\mathcal{A}'$. *Then the functor*

$$\mathcal{A} \amalg \mathcal{A}'' \longrightarrow \mathcal{B} \amalg_{\mathcal{A}'} \mathcal{B}'$$

inherits a canonical relative left Calabi–Yau structure.

As an application of this result, we construct relative left Calabi–Yau structures on topological Fukaya categories of punctured framed Riemann surfaces.

We would like to mention that this work has a sequel [BD18] in which we relate left Calabi–Yau structures to derived symplectic geometry in the sense of [PTVV13]. We announce the following main result of that work.

Theorem 1.2. *Let k be a field of characteristic* 0.

(1) *Let $\mathcal{A}$ be a k-linear dg category of finite type. Then an n-dimensional left Calabi–Yau structure on $\mathcal{A}$ determines a canonical $(2-n)$-shifted symplectic form on the derived moduli stack $\mathcal{M}_{\mathcal{A}}$ of pseudo-perfect modules.*

(2) *Let $f : \mathcal{A} \to \mathcal{B}$ be a functor of k-linear dg categories of finite type. Assume that f carries an $(n+1)$-dimensional left Calabi–Yau structure so that the corresponding negative cyclic class on $\mathcal{A}$ determines an n-dimensional left Calabi–Yau structure. Then the induced pullback morphism of derived stacks*

$$f^* : \mathcal{M}_{\mathcal{B}} \longrightarrow \mathcal{M}_{\mathcal{A}}$$

carries a canonical Lagrangian structure.

Note that this is a variant of a statement announced in [Toë14, 5.3]. However, the result stated in [Toë14] uses right Calabi–Yau structures. The use of left Calabi–Yau structures in Theorem 1.2 allows for applications to finite type categories which are not necessarily proper. For example, in the context of topological Fukaya categories, Theorems 7.2 and 1.2 imply the following statement.

Theorem 1.3. *Let (S, M) be a stable marked surface with framing on $S\backslash M$, and let $F(S, M)$ denote its topological Fukaya category. Then pullback along the boundary functor induces a morphism of derived stacks*

$$i^* : \mathcal{M}_{F(S,M)} \longrightarrow \prod_{\pi_0(\partial S \backslash M)} \mathcal{M}_{\underline{k}}$$

where the right-hand side carries a 2-shifted symplectic structure and i^* has a Lagrangian structure. In particular, if ∂S is empty, then $\mathcal{M}_{F(S,M)}$ has a 1-shifted symplectic structure.

Here it is crucial to use versions of topological Fukaya categories which arise as global sections of cosheaves of dg categories (as opposed to sheaves) since these are of finite type.

In light of Theorem 1.2 one can interpret the theory of absolute (respectively relative) left Calabi–Yau structures as a noncommutative predual of the geometric theory of shifted symplectic (respectively Lagrangian) structures. For example, Theorem 1.1 is a predual of [Cal15, Theorem 4.4].

We provide an outline of the contents of this work. In § 2, we introduce the technical context of this work: derived Morita theory for dg categories. In §§ 3 and 4 we give a detailed account of the absolute and relative Calabi–Yau structures sketched above. Section 5 provides examples of Calabi–Yau structures in topology, algebraic geometry and representation theory. In § 6 we provide a proof of the main result on the composition of Calabi–Yau cospans. The final section, § 7, contains the applications to topological Fukaya categories of surfaces.

We conclude with a discussion of relations to the previously existing literature. The concept of a boundary algebra introduced in [Sei12] as well as the notion of a pre-CY structure introduced in [KV13] are very close in spirit to the relative Calabi–Yau structures studied here. In fact, as we were informed by the authors, Kontsevich and Vlassopoulos had contemplated a modified definition very close to ours with results similar to the ones in this work. The appearance of left and right Calabi–Yau structures with applications to mirror symmetry is discussed in [GPS15] where the terminology of 'smooth' and 'proper' Calabi–Yau structures is used instead. Techniques similar to ours to construct Calabi–Yau structures on Fukaya-type categories appear in the recent work [ST16].

2. Morita theory of differential graded categories

We introduce some basic ingredients of Morita theory of differential graded categories which will form the technical context for this work.

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2.1 Modules over differential graded categories

Let k be a field. We denote by $\mathrm{Ch}(k)$ the category of unbounded cochain complexes of vector spaces over k equipped with its usual monoidal structure. A *differential graded (dg) category* is a category enriched over $\mathrm{Ch}(k)$ equipped with its usual monoidal structure. We refer the reader to [Kel82] for the foundations of enriched category theory and to [Toë07] for more details on derived Morita theory. Given dg categories $\mathcal{A}$, $\mathcal{B}$, there is a dg category

$$\mathcal{A} \otimes \mathcal{B}$$

called the *tensor product* of $\mathcal{A}$ and $\mathcal{B}$. A *dg functor* $\mathcal{A} \to \mathcal{B}$ of dg categories is defined to be a $\mathrm{Ch}(k)$-enriched functor. The collection of functors from $\mathcal{A}$ to $\mathcal{B}$ organize into a dg category

$$\underline{\mathrm{Fun}}(\mathcal{A}, \mathcal{B})$$

which is adjoint to the above tensor product.

Given a dg category $\mathcal{A}$, we introduce the dg category

$$\mathrm{Mod}_{\mathcal{A}} = \underline{\mathrm{Fun}}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ch}(k))$$

of *right modules over* $\mathcal{A}$. We will mostly use right modules, but it is notationally convenient to further introduce the dg category

$$\mathrm{Mod}^{\mathcal{A}} = \underline{\mathrm{Fun}}(\mathcal{A}, \mathrm{Ch}(k))$$

of *left modules over* $\mathcal{A}$, and, given another dg category $\mathcal{B}$, the dg category

$$\mathrm{Mod}^{\mathcal{A}}_{\mathcal{B}} = \underline{\mathrm{Fun}}(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathrm{Ch}(k)) \cong \underline{\mathrm{Fun}}(\mathcal{A}, \mathrm{Mod}_{\mathcal{B}})$$

of $\mathcal{A}$-$\mathcal{B}$-*bimodules*. There is a canonical dg functor

$$\mathcal{A} \longrightarrow \mathrm{Mod}_{\mathcal{A}}, \quad a \mapsto \mathcal{A}(-, a)$$

given by the $\mathrm{Ch}(k)$-enriched Yoneda embedding. We set $\mathcal{A}^e = \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}$, and call the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{A} : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathrm{Mod}_k, \quad (a, a') \mapsto \mathcal{A}(a', a)$$

the *diagonal bimodule*. Note that, via the equivalence $\mathcal{A}^e \simeq (\mathcal{A}^e)^{\mathrm{op}}$, we may identify left and right $\mathcal{A}^e$-modules. In what follows, we will leave this distinction implicit and refer to either of these modules as $\mathcal{A}$-bimodules.

The category $\mathrm{Mod}_{\mathcal{A}}$ admits a natural cofibrantly generated $\mathrm{Ch}(k)$-model structure in the sense of [Hov99]: it is obtained from the projective model structure on $\mathrm{Ch}(k)$ by defining weak equivalences and fibrations pointwise.

Hochschild homology. Let $\mathcal{A}$ be a dg category. We define the *Hochschild complex*

$$\mathrm{CC}_\bullet(\mathcal{A}) = \mathcal{A} \otimes^L_{\mathcal{A}^e} \mathcal{A}$$

where $\mathcal{A}$ denotes the diagonal $\mathcal{A}$-bimodule. If we use the bar resolution of $\mathcal{A}$ as a particular choice of cofibrant replacement, then the right-hand side complex becomes the cyclic bar construction. This complex arises as the realization of a cyclic object in $\mathrm{Ch}(k)$ which equips $\mathrm{CC}_\bullet(\mathcal{A})$ with an action of the circle S^1 (cf. [Hoy15]). This model further exhibits an explicit functoriality: a functor $f : \mathcal{A} \to \mathcal{B}$ induces an S^1-equivariant morphism

$$\mathrm{CC}_\bullet(\mathcal{A}) \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}).$$

In virtue of the circle action, we obtain the *cyclic complex*

$$\mathrm{CC}_\bullet(\mathcal{A})_{S^1} = (\mathrm{CC}_\bullet(\mathcal{A})[u^{-1}], b + uB)$$

by passing to homotopy orbits and the *negative cyclic complex*

$$\mathrm{CC}_\bullet(\mathcal{A})^{S^1} = (\mathrm{CC}_\bullet(\mathcal{A})[[u]], b + uB)$$

by passing to homotopy fixed points. As explained in [Hoy15], the circle action is captured algebraically in terms of the structure of a mixed complex so that the above orbit and fixed point constructions can be computed by the well-known complexes (cf. [Kas87, Lod13]).

Given a functor $f : \mathcal{A} \to \mathcal{B}$ of dg categories, we define the *relative Hochschild complex* $\mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A})$ as the cofiber (or cone) of the morphism $\mathrm{CC}_\bullet(\mathcal{A}) \to \mathrm{CC}_\bullet(\mathcal{B})$. Similarly, we obtain the relative cyclic complex $\mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A})_{S^1}$ and the relative negative cyclic complex $\mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A})^{S^1}$.

Derived ∞-categories. It will be convenient to formulate some of the constructions below in terms of ∞-categories. Given a dg category $\mathcal{A}$, let $\mathrm{Mod}^\circ_\mathcal{A}$ denote the full dg subcategory of $\mathrm{Mod}_\mathcal{A}$ spanned by the cofibrant objects. We call the dg nerve

$$\mathcal{D}(\mathrm{Mod}_\mathcal{A}) = \mathrm{N}_{\mathrm{dg}}(\mathrm{Mod}^\circ_\mathcal{A})$$

the derived ∞-category of $\mathcal{A}$-modules (cf. [Lur11b]).

Morita localization. A dg functor $f : \mathcal{A} \to \mathcal{B}$ is called a *quasi-equivalence* if the following hold:

(1) the functor $H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is an equivalence of categories;
(2) for every pair (a, a') of objects in $\mathcal{A}$, the map

$$\mathcal{A}(a, a') \to \mathcal{B}(f(a), f(a'))$$

is a quasi-isomorphism of complexes.

Given a dg category $\mathcal{A}$, we define the dg category $\mathrm{Perf}_\mathcal{A}$ of *perfect $\mathcal{A}$-modules* as the full dg subcategory of $\mathrm{Mod}_\mathcal{A}$ spanned by those cofibrant objects which are compact in $\mathrm{Ho}(\mathrm{Mod}_\mathcal{A})$. Here, an object M in $\mathrm{Mod}_\mathcal{A}$ is called *compact*, if the functor

$$\underline{\mathrm{Hom}}_\mathcal{A}(M, -) : \mathrm{Mod}_\mathcal{A} \longrightarrow \mathrm{Mod}_k$$

commutes with filtered homotopy colimits. In fact, to show that $M \in \mathrm{Mod}_\mathcal{A}$ is compact it is enough to check that the functor $\underline{\mathrm{Hom}}_\mathcal{A}(M, -)$ preserves arbitrary direct sums. Furthermore, it is known that the compact objects in $\mathrm{Mod}_\mathcal{A}$ are precisely the homotopy retracts of finite colimits of representable modules. A dg functor $f : \mathcal{A} \to \mathcal{B}$ induces via enriched left Kan extension a functor

$$f_! : \mathrm{Perf}_\mathcal{A} \longrightarrow \mathrm{Perf}_\mathcal{B}.$$

We say f is a *Morita equivalence* if the functor $f_!$ is a quasi-equivalence. We are usually interested in dg categories up to quasi-equivalence (respectively Morita equivalence) and will implement this by working with the ∞-categories $\mathrm{L}_{\mathrm{eq}}(\mathcal{C}at_{\mathrm{dg}}(k))$ (respectively $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}(k))$) obtained by localizing $\mathcal{C}at_{\mathrm{dg}}(k)$ along the respective collection of morphisms.

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2.2 Morita theory

Let $\mathcal{A}$, $\mathcal{B}$ be dg categories and let $M \in \operatorname{Mod}_{\mathcal{B}}^{\mathcal{A}}$ be a cofibrant bimodule. We denote by

$$-\otimes_{\mathcal{A}} M : \operatorname{Mod}_{\mathcal{A}} \longrightarrow \operatorname{Mod}_{\mathcal{B}}$$

the $\operatorname{Ch}(k)$-enriched left Kan extension of $M : \mathcal{A} \to \operatorname{Mod}_{\mathcal{B}}$ along the Yoneda embedding $\mathcal{A} \to \operatorname{Mod}_{\mathcal{A}}$. We further introduce the dg functor $\underline{\operatorname{Hom}}_{\mathcal{B}}(M, -) : \operatorname{Mod}_{\mathcal{B}} \to \operatorname{Mod}_{\mathcal{A}}$ given by the composite

$$\operatorname{Mod}_{\mathcal{B}} \xrightarrow{M^{\mathrm{op}} \otimes \mathbf{1}} \underline{\operatorname{Fun}}(\mathcal{A}^{\mathrm{op}}, \operatorname{Mod}_{\mathcal{B}}^{\mathrm{op}} \otimes \operatorname{Mod}_{\mathcal{B}}) \xrightarrow{\underline{\operatorname{Hom}}_{\mathcal{B}}(-,-)} \operatorname{Mod}_{\mathcal{A}},$$

obtaining a $\operatorname{Ch}(k)$-enriched Quillen adjunction

$$-\otimes_{\mathcal{A}} M : \operatorname{Mod}_{\mathcal{A}} \longleftrightarrow \operatorname{Mod}_{\mathcal{B}} : \underline{\operatorname{Hom}}_{\mathcal{B}}(M, -).$$

Concretely, the dg functor $\underline{\operatorname{Hom}}_{\mathcal{B}}(M, -) : \operatorname{Mod}_{\mathcal{B}} \to \operatorname{Mod}_{\mathcal{A}}$ takes a module $N \in \operatorname{Mod}_{\mathcal{B}}$ to the $\mathcal{A}$-module $a \mapsto \operatorname{Hom}_{\mathcal{B}}(M(a), N)$. We call the bimodule $M^{\vee} : \mathcal{B} \to \operatorname{Mod}_{\mathcal{A}}$ given by the restriction of $\underline{\operatorname{Hom}}_{\mathcal{B}}(M, -)$ along the enriched Yoneda embedding $\mathcal{B} \longrightarrow \operatorname{Mod}_{\mathcal{B}}$ the *right dual of* M. By the universal property of its enriched left Kan extension

$$-\otimes_{\mathcal{B}} M^{\vee} : \operatorname{Mod}_{\mathcal{B}} \longrightarrow \operatorname{Mod}_{\mathcal{A}},$$

we obtain a canonical natural transformation

$$\eta : -\otimes_{\mathcal{B}} M^{\vee} \longrightarrow \underline{\operatorname{Hom}}_{\mathcal{B}}(M, -).$$

Definition 2.1. The bimodule M is called *right dualizable* if, for every cofibrant $\mathcal{B}$-module N, the morphism $\eta(N)$ in $\operatorname{Mod}_{\mathcal{A}}$ is a weak equivalence.

Remark 2.2. A bimodule $M \in \operatorname{Mod}_{\mathcal{B}}^{\mathcal{A}}$ is right dualizable if and only if, for every $a \in \mathcal{A}$, the right $\mathcal{B}$-module $M(a)$ is perfect.

Remark 2.3. A right dualizable bimodule $M \in \operatorname{Mod}_{\mathcal{B}}^{\mathcal{A}}$ induces a $\mathcal{D}(\operatorname{Ch}(k))$-enriched adjunction of derived categories

$$-\otimes_{\mathcal{A}}^{L} M : \mathcal{D}(\operatorname{Mod}_{\mathcal{A}}) \longleftrightarrow \mathcal{D}(\operatorname{Mod}_{\mathcal{B}}) : -\otimes_{\mathcal{B}}^{L} M^{\vee}$$

with unit and counit induced via Kan extensions from bimodule morphisms

$$\mathcal{A} \longrightarrow M \otimes_{\mathcal{B}}^{L} M^{\vee}$$

and

$$M^{\vee} \otimes_{\mathcal{A}}^{L} M \longrightarrow \mathcal{B},$$

respectively.

Dually, we may consider $M \in \operatorname{Mod}_{\mathcal{B}}^{\mathcal{A}}$ as a dg functor

$$M : \mathcal{B}^{\mathrm{op}} \longrightarrow \operatorname{Mod}^{\mathcal{A}}.$$

By the above construction, we obtain a $\operatorname{Ch}(k)$-enriched Quillen adjunction

$$M \otimes_{\mathcal{B}} - : \operatorname{Mod}^{\mathcal{B}} \longleftrightarrow \operatorname{Mod}^{\mathcal{A}} : \underline{\operatorname{Hom}}^{\mathcal{A}}(M, -)$$

and call the bimodule ${}^{\vee}M \in \operatorname{Mod}_{\mathcal{A}}^{\mathcal{B}}$ given by restricting $\underline{\operatorname{Hom}}^{\mathcal{A}}(M, -)$ along $\mathcal{A}^{\mathrm{op}} \to \operatorname{Mod}^{\mathcal{A}}$ the *left dual of* M. From the universal property of left Kan extension, we obtain a canonical natural transformation

$$\xi : {}^{\vee}M \otimes_{\mathcal{A}} - \longrightarrow \underline{\operatorname{Hom}}^{\mathcal{A}}(M, -). \tag{2.4}$$

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Definition 2.5. The bimodule $M \in \mathrm{Mod}^{\mathcal{A}}_{\mathcal{B}}$ is called *left dualizable* if, for every cofibrant $N \in \mathrm{Mod}^{\mathcal{A}}$, the morphism $\xi(N)$ is a weak equivalence in $\mathrm{Mod}^{\mathcal{B}}$.

Remark 2.6. A bimodule $M \in \mathrm{Mod}^{\mathcal{A}}_{\mathcal{B}}$ is left dualizable if and only if, for every $b \in \mathcal{B}$, the left $\mathcal{A}$-module $M(b)$ is perfect.

Remark 2.7. A left dualizable bimodule $M \in \mathrm{Mod}^{\mathcal{A}}_{\mathcal{B}}$ induces a $\mathcal{D}(\mathrm{Ch}(k))$-enriched adjunction of derived categories

$$M \otimes^L_{\mathcal{B}} - : \mathcal{D}(\mathrm{Mod}^{\mathcal{B}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}^{\mathcal{A}}) : {}^{\vee}M \otimes^L_{\mathcal{A}} -$$

with unit and counit induced via Kan extensions from bimodule morphisms

$$\mathcal{B} \longrightarrow {}^{\vee}M \otimes^L_{\mathcal{A}} M$$

and

$$M \otimes^L_{\mathcal{B}} {}^{\vee}M \longrightarrow \mathcal{A},$$

respectively.

Proposition 2.1. *Let $M \in \mathrm{Mod}^{\mathcal{A}}_{\mathcal{B}}$ be a cofibrant bimodule.*

(1) *Assume that M is right dualizable. Then the cofibrant replacement $Q(M^{\vee}) \in \mathrm{Mod}^{\mathcal{B}}_{\mathcal{A}}$ of the right dual of M is left dualizable and its left dual is canonically equivalent to M.*

(2) *Assume that M is left dualizable. Then the cofibrant replacement $Q({}^{\vee}M) \in \mathrm{Mod}^{\mathcal{B}}_{\mathcal{A}}$ of the left dual of M is right dualizable and its right dual is canonically equivalent to M.*

Proof. We give an argument for statement (1). We observe that the unit and counit morphisms for the adjunction

$$-\otimes^L_{\mathcal{A}} M : \mathcal{D}(\mathrm{Mod}_{\mathcal{A}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{B}}) : R\underline{\mathrm{Hom}}_{\mathcal{B}}(M, -)$$

from Remark 2.3 can be interpreted as unit and counit morphisms for the adjunction

$$M^{\vee} \otimes^L_{\mathcal{A}} - : \mathcal{D}(\mathrm{Mod}^{\mathcal{A}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}^{\mathcal{B}}) : R\underline{\mathrm{Hom}}^{\mathcal{B}}(M^{\vee}, -)$$

from Remark 2.7. The statements now follow from the uniqueness of right adjoints. □

Corollary 2.2. *Let $M \in \mathrm{Mod}^{\mathcal{A}}_{\mathcal{B}}$ be a cofibrant right dualizable bimodule. Then there are $\mathcal{D}(\mathrm{Ch}(k))$-enriched adjunctions*

$$-\otimes^L_{\mathcal{A}} M : \mathcal{D}(\mathrm{Mod}_{\mathcal{A}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{B}}) : -\otimes^L_{\mathcal{B}} M^{\vee}$$

and

$$M^{\vee} \otimes^L_{\mathcal{A}} - : \mathcal{D}(\mathrm{Mod}^{\mathcal{A}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}^{\mathcal{B}}) : M \otimes^L_{\mathcal{B}} -.$$

Similarly, let $M \in \mathrm{Mod}^{\mathcal{A}}_{\mathcal{B}}$ be a left dualizable bimodule. Then there are $\mathcal{D}(\mathrm{Ch}(k))$-enriched adjunctions

$$M \otimes^L_{\mathcal{B}} - : \mathcal{D}(\mathrm{Mod}^{\mathcal{B}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}^{\mathcal{A}}) : {}^{\vee}M \otimes^L_{\mathcal{A}} -$$

and

$$-\otimes^L_{\mathcal{B}} {}^{\vee}M : \mathcal{D}(\mathrm{Mod}_{\mathcal{B}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{A}}) : -\otimes^L_{\mathcal{A}} M.$$

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Example 2.8. The dg category $\mathcal{A}$ is called *(locally) proper* if the diagonal bimodule, considered as an object of $\mathrm{Mod}_k^{\mathcal{A}^e}$, is right dualizable. Concretely, $\mathcal{A}$ is proper if $\mathcal{A}(a, a') \in \mathrm{Perf}_k$ for all $(a, a') \in \mathcal{A}^e$. We denote the right dual of $\mathcal{A}$ by $\mathcal{A}^*$. Assuming $\mathcal{A}$ is proper, we have, by Corollary 2.2, adjunctions

$$-\otimes^L_{\mathcal{A}^e}\mathcal{A} : \mathcal{D}(\mathrm{Mod}_{\mathcal{A}^e}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_k) : -\otimes^L_k \mathcal{A}^*$$

and

$$\mathcal{A}^* \otimes^L_{\mathcal{A}^e} - : \mathcal{D}(\mathrm{Mod}^{\mathcal{A}^e}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}^k) : \mathcal{A} \otimes^L_k -.$$

In particular, we obtain canonical equivalences in $\mathcal{D}(\mathrm{Ch}(k))$

$$R\underline{\mathrm{Hom}}_k(\mathcal{A} \otimes^L_{\mathcal{A}^e} \mathcal{A}, k) \xrightarrow{\simeq} R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^*) \tag{2.9}$$

and

$$R\underline{\mathrm{Hom}}_k(\mathcal{A}^* \otimes^L_{\mathcal{A}^e} \mathcal{A}, k) \xrightarrow{\simeq} R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A})$$

giving descriptions of the k-linear dual of Hochschild homology and of Hochschild cohomology, respectively, in terms of $\mathcal{A}^*$. In this context, we will also refer to $\mathcal{A}^*$ as the *dualizing bimodule*.

Example 2.10. Let $\mathcal{A}$ be a dg category. The dg category $\mathcal{A}$ is called *smooth* if the diagonal bimodule, considered as an object $\mathcal{A} \in \mathrm{Mod}_k^{\mathcal{A}^e}$, is left dualizable. Concretely, $\mathcal{A}$ is smooth if the diagonal bimodule $\mathcal{A}$ is perfect as a bimodule. We denote the left dual of $\mathcal{A}$ by $\mathcal{A}^!$. Assuming $\mathcal{A}$ is smooth, we have, by Corollary 2.2, adjunctions

$$\mathcal{A} \otimes^L_k - : \mathcal{D}(\mathrm{Mod}^k) \longleftrightarrow \mathcal{D}(\mathrm{Mod}^{\mathcal{A}^e}) : \mathcal{A}^! \otimes^L_{\mathcal{A}^e} -$$

and

$$-\otimes^L_k \mathcal{A}^! : \mathcal{D}(\mathrm{Mod}_k) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{A}^e}) : -\otimes^L_{\mathcal{A}^e} \mathcal{A}.$$

In particular, we obtain canonical equivalences in $\mathcal{D}(\mathrm{Ch}(k))$

$$R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}) \xrightarrow{\simeq} \mathcal{A}^! \otimes^L_{\mathcal{A}^e} \mathcal{A}$$

and

$$R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}^!, \mathcal{A}) \xrightarrow{\simeq} \mathcal{A} \otimes^L_{\mathcal{A}^e} \mathcal{A} \tag{2.11}$$

providing descriptions of Hochschild cohomology and homology, respectively, in terms of $\mathcal{A}^!$. We will refer to $\mathcal{A}^!$ as the *inverse dualizing bimodule*.

Remark 2.12. Note that the map (2.9) and the inverse of the map (2.11) can be directly constructed without the assumption on $\mathcal{A}$ to be proper or smooth, respectively. However, the context given by the adjunctions from which these maps arise via Corollary 2.2 relies on $\mathcal{A}$ being proper and smooth, respectively. Since this context is important in what follows, we will consider the maps (2.9) (respectively (2.11)) only if $\mathcal{A}$ is proper (respectively smooth).

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2.3 Duality

Let $\mathcal{A}$ be a smooth dg category. An object $p \in \mathcal{A}$ is called *locally perfect* if, for every $a \in \mathcal{A}$, the mapping complex $\mathcal{A}(a,p)$ is perfect. Let $\mathcal{P} \subset \mathcal{A}$ be a full dg subcategory of $\mathcal{A}$ spanned by some collection of locally perfect objects. Consider the bimodule

$$\mathcal{A}_{\mathcal{P}} : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{P} \longrightarrow \mathrm{Mod}_k,\ (a,p) \mapsto \mathcal{A}(a,p)$$

as an object of $\mathrm{Mod}_{\mathcal{A}}^{\mathcal{P}}$. Since $\mathcal{P}$ consists of locally perfect objects, the bimodule $\mathcal{A}_{\mathcal{P}}$ is right dualizable and we denote its right dual by $\mathcal{A}_{\mathcal{P}}^* \in \mathrm{Mod}_{\mathcal{P}}^{\mathcal{A}}$ so that we have an adjunction

$$-\otimes^L_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{P}} \mathcal{A}_{\mathcal{P}} : \mathcal{D}(\mathrm{Mod}_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{P}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_k) : - \otimes^L_k \mathcal{A}_{\mathcal{P}}^*.$$

On the other hand, since $\mathcal{A}$ is smooth, we have an adjunction

$$-\otimes^L_k \mathcal{A}^! : \mathcal{D}(\mathrm{Mod}_k) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{A}^e}) : -\otimes^L_{\mathcal{A}^e} \mathcal{A}$$

where $\mathcal{A}^! \in \mathrm{Mod}_{\mathcal{A}}^{\mathcal{A}}$ is the inverse dualizing bimodule. We denote by ${}_{\mathcal{P}}\mathcal{A}^!$ the restriction of $\mathcal{A}^! : \mathcal{A} \to \mathrm{Mod}_{\mathcal{A}}$ along $\mathcal{P} \subset \mathcal{A}$.

Proposition 2.3. *Let $\mathcal{A}$ be a smooth dg category.*

(1) *The above bimodules $\mathcal{A}_{\mathcal{P}}^*$ and ${}_{\mathcal{P}}\mathcal{A}^!$ form an adjunction*

$$-\otimes^L_P {}_{\mathcal{P}}\mathcal{A}^! : \mathcal{D}(\mathrm{Mod}_{\mathcal{P}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{A}}) : -\otimes^L_{\mathcal{A}} \mathcal{A}_{\mathcal{P}}^*.$$

(2) *Assume in addition that $\mathcal{A}$ is proper. Then we may set $\mathcal{P} = \mathcal{A}$ and the adjunction becomes an equivalence*

$$-\otimes^L_{\mathcal{A}} \mathcal{A}^! : \mathcal{D}(\mathrm{Mod}_{\mathcal{A}}) \overset{\simeq}{\longleftrightarrow} \mathcal{D}(\mathrm{Mod}_{\mathcal{A}}) : - \otimes^L_{\mathcal{A}} \mathcal{A}^*.$$

Proof. To show (1), we may describe $-\otimes^L_{\mathcal{P}} {}_{\mathcal{P}}\mathcal{A}^!$ as a Morita composite of the functors

$$\mathcal{P} \otimes_k \mathcal{A}^! \in \mathrm{Mod}^{\mathcal{P}}_{\mathcal{P}\otimes\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A}}$$

and

$${}_{\mathcal{P}}\mathcal{A} \otimes_k \mathcal{A} \in \mathrm{Mod}_{\mathcal{A}}^{\mathcal{P}\otimes\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A}}$$

so that

$$-\otimes^L_{\mathcal{P}} {}_{\mathcal{P}}\mathcal{A}^! \simeq -\otimes^L_{\mathcal{P}} (\mathcal{P} \otimes_k \mathcal{A}^!) \otimes^L_{\mathcal{P}\otimes\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A}} ({}_{\mathcal{P}}\mathcal{A} \otimes_k \mathcal{A}).$$

Passing to right adjoints, we obtain that

$$\begin{aligned}(-\otimes^L_{\mathcal{P}} {}_{\mathcal{P}}\mathcal{A}^!)^R &\simeq \otimes^L_{\mathcal{A}} ({}_{\mathcal{P}}\mathcal{A} \otimes_k \mathcal{A})^R \otimes^L_{\mathcal{P}\otimes\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A}} (\mathcal{P} \otimes_k \mathcal{A}^!)^R \\ &\simeq \otimes^L_{\mathcal{A}} (\mathcal{A}^*_{\mathcal{P}} \otimes_k \mathcal{A}) \otimes^L_{\mathcal{P}\otimes\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A}} (\mathcal{P} \otimes_k \mathcal{A}) \\ &\simeq \otimes^L_{\mathcal{A}} \mathcal{A}^*_{\mathcal{P}}\end{aligned}$$

proving the claim. Statement (2) follows from a similar calculation. □

Example 2.13. Let $\mathcal{A}$ be a smooth dg category, let $a \in \mathcal{A}$ be any object, and let $p \in \mathcal{P}$ be a locally perfect object. Then we have an equivalence

$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}^!(-,p), \mathcal{A}(-,a)) \simeq R\underline{\mathrm{Hom}}_{\mathcal{P}}(\mathcal{P}(-,p), \mathcal{A}^*_{\mathcal{P}}(-,a)) \simeq R\underline{\mathrm{Hom}}_k(\mathcal{A}(a,p), k) \qquad (2.14)$$

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in $\mathcal{D}(\mathrm{Ch}(k))$. Assume, in addition, that $\mathcal{A}$ is proper. Then, restricting to compact objects, we obtain inverse autoequivalences

$$-\otimes^L_{\mathcal{A}}\mathcal{A}^! : \mathrm{Perf}_{\mathcal{A}} \longleftrightarrow \mathrm{Perf}_{\mathcal{A}} : -\otimes^L_{\mathcal{A}} \mathcal{A}^*$$

so that, for any pair of perfect modules M, N, we have

$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N \otimes_{\mathcal{A}} \mathcal{A}^*) \simeq R\underline{\mathrm{Hom}}_k(R\underline{\mathrm{Hom}}_{\mathcal{A}}(N, M), k).$$

In this situation, the autoequivalence $\mathcal{A}^*$ is known as the Serre functor so that $\mathcal{A}^!$ becomes the inverse of the Serre functor. In a suitable geometric context, this functor can be described in terms of a dualizing complex which explains the terminology *(inverse) dualizing bimodule.*

3. Absolute Calabi–Yau structures

3.1 Right Calabi–Yau structures

Let $\mathcal{A}$ be a proper dg category so that, by (2.9), we have an equivalence

$$\Psi : R\underline{\mathrm{Hom}}_k(\mathcal{A} \otimes^L_{\mathcal{A}^e} \mathcal{A}, k) \xrightarrow{\simeq} R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^*). \tag{3.1}$$

Definition 3.2. An n-dimensional *right Calabi–Yau structure* on $\mathcal{A}$ consists of a map of complexes

$$\widetilde{\omega} : \mathrm{CC}_\bullet(\mathcal{A})_{S^1} \longrightarrow k[-n]$$

such that the corresponding morphism of $\mathcal{A}$-bimodules

$$\Psi(\omega) : \mathcal{A}[n] \longrightarrow \mathcal{A}^*$$

is a weak equivalence. Here ω denotes the pullback of $\widetilde{\omega}$ along $\mathrm{CC}_\bullet(\mathcal{A}) \to \mathrm{CC}_\bullet(\mathcal{A})_{S^1}$.

Remark 3.3. A right Calabi–Yau structure identifies the diagonal bimodule $\mathcal{A}$ up to shift with its right dual $\mathcal{A}^*$.

3.2 Left Calabi–Yau structures

Let $\mathcal{A}$ be a smooth dg category and consider the equivalence

$$\Phi : \mathcal{A} \otimes^L_{\mathcal{A}^e} \mathcal{A} \xrightarrow{\simeq} R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}^!, \mathcal{A}) \tag{3.4}$$

from (2.11).

Definition 3.5. Let $\mathcal{A}$ be a smooth dg category. An n-dimensional *left Calabi–Yau structure* on $\mathcal{A}$ consists of a map of complexes

$$\widetilde{[\mathcal{A}]} : k[n] \longrightarrow \mathrm{CC}_\bullet(\mathcal{A})^{S^1}$$

such that the corresponding morphism of $\mathcal{A}$-bimodules

$$\Phi([\mathcal{A}]) : \mathcal{A}^! \longrightarrow \mathcal{A}[-n]$$

is a weak equivalence. Here $[\mathcal{A}]$ denotes the postcomposition of $\widetilde{[\mathcal{A}]}$ with $\mathrm{CC}_\bullet(\mathcal{A})^{S^1} \to \mathrm{CC}_\bullet(\mathcal{A})$.

Remark 3.6. A left Calabi–Yau structure identifies the diagonal bimodule $\mathcal{A}$ up to shift with its left dual $\mathcal{A}^!$.

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Given a dg category $\mathcal{A}$ and a full dg subcategory $\mathcal{P} \subset \mathcal{A}$ spanned by locally perfect objects, we obtain a dg functor

$$\mathcal{A}^{\mathrm{op}} \otimes \mathcal{P} \longrightarrow \mathrm{Perf}_k,\ (a,p) \mapsto \mathcal{A}(a,p).$$

Applying $\mathrm{CC}_\bullet(-)$, we obtain an adjoint morphism of complexes

$$\mathrm{CC}_\bullet(\mathcal{A}) \simeq \mathrm{CC}_\bullet(\mathcal{A}^{\mathrm{op}}) \longrightarrow R\underline{\mathrm{Hom}}(\mathrm{CC}_\bullet(\mathcal{P}), \mathrm{CC}_\bullet(\mathrm{Perf}_k)) \simeq R\underline{\mathrm{Hom}}_k(\mathrm{CC}_\bullet(\mathcal{P}), k)$$

which is compatible with the circle actions (it can be realized as a map of cyclic complexes), so that upon passing to homotopy fixed points, we obtain a map

$$\widetilde{\Theta} : \mathrm{CC}_\bullet(\mathcal{A})^{S^1} \longrightarrow R\underline{\mathrm{Hom}}_k(\mathrm{CC}_\bullet(\mathcal{P})_{S^1}, k)$$

which is part of the following commutative square.

$$\begin{array}{ccc} \mathrm{CC}_\bullet(\mathcal{A})^{S^1} & \xrightarrow{\widetilde{\Theta}} & R\underline{\mathrm{Hom}}_k(\mathrm{CC}_\bullet(\mathcal{P})_{S^1}, k) \\ \downarrow & & \downarrow \\ \mathrm{CC}_\bullet(\mathcal{A}) & \xrightarrow{\Theta} & R\underline{\mathrm{Hom}}_k(\mathrm{CC}_\bullet(\mathcal{P}), k) \end{array} \tag{3.7}$$

We now provide a relation between left and right Calabi–Yau structures. Various incarnations of this result have been established in [GPS15] and [KVdB11].

Theorem 3.1. *Let $\mathcal{A}$ be a smooth dg category equipped with an n-dimensional left Calabi–Yau structure $[\widetilde{\mathcal{A}}]$. Let $\mathcal{P} \subset \mathcal{A}$ be a full dg subcategory spanned by a set of locally perfect objects. Then the map $\widetilde{\Theta}([\widetilde{\mathcal{A}}])$ provides an n-dimensional right Calabi–Yau structure on $\mathcal{P}$.*

Proof. Using the equivalences (3.4) and (3.1), we may augment (3.7) by the following square.

$$\begin{array}{ccc} \mathrm{CC}_\bullet(\mathcal{A}) & \xrightarrow{\Theta} & R\underline{\mathrm{Hom}}_k(\mathrm{CC}_\bullet(\mathcal{P}), k) \\ \simeq \downarrow \Phi & & \simeq \downarrow \Psi \\ R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}^!, \mathcal{A}) & \xrightarrow{\Theta'} & R\underline{\mathrm{Hom}}_{\mathcal{P}^e}(\mathcal{P}, \mathcal{P}^*) \end{array}$$

The map Θ' admits the following description. Consider the dg functor D given by the composite

$$\mathrm{Mod}_{\mathcal{A}}^{\mathcal{A}} \longrightarrow \mathrm{Mod}_{\mathcal{A}}^{\mathcal{P}} \overset{M \mapsto M^\vee}{\longrightarrow} (\mathrm{Mod}_{\mathcal{P}}^{\mathcal{A}})^{\mathrm{op}} \longrightarrow (\mathrm{Mod}_{\mathcal{P}}^{\mathcal{P}})^{\mathrm{op}}$$

where the first and last functors are given by restriction along $\mathcal{P} \subset \mathcal{A}$. We obtain an induced $\mathcal{D}(\mathrm{Ch}(k))$-enriched functor

$$LD : \mathcal{D}(\mathrm{Mod}_{\mathcal{A}^e}) \longrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{P}^e})^{\mathrm{op}}.$$

Explicitly, this functor associates to an $\mathcal{A}$-bimodule M, the $\mathcal{P}$-bimodule given by

$$(p, p') \mapsto R\underline{\mathrm{Hom}}_{\mathcal{A}}(M(-,p'), \mathcal{A}(-,p)).$$

Therefore, the functor LD maps the diagonal bimodule $\mathcal{A}$ to the diagonal bimodule $\mathcal{P}$, and, by (2.14), the inverse dualizing bimodule $\mathcal{A}^!$ to the dualizing bimodule $\mathcal{P}^*$. An explicit calculation

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shows that the map Θ' is the map induced by LD on mapping complexes. In particular, Θ' preserves equivalences: the equivalence

$$\Phi([\mathcal{A}]) : \mathcal{A}^! \xrightarrow{\simeq} \mathcal{A}[-n]$$

maps to an equivalence

$$\Theta'(\Phi([\mathcal{A}])) : \mathcal{P} \xrightarrow{\simeq} \mathcal{P}^*[-n]$$

showing that $\widetilde{\Theta}(\widetilde{[\mathcal{A}]})$ is indeed a right Calabi–Yau structure. □

Remark 3.8. Essentially all examples of right Calabi–Yau structures in this work are induced from left Calabi–Yau structures via the construction of Theorem 3.1. Therefore, it seems that smooth (or even finite type) dg categories equipped with left Calabi–Yau structures should be considered as the fundamental objects.

4. Relative Calabi–Yau structures

4.1 Relative right Calabi–Yau structures

Let $f : \mathcal{A} \to \mathcal{B}$ be a dg functor of proper dg categories. We have an induced morphism

$$\mathrm{CC}_\bullet(f)_{S^1} : \mathrm{CC}_\bullet(\mathcal{A})_{S^1} \longrightarrow \mathrm{CC}_\bullet(\mathcal{B})_{S^1}$$

defined explicitly in terms of cyclic bar constructions. We abbreviate $R\underline{\mathrm{Hom}}_k(-, k)$ by $(-)^*$. Using (2.11), we obtain a coherent diagram

$$\begin{array}{ccccc} {\mathrm{CC}_\bullet(\mathcal{A})_{S^1}}^* & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{A})^* & \xrightarrow[\simeq]{\Psi_{\mathcal{A}}} & R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^*) \\ \uparrow {\scriptstyle \mathrm{CC}_\bullet(f)_{S^1}{}^*} & & \uparrow {\scriptstyle \mathrm{CC}_\bullet(f)^*} & & \uparrow {\scriptstyle \Psi_f} \\ {\mathrm{CC}_\bullet(\mathcal{B})_{S^1}}^* & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{B})^* & \xrightarrow[\simeq]{\Psi_{\mathcal{B}}} & R\underline{\mathrm{Hom}}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^*) \end{array} \tag{4.1}$$

in $\mathcal{D}(\mathrm{Ch}(k))$. We give an explicit description of the map Ψ_f. Consider the $\mathrm{Ch}(k)$-enriched Quillen adjunction

$$F_! : \mathrm{Mod}_{\mathcal{A}^e} \longleftrightarrow \mathrm{Mod}_{\mathcal{B}^e} : F^* \tag{4.2}$$

with $F = f^{\mathrm{op}} \otimes f$. We introduce the morphism

$$u : \mathcal{A} \longrightarrow F^*\mathcal{B} \tag{4.3}$$

in $\mathcal{D}(\mathrm{Mod}_{\mathcal{A}^e})$ and its derived left adjoint

$$c : LF_!\mathcal{A} \longrightarrow \mathcal{B} \tag{4.4}$$

in $\mathcal{D}(\mathrm{Mod}_{\mathcal{B}^e})$.

Remark 4.5. The morphisms u and c represent unit and counit, respectively, of the derived adjunction

$$Lf_! : \mathcal{D}(\mathrm{Mod}_{\mathcal{A}}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_{\mathcal{B}}) : Rf^*.$$

Proposition 4.1. *The morphism Ψ_f is given by the composite*

$$R\underline{\mathrm{Hom}}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^*) \xrightarrow{F^*} R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(F^*\mathcal{B}, (F^*\mathcal{B})^*) \xrightarrow{u} R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^*)$$

where we implicitly use the canonical identification $F^*(\mathcal{B}^*) \simeq (F^*\mathcal{B})^*$.

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Proof. The proposition follows from an explicit calculation using the bar resolution. □

Let $\widetilde{\omega} : \mathrm{CC}_\bullet(\mathcal{B},\mathcal{A})_{S^1} \to k[-n+1]$ be a morphism of complexes. We interpret ω as a coherent diagram

$$\begin{array}{ccc} k[n-1] & \longrightarrow & {\mathrm{CC}_\bullet(\mathcal{B})_{S^1}}^* \\ \downarrow & & \downarrow {\scriptstyle {\mathrm{CC}_\bullet(f)_{S^1}}^*} \\ 0 & \longrightarrow & {\mathrm{CC}_\bullet(\mathcal{A})_{S^1}}^* \end{array}$$

in $\mathcal{D}(\mathrm{Ch}(k))$. By forming the composite with (4.1), we obtain the coherent diagram

$$\begin{array}{ccc} k[n-1] & \longrightarrow & R\underline{\mathrm{Hom}}_{\mathcal{B}^e}(\mathcal{B},\mathcal{B}^*) \\ \downarrow & & \downarrow {\scriptstyle \Psi_f} \\ 0 & \longrightarrow & R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A},\mathcal{A}^*) \end{array}$$

from which we extract the datum of a morphism $\xi : \mathcal{B}[n-1] \to \mathcal{B}^*$ together with a chosen zero homotopy of the morphism $\Psi_f(\xi) : \mathcal{A}[n-1] \to \mathcal{A}^*$. By Proposition 4.1, this morphism can be identified with the composite

$$\mathcal{A}[n-1] \xrightarrow{u} F^*\mathcal{B}[n-1] \xrightarrow{F^*\xi} (F^*\mathcal{B})^* \xrightarrow{c} \mathcal{A}^*$$

so that the chosen zero homotopy induces the dashed arrows which make the diagram

$$\begin{array}{ccccc} \mathcal{A}[n-1] & \xrightarrow{u} & F^*\mathcal{B}[n-1] & \longrightarrow & \mathrm{cof}(u) \\ \downarrow {\scriptstyle \xi'} & & \downarrow {\scriptstyle \xi} & & \downarrow {\scriptstyle \xi''} \\ \mathrm{fib}(u^*) & \longrightarrow & (F^*\mathcal{B})^* & \xrightarrow{u^*} & \mathcal{A}^* \end{array} \tag{4.6}$$

in $\mathcal{D}(\mathrm{Mod}_{\mathcal{A}^e})$ coherent.

Definition 4.7. An n-dimensional *right Calabi–Yau structure* on f consists of a morphism

$$\widetilde{\omega_{(\mathcal{B},\mathcal{A})}} : \mathrm{CC}_\bullet(\mathcal{B},\mathcal{A})_{S^1} \longrightarrow k[-n+1]$$

such that all vertical morphisms in the corresponding diagram (4.6) are equivalences in $\mathcal{D}(\mathrm{Mod}_{\mathcal{A}^e})$.

Example 4.8. Let $\mathcal{A}$ be a proper dg category and consider the zero functor $f : \mathcal{A} \to 0$. Then an n-dimensional right Calabi–Yau structure on f translates to a morphism

$$\widetilde{\omega_{\mathcal{A}}} : \mathrm{CC}_\bullet(\mathcal{A})_{S^1} \longrightarrow k[-n]$$

such that the vertical maps in

$$\begin{array}{ccccc} \mathcal{A}[-n+1] & \longrightarrow & 0 & \longrightarrow & \mathcal{A}[-n] \\ \downarrow {\scriptstyle \xi'} & & \downarrow & & \downarrow {\scriptstyle \xi''} \\ \mathcal{A}^*[1] & \longrightarrow & 0 & \longrightarrow & \mathcal{A}^* \end{array}$$

are equivalences. This datum, however, is equivalent to an absolute right Calabi–Yau structure on $\mathcal{A}$.

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4.2 Relative left Calabi–Yau structures

Let $f : \mathcal{A} \to \mathcal{B}$ be a functor of smooth dg categories. We have an induced morphism

$$\mathrm{CC}_\bullet(f)^{S^1} : \mathrm{CC}_\bullet(\mathcal{A})^{S^1} \longrightarrow \mathrm{CC}_\bullet(\mathcal{B})^{S^1}$$

defined explicitly in terms of cyclic bar constructions. Using (2.11), we obtain a coherent diagram

$$\begin{array}{ccccc} \mathrm{CC}_\bullet(\mathcal{A})^{S^1} & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{A}) & \xrightarrow[\simeq]{\Phi_{\mathcal{A}}} & R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}^!, \mathcal{A}) \\ \downarrow{\scriptstyle \mathrm{CC}_\bullet(f)^{S^1}} & & \downarrow{\scriptstyle \mathrm{CC}_\bullet(f)} & & \downarrow{\scriptstyle \Phi_f} \\ \mathrm{CC}_\bullet(\mathcal{B})^{S^1} & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{B}) & \xrightarrow[\simeq]{\Phi_{\mathcal{B}}} & R\underline{\mathrm{Hom}}_{\mathcal{B}^e}(\mathcal{B}^!, \mathcal{B}) \end{array} \tag{4.9}$$

in $\mathcal{D}(\mathrm{Ch}(k))$. Just like for proper dg categories, we give an explicit description of the map Φ_f which will now involve the *counit morphism*

$$c : LF_!\mathcal{A} \longrightarrow \mathcal{B}$$

in $\mathcal{D}(\mathrm{Mod}_{\mathcal{B}^e})$ from (4.4).

Lemma 4.2. *Let $M \in \mathrm{Perf}_{\mathcal{A}^e}$ be a perfect bimodule. Then there is a canonical equivalence*

$$\delta : LF_!(M^!) \simeq (LF_!M)^!$$

in $\mathcal{D}(\mathrm{Mod}_{\mathcal{B}^e})$.

Proof. We may interpret M as a left dualizable module $M \in \mathrm{Mod}_k^{\mathcal{A}^e}$. Therefore, using (2.4), we have equivalences

$$LF_!(M^!) \simeq M^! \otimes^L_{\mathcal{A}^e} F^*\mathcal{B}^e \overset{\xi}{\simeq} R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(M, F^*\mathcal{B}^e) \simeq R\underline{\mathrm{Hom}}_{\mathcal{B}^e}(LF_!M, \mathcal{B}^e) \simeq (LF_!M)^!. \qquad \square$$

Proposition 4.3. *The morphism Φ_f is given by the composite*

$$R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}^!, \mathcal{A}) \xrightarrow{LF_!} R\underline{\mathrm{Hom}}_{\mathcal{A}^e}((LF_!\mathcal{A})^!, LF_!\mathcal{A}) \xrightarrow{c} R\underline{\mathrm{Hom}}_{\mathcal{B}^e}(\mathcal{B}^!, \mathcal{B})$$

where we implicitly use the identification $\delta : LF_!(\mathcal{A}^!) \simeq (LF_!\mathcal{A})^!$ from Lemma 4.2.

Proof. Follows from an explicit calculation using the bar resolution. $\square$

Let $\sigma : k[n] \to \mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A})^{S^1}$ be a negative cyclic cycle. We may interpret σ as a coherent diagram

$$\begin{array}{ccc} k[n-1] & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{A})^{S^1} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{CC}_\bullet(f)^{S^1}} \\ 0 & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{B})^{S^1} \end{array}$$

in $\mathcal{D}(\mathrm{Ch}(k))$. By forming the composite with (4.9), we obtain the coherent diagram

$$\begin{array}{ccc} k[n-1] & \longrightarrow & R\underline{\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}^!, \mathcal{A}) \\ \downarrow & & \downarrow{\scriptstyle \Phi_f} \\ 0 & \longrightarrow & R\underline{\mathrm{Hom}}_{\mathcal{B}^e}(\mathcal{B}^!, \mathcal{B}) \end{array}$$

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from which we extract the datum of a morphism $\xi : \mathcal{A}^! \to \mathcal{A}[-n+1]$ together with a chosen zero homotopy of the morphism $\Phi_f(\xi) : \mathcal{B}^! \to \mathcal{B}[-n+1]$. By Proposition 4.3, this morphism can be identified with the composite

$$\mathcal{B}^! \xrightarrow{c^!} (LF_!\mathcal{A})^! \simeq LF_!(\mathcal{A}^!) \xrightarrow{LF_!(\xi)} LF_!(\mathcal{A}[-n+1]) \xrightarrow{c} \mathcal{B}[-n+1]$$

so that the chosen zero homotopy induces the dashed arrows which make the diagram

$$\begin{array}{ccccc} \mathcal{B}^! & \xrightarrow{c^!} & (LF_!\mathcal{A})^! & \longrightarrow & \mathrm{cof}(c^!) \\ \Big\downarrow{\scriptstyle \xi'} & & \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \xi''} \\ \mathrm{fib}(c) & \longrightarrow & LF_!\mathcal{A}[-n+1] & \xrightarrow{c} & \mathcal{B}[-n+1] \end{array} \tag{4.10}$$

in $\mathcal{D}(\mathrm{Mod}_{\mathcal{B}^e})$ coherent.

Definition 4.11. An n-dimensional *left Calabi–Yau structure* on f consists of a morphism

$$\widetilde{[\mathcal{B}, \mathcal{A}]} : k[n] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A})^{S^1}$$

such that all vertical morphisms in the corresponding diagram (4.10) are equivalences in $\mathcal{D}(\mathrm{Mod}_{\mathcal{B}^e})$.

Example 4.12. Let $\mathcal{B}$ be a smooth dg category. Consider the zero functor $f : 0 \to \mathcal{B}$ (up to Morita equivalence we may assume that $\mathcal{B}$ has a zero object). Then an n-dimensional relative left Calabi–Yau structure on f translates to a morphism

$$\widetilde{[\mathcal{B}]} : k[n] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B})^{S^1}$$

such that the vertical maps in

$$\begin{array}{ccccc} \mathcal{B}^! & \longrightarrow & 0 & \longrightarrow & \mathcal{B}^![1] \\ \Big\downarrow{\scriptstyle \xi'} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \xi''} \\ \mathcal{B}[-n] & \longrightarrow & 0 & \longrightarrow & \mathcal{B}[-n+1] \end{array}$$

are equivalences. This datum, however, is equivalent to an absolute left Calabi–Yau structure on $\mathcal{B}$.

We conclude this section, by noting that there is a relative variant of Theorem 3.1: a relative left Calabi–Yau structure on a functor $f : \mathcal{A} \to \mathcal{B}$ of smooth dg categories induces a right Calabi–Yau structure on the Morita dual functor $f^\vee : \mathcal{B}^\vee \to \mathcal{A}^\vee$.

5. Examples

5.1 Topology

The material in this section is inspired by various parts of [Lur11a] and generalizes results of [CG15].

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5.1.1 *Poincaré complexes.* We say a topological space Y is of *finite type* if it is homotopy equivalent to a homotopy retract of a finite CW complex. Let Y be of finite type. We define the functor

$$\mathrm{dg} : \mathcal{C}at_\infty \xrightarrow{\mathfrak{C}} \mathcal{C}at_\Delta \xrightarrow{\mathrm{N}} \mathcal{C}at_{\mathrm{dg}}(k) \tag{5.1}$$

given as the composite of the left Quillen functor $\mathfrak{C}$ from [Lur09a] and the functor N of applying normalized Moore chains to the mapping spaces. We set

$$\mathcal{L}(Y) := \mathrm{dg}(\mathrm{Sing}(Y)).$$

Remark 5.2. The objects of $\mathcal{L}(Y)$ can be identified with the points of Y, and the mapping complex between objects $y, y' \in Y$ is quasi-isomorphic to the complex $C_\bullet(P_{y,y'}Y)$ of singular chains on the space $P_{y,y'}Y$ of paths in Y from y to y'. In particular, if Y is path connected, then the choice of any point $y \in Y$ determines a quasi-equivalence of dg categories

$$\mathcal{L}(Y) \simeq C_\bullet(\Omega_y Y)$$

where the right-hand dg algebra of chains on the based loop space is interpreted as a dg category with one object.

Example 5.3. The choice of any point x on the circle S^1 provides a quasi-equivalence

$$\mathrm{dg}(\mathrm{Sing}(S^1)) \simeq k\mathbb{Z}$$

where $k\mathbb{Z} = k[t, t^{-1}]$ denotes the group algebra of the group $\mathbb{Z} = \pi_1(S^1, x)$.

Remark 5.4. The functor dg is weakly equivalent to the left adjoint of the Quillen adjunction

$$\mathrm{dg}' : \mathcal{C}at_\infty \longleftrightarrow \mathcal{C}at_{\mathrm{dg}}(k) : \mathrm{N}_{\mathrm{dg}}$$

where N_{dg} denotes the differential graded nerve of [Lur11b]. From an enriched variant of the adjunction (5.1), we obtain an equivalence of ∞-categories

$$\mathcal{D}(\mathrm{Mod}_{\mathcal{L}(Y)}) \simeq \mathrm{Fun}_\infty(\mathrm{Sing}(Y), \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(k)))$$

so that we are naturally led to interpret $\mathcal{L}(Y)$-modules as ∞-local systems of complexes of k-vector spaces on Y.

Proposition 5.1. *The dg category $\mathcal{L}(Y)$ is of finite type. In particular, it is smooth.*

Proof. By construction, the functor dg commutes with homotopy colimits and takes retracts to retracts. The space Y is of finite type and so expressible as a retract of a finite homotopy colimit of the constant diagram with value pt. Finite type dg categories are stable under retracts and finite homotopy colimits [TV07] so that it suffices to check the statement of the proposition for $Y = \mathrm{pt}$, where it is apparent. □

The constant map $\pi : Y \to \mathrm{pt}$ induces a dg module

$$k_Y : \mathcal{L}(Y) \xrightarrow{\mathrm{dg}(\pi)} k \subset \mathrm{Mod}_k$$

which corresponds to the constant local system on Y with value k. We have a corresponding adjunction

$$- \otimes^L_{\mathcal{L}(Y)} k_Y : \mathcal{D}(\mathrm{Mod}_{\mathcal{L}(Y)}) \longleftrightarrow \mathcal{D}(\mathrm{Mod}_k) : R\underline{\mathrm{Hom}}_k(k_Y, -)$$

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and

$$k_Y \otimes_k^L - : \mathcal{D}(\mathrm{Mod}^k) \longleftrightarrow \mathcal{D}(\mathrm{Mod}^{\mathcal{L}(Y)}) : R\underline{\mathrm{Hom}}^{\mathcal{L}(Y)}(k_Y, -).$$

The functors

$$C_\bullet(Y, -) := - \otimes_{\mathcal{L}(Y)}^L k_Y,$$
$$C^\bullet(Y, -) := R\underline{\mathrm{Hom}}^{\mathcal{L}(Y)}(k_Y, -)$$

define homology, respectively cohomology, of Y with ∞-local systems as coefficients. We denote by

$$\zeta_Y \in \mathrm{Mod}_{\mathcal{L}(Y)}$$

the left dual of a cofibrant replacement $Q(k_Y)$ of $k_Y \in \mathrm{Mod}_k^{\mathcal{L}(Y)}$.

Proposition 5.2 (Poincaré duality). *The module $Q(k_Y)$ is left dualizable so that the canonical map*

$$\zeta_Y \otimes_{\mathcal{L}(Y)}^L - \longrightarrow C^\bullet(Y, -) \tag{5.5}$$

is an equivalence.

Proof. It is immediate that k_Y is locally perfect and hence right dualizable. But since, by Proposition 5.1, $\mathcal{L}(Y)$ is smooth, every cofibrant locally perfect $\mathcal{L}(Y)$-module is perfect [TV07]. Therefore, $Q(k_Y)$ is left dualizable. $\square$

Remark 5.6. Evaluating (5.5) at k_Y, we obtain an equivalence

$$C_\bullet(Y, \zeta_Y) \simeq C^\bullet(Y, k_Y)$$

which can be understood as a version of Poincaré duality.

Remark 5.7. By definition, the functor ζ_Y is given as the restriction of the dg functor $C^\bullet(Y, -)$ along $\mathcal{L}(Y)^{\mathrm{op}} \to \mathrm{Mod}^{\mathcal{L}(Y)}$. In particular, for a point $y \in Y$, the complex $\zeta_Y(y)$ is the cohomology of the ∞-local system on Y which assigns to a point y' the complex $C_\bullet(P_{y,y'}Y)$ (cf. [Lur11a]).

We define dg functors

$$\mathcal{L}(Y) \otimes \mathcal{L}(Y) \xrightarrow{\nabla} \mathrm{N}(\mathfrak{C}(\mathrm{Sing}(Y)) \times \mathfrak{C}(\mathrm{Sing}(Y))) \xleftarrow{\Delta} \mathcal{L}(Y)$$

where the functor ∇ is given by applying the Eilenberg–Zilber construction to mapping complexes, and the functor Δ is induced by the diagonal map $Y \to Y \times Y$. We obtain a dg functor

$$- \otimes_Y - : \mathrm{Mod}_{\mathcal{L}(Y)} \otimes \mathrm{Mod}_{\mathcal{L}(Y)} \xrightarrow{\Delta^* \nabla_!} \mathrm{Mod}_{\mathcal{L}(Y)}$$

which we call the *internal tensor product*. We may consider the restriction of this dg functor to $\mathcal{L}(Y) \otimes \mathcal{L}(Y)$ as a module $M_Y \in \mathrm{Mod}_{\mathcal{L}(Y)^{\mathrm{op}} \otimes \mathcal{L}(Y)}^{\mathcal{L}(Y)}$ and obtain, via enriched Kan extension, another dg functor

$$- \otimes_{\mathcal{L}(Y)} M_Y : \mathrm{Mod}_{\mathcal{L}(Y)} \longrightarrow \mathrm{Mod}_{\mathcal{L}(Y)^e}.$$

Remark 5.8. If the local system ζ_Y is invertible with respect to the tensor product $\otimes_Y$ on $\mathcal{D}(\mathrm{Mod}_{\mathcal{L}(Y)})$, then we call ζ_Y the k-linear *Spivak normal fibration*. In this case, the space Y is therefore a topological analogue of a Gorenstein variety.

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By Corollary 2.2 and Proposition 5.2, we have an adjunction

$$\zeta_Y \otimes^L_k - : \mathcal{D}(\mathrm{Mod}^k) \longleftrightarrow \mathcal{D}(\mathrm{Mod}^{\mathcal{L}(Y)}) : k_Y \otimes^L_{\mathcal{L}(Y)} -$$

which provides us with a canonical equivalence

$$\psi : C_\bullet(Y, k_Y) \xrightarrow{\simeq} R\underline{\mathrm{Hom}}_{\mathcal{L}(Y)}(\zeta_Y, k_Y).$$

Here, we slightly abuse notation and also denote the *right* $\mathcal{L}(Y)$-module

$$\mathcal{L}(Y)^{\mathrm{op}} \xrightarrow{\mathrm{dg}(\pi)^{\mathrm{op}}} k^{\mathrm{op}} = k \subset \mathrm{Mod}_k$$

by k_Y.

Definition 5.9. A class
$$[Y] \in H^{-n}(C_\bullet(Y, k_Y)) \cong H_n(Y, k)$$
is called a *fundamental class* if the morphism

$$\psi([Y]) : \zeta_Y \longrightarrow k_Y[-n]$$

is an equivalence of $\mathcal{L}(Y)$-modules. A pair $(Y, [Y])$ of a space equipped with a fundamental class is called a *Poincaré complex*.

Remark 5.10. Let $(Y, [Y])$ be a Poincaré complex. Combining the equivalence $\psi([Y])$ with (5.5) we obtain, for every i, isomorphisms

$$H^i(Y, k) \cong H_{n-i}(Y, k)$$

recovering classical Poincaré duality.

Example 5.11. A closed topological manifold with chosen orientation provides an example of a Poincaré complex.

We will now explain how a fundamental class for Y gives rise to a left Calabi–Yau structure on the dg category $\mathcal{L}(Y)$.

Proposition 5.3. *Let Y be a topological space of finite type. Then the following hold.*

(1) *We have canonical equivalences*

$$k_Y \otimes^L_{\mathcal{L}(Y)} M_Y \simeq \mathcal{L}(Y), \quad \zeta_Y \otimes^L_{\mathcal{L}(Y)} M_Y \simeq \mathcal{L}(Y)^!.$$

(2) *There is an S^1-equivariant equivalence*

$$j : \mathrm{CC}_\bullet(\mathcal{L}(Y)) \xrightarrow{\simeq} C_\bullet(LY, k)$$

where LY denotes the free loop space of Y with circle action given by loop rotation.

(3) *The composite*

$$\alpha : C_\bullet(Y, k) \simeq R\underline{\mathrm{Hom}}_{\mathcal{L}(Y)}(\zeta_Y, k_Y) \xrightarrow{-\otimes^L_{\mathcal{L}(Y)} M_Y} R\underline{\mathrm{Hom}}_{\mathcal{L}(Y)^e}(\mathcal{L}(Y)^!, \mathcal{L}(Y)) \overset{j}{\simeq} C_\bullet(LY, k)$$

can be identified with the natural map induced by the inclusion of Y as constant loops in LY.

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Proof. Arguing for each connected component, we may assume $X = BG$ where G is a topological group. The first statement then follows by explicit calculation. Statement (2) is shown in [Jon87] and (3) follows by a similar argument. □

Theorem 5.4 [CG15]. *Let $(Y, [Y])$ be a Poincaré complex. Then the dg category $\mathcal{L}(Y)$ is equipped with a canonical left Calabi–Yau structure.*

Proof. Since the map α from Proposition 5.3 is S^1-invariant, it induces a canonical map

$$\alpha : C_\bullet(Y, k) \longrightarrow \mathrm{CC}_\bullet(\mathcal{L}(Y))^{S^1}.$$

Define $[\mathcal{L}(Y)] = \alpha([Y])$. The corresponding map $\mathcal{L}(Y)^! \to \mathcal{L}(Y)[-n]$ is an equivalence since it is the image of the equivalence $\zeta_Y \to k_Y[-n]$ under the dg functor $- \otimes_{\mathcal{L}(Y)} M_Y$. □

5.1.2 *Poincaré pairs.* We provide a generalization of Theorem 5.4 in the relative context. Let X, Y be topological spaces of finite type, and let $\varphi : X \to Y$ a continuous map. Applying $\mathrm{dg}(\mathrm{Sing}(-))$ to φ, we obtain a dg functor

$$f : \mathcal{L}(X) \longrightarrow \mathcal{L}(Y).$$

Lemma 5.5. *There is a canonical equivalence $Lf_!({}^\vee k_X) \simeq {}^\vee(Lf_! k_Y)$.*

The equivalence $k_X \to f^* k_Y$ has an adjoint map

$$\gamma : Lf_! k_X \longrightarrow k_Y$$

with left dual

$${}^\vee\gamma : \zeta_Y \longrightarrow Lf_! \zeta_X.$$

Lemma 5.6. *The composite*

$$C_\bullet(X; k) \simeq R\underline{\mathrm{Hom}}_{\mathcal{L}(X)}(\zeta_X, k_X) \xrightarrow{Lf_!} R\underline{\mathrm{Hom}}_{\mathcal{L}(Y)}(Lf_!\zeta_X, Lf_! k_X) \xrightarrow{\gamma\circ - \circ {}^\vee\gamma} R\underline{\mathrm{Hom}}_{\mathcal{L}(Y)}(\zeta_Y, k_Y) \simeq C_\bullet(Y; k)$$

can be identified with the map $C_\bullet(\varphi; k)$.

As a consequence of Lemma 5.6, the choice of a class $[Y, X] \in H_n(Y, X; k)$ gives a coherent diagram in $\mathcal{D}(\mathrm{Mod}_{\mathcal{L}(Y)})$ of the following form.

$$\begin{array}{ccccc} \zeta_Y & \xrightarrow{{}^\vee\alpha} & Lf_!\zeta_X & \longrightarrow & \mathrm{cof} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{fib} & \longrightarrow & Lf_! k_X[-n+1] & \xrightarrow{\alpha} & k_Y \end{array} \tag{5.12}$$

Definition 5.13. We call $[Y, X] \in H_n(Y, X; k)$ a *relative fundamental class* if all vertical maps in (5.12) are equivalences. A pair $(X \to Y, [Y, X])$ consisting of a continuous map of finite type topological spaces and a relative fundamental class is called a *Poincaré pair*.

Example 5.14. A relative fundamental class for $\emptyset \to Y$ can be identified with an absolute fundamental class for Y.

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Example 5.15. An example of a Poincaré pair is given by a compact oriented manifold Y with boundary X.

Theorem 5.7. *Let* $(X \to Y, [Y, X])$ *be a Poincaré pair. Then the corresponding dg functor*

$$f : \mathcal{L}(X) \longrightarrow \mathcal{L}(Y)$$

of linearizations carries a canonical relative left Calabi–Yau structure.

Proof. This follows from Proposition 5.3 and Lemma 5.6 by applying the dg functor

$$- \otimes_{\mathcal{L}(Y)} M_Y : \mathrm{Mod}_{\mathcal{L}(Y)} \longrightarrow \mathrm{Mod}_{\mathcal{L}(Y)^{\mathrm{op}} \otimes \mathcal{L}(Y)}.$$ □

5.2 Algebraic geometry

In this subsection, we give examples of (relative) left Calabi–Yau structures coming from anticanonical divisors. We assume all schemes are separated and of finite type over a field k, which for simplicity we take to be of characteristic zero, although the assumption on the characteristic seems to be unnecessary.

5.2.1 *Background on ind-coherent sheaves.* In this subsection, we briefly review the theory of ind-coherent sheaves, following [GR17]. Among other things, the theory of ind-coherent sheaves provides a functorially defined dualizing complex, which we can use to give a geometric computation of Hochschild chains for the dg category of coherent sheaves and a geometric computation for the map on Hochschild chains induced by pushforward along a proper morphism. We shall only need the 'elementary' parts of the theory, which can be developed using only basic facts about ind-completion of ∞-categories and the notion of dualizable object in a monoidal ∞-category. If the reader is willing to make smoothness assumptions, then ind-coherent sheaves can be identified with (the dg derived category of) quasi-coherent sheaves, simplifying the formalism.

Given a separated scheme X of finite type over a field k, let $\mathrm{Perf}(X)$ denote the dg category of perfect complexes on X and $\mathrm{QCoh}(X)$ the dg unbounded derived category of quasi-coherent sheaves. It is known that $\mathrm{QCoh}(X)$ is compactly generated and that the compact objects in $\mathrm{QCoh}(X)$ are exactly the perfect complexes $\mathrm{Perf}(X)$ (cf. [TT90]), so that we have identifications

$$\mathrm{QCoh}(X) \simeq \mathrm{Ind}(\mathrm{Perf}(X)) \simeq \mathrm{Mod}_{\mathrm{Perf}(X)}. \tag{5.16}$$

For every morphism $f : X \to Y$, we have an adjunction

$$f^* : \mathrm{QCoh}(Y) \leftrightarrow \mathrm{QCoh}(X) : f_*. \tag{5.17}$$

Note that since f^* sends perfects to perfects and hence compacts to compacts, the right adjoint f_* preserves colimits.

Let $\mathrm{Coh}(X)$ denote the full dg subcategory of $\mathrm{QCoh}(X)$ spanned by objects with bounded coherent cohomology sheaves. By definition, ind-coherent sheaves on X are obtained by ind-completion of $\mathrm{Coh}(X)$:

$$\mathrm{IndCoh}(X) := \mathrm{Ind}(\mathrm{Coh}(X)) \simeq \mathrm{Mod}_{\mathrm{Coh}(X)}. \tag{5.18}$$

For smooth X, we have $\mathrm{Coh}(X) = \mathrm{Perf}(X)$ and so $\mathrm{IndCoh}(X) = \mathrm{QCoh}(X)$. For singular X, we have proper inclusions $\mathrm{Perf}(X) \subset \mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$. Ind-completion along the first and second inclusion gives an adjunction

$$\Xi_X : \mathrm{QCoh}(X) \leftrightarrow \mathrm{IndCoh}(X) : \Psi_X \tag{5.19}$$

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in which the left adjoint is fully faithful and hence Ψ_X realizes $\mathrm{QCoh}(X)$ as a colocalization of $\mathrm{IndCoh}(X)$. There is moreover a natural t-structure on $\mathrm{IndCoh}(X)$ for which Ψ_X is t-exact and, for every n, the restricted functor $\Psi_X : \mathrm{IndCoh}(X)^{\geqslant n} \to \mathrm{QCoh}(X)^{\geqslant n}$ is an equivalence. Taking the union over all n, we have $\mathrm{IndCoh}(X)^+ \simeq \mathrm{QCoh}(X)^+$. The categories $\mathrm{IndCoh}(X)$ and $\mathrm{QCoh}(X)$ therefore differ only in their 'tails' at $-\infty$.

Given a morphism $f : X \to Y$, we can restrict the functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ to $\mathrm{Coh}(X)$, obtaining a functor $f_* : \mathrm{Coh}(X) \to \mathrm{QCoh}(X)^+ \simeq \mathrm{IndCoh}(X)^+ \subset \mathrm{IndCoh}(X)$. Passing to ind-completion and slightly abusing notation, we obtain a functor $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$. When $f : X \to Y$ is proper, we have an adjunction

$$f_* : \mathrm{IndCoh}(X) \leftrightarrow \mathrm{IndCoh}(Y) : f^!. \tag{5.20}$$

Note that since f is proper, f_* sends compact objects to compact objects ($f_* \mathrm{Coh}(X) \subseteq \mathrm{Coh}(Y)$), hence the right adjoint $f^!$ preserves colimits. Note in particular that since we are working with separated schemes, the diagonal morphism $\Delta : X \to X \times X$ is a closed immersion, so that the functor $\Delta^!$ is right adjoint to Δ_*.

Like quasi-coherent sheaves, ind-coherent sheaves enjoy good monoidal properties. In particular, there is a natural equivalence

$$\boxtimes : \mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X \times Y) \tag{5.21}$$

which, for $F \in \mathrm{Coh}(X)$, $G \in \mathrm{Coh}(Y)$, is given by the usual formula

$$F \boxtimes G = \pi_X^* F \otimes^L_{\mathcal{O}_X} \pi_Y^* G$$

while (5.21) is obtained by ind-extension (see [GR17, II.2.6.3]).

Using the equivalence 5.21, we construct a pairing

$$\mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(X) \simeq \mathrm{IndCoh}(X \times X) \xrightarrow{\Delta^!} \mathrm{IndCoh}(X) \xrightarrow{p_*} \mathrm{Mod}_k \tag{5.22}$$

where $p : X \to \mathrm{pt} = \mathrm{Spec}(k)$ is the structure map and we use the equivalence $\mathrm{IndCoh}(\mathrm{pt}) \simeq \mathrm{Mod}_k$. One can show that the pairing in 5.22 is nondegenerate in the symmetric monoidal ∞-category of presentable dg categories with colimit preserving dg functors. The copairing is given by the functor

$$\mathrm{Mod}_k \xrightarrow{\Delta_* p^!} \mathrm{IndCoh}(X \times X) \tag{5.23}$$

where $p^! : \mathrm{Mod}_k \to \mathrm{IndCoh}(X)$ sends k to the dualizing complex ω_X (see [GR17, II.2.4.3], which also uses this duality pairing to give an 'elementary' construction of the functor $f^!$ for an arbitrary morphism $f : X \to Y$).

From the above discussion, we obtain the following computation of Hochschild chains of $\mathrm{Coh}(X)$.

Lemma 5.8. $\mathrm{CC}_\bullet(\mathrm{Coh}(X)) \simeq R\Gamma(X, \Delta^! \Delta_* \omega_X) \simeq R\mathrm{Hom}_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \omega_X)$.

Proof. In general, the Hochschild chains of a small dg category $\mathcal{A}$ can be computed as the composition of the functor $\mathrm{Mod}_k \to \mathrm{Mod}_{\mathcal{A}^e}$ sending k to the diagonal bimodule $\mathcal{A}$, and the functor $\mathrm{Mod}_{\mathcal{A}^e} \to \mathrm{Mod}_k$ given as left Kan extension of the Yoneda pairing $\mathcal{A}^e \to \mathrm{Mod}_k$. Note that under the equivalence $\mathrm{Mod}_{\mathcal{A}^{\mathrm{op}}} \otimes \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{A}^e}$, the Yoneda pairing exhibits $\mathrm{Mod}_{\mathcal{A}^{\mathrm{op}}}$ as the dual of $\mathrm{Mod}_{\mathcal{A}}$. Letting $\mathcal{A} = \mathrm{Coh}(X)$ and using the duality pairing 5.22, we obtain an identification

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of $\mathrm{Mod}_{\mathrm{Coh}(X)^{\mathrm{op}}}$ with $\mathrm{Mod}_{\mathrm{Coh}(X)} \simeq \mathrm{IndCoh}(X)$. Under this identification, the diagonal bimodule corresponds to $\Delta_*\omega_X$, and we obtain the isomorphism $\mathrm{CC}_\bullet(\mathrm{Coh}(X)) \simeq R\Gamma(X, \Delta^!\Delta_*\omega_X)$. The isomorphism $R\Gamma(X, \Delta^!\Delta_*\omega_X) \simeq R\mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X)$ follows from properness of the diagonal morphism, which gives adjointness between Δ_* and $\Delta^!$. □

The next lemma describes the functoriality of Lemma 5.8 for proper morphisms, and is simply a translation of Proposition 4.3.

Lemma 5.9. *Let $f : X \to Y$ be a proper morphism, so that we have a morphism $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ and an induced morphism $\mathrm{CC}_\bullet(f_*) : \mathrm{CC}_\bullet(\mathrm{Coh}(X)) \to \mathrm{CC}_\bullet(\mathrm{Coh}(Y))$. Then under the identifications provided by Lemma 5.8, the induced morphism $\mathrm{CC}_\bullet(f_*)$ is given as the composition*

$$R\mathrm{Hom}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X) \longrightarrow R\mathrm{Hom}(\Delta_*f_*\mathcal{O}_X, \Delta_*f_*\omega_X) \longrightarrow R\mathrm{Hom}(\Delta_*\mathcal{O}_Y, \Delta_*\omega_Y),$$

where the first arrow is given by pushforward along $f \times f$ and the natural isomorphism $(f\times f)_*\Delta_* \simeq \Delta_* f_*$ and the second arrow is given by pre-composition with Δ_* applied to the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_Y$ and post-composition with Δ_* applied to the natural counit map $f_*\omega_X \simeq f_*f^!\omega_Y \to \omega_Y$.

In some cases arising in classical algebraic geometry and representation theory, we have vanishing of Hochschild homology above a certain degree. The following lemma describes the consequence of this for negative cyclic homology and provides a means of constructing S^1-equivariance data for left Calabi–Yau structures.

Lemma 5.10.

(1) *Let $\mathcal{A}$ be a small dg category and suppose $HH_i(\mathcal{A}) = 0$ for $i > d$. Then the natural map $HC_i^-(\mathcal{A}) \to HH_i(\mathcal{A})$ is an isomorphism for $i \geqslant d$.*
(2) *Let $f : \mathcal{A} \to \mathcal{B}$ be a dg functor between small dg categories and suppose that $HH_i(\mathcal{A}) = 0$ for $i > d-1$ and $HH_i(\mathcal{B}) = 0$ for $i > d$. Then the map $HC_d^-(\mathcal{B}, \mathcal{A}) \to HH_d(\mathcal{B}, \mathcal{A})$ is an isomorphism.*

Proof. Using $\mathrm{CC}_\bullet(\mathcal{A})[[u]]$ with differential $b + Bu$ as our model for the negative cyclic complex $\mathrm{CC}_\bullet(\mathcal{A})^{S^1}$ (cf. [Hoy15]), we obtain a filtration $\cdots \subset u^2\mathrm{CC}_\bullet(\mathcal{A})^{S^1} \subset u\mathrm{CC}_\bullet(\mathcal{A})^{S^1} \subset \mathrm{CC}_\bullet(\mathcal{A})^{S^1}$. Inspecting the associated spectral sequence immediately gives the first part of the lemma and the second part follows immediately by looking at the map between long exact sequences of negative cyclic and Hochschild homology.

Equivalently, and very concretely, suppose a class in $HC_i^-(\mathcal{A})$ is represented by a cycle $c = \sum_{j=0}^\infty c_{i+2j}u^j$ in $\mathrm{CC}_\bullet(\mathcal{A})[[u]]$, so that we have relations $bc_i = 0$ and $bc_{i+2j} + Bc_{i+2j-2} = 0$ for $j > 0$ in $\mathrm{CC}_\bullet(\mathcal{A})$.

First, to show $HC_i^-(\mathcal{A}) = 0$ for $i > d$, we must integrate the cycle c. The assumption that $HH_i(\mathcal{A}) = 0$ for $i > d$ ensures that this can be done order by order.

Next, let $i = d$. We want to show that the map $HC_d^-(\mathcal{A}) \to HH_d(\mathcal{A})$ is an isomorphism. If $c = \sum_{j=0}^\infty c_{d+2j}u^j$ represents a class in the kernel of $HC_d^-(\mathcal{A}) \to HH_d(\mathcal{A})$, then the constant coefficient c_d of c must be b-exact, and so, replacing c with a homologous cycle if necessary, we may assume that the constant coefficient of c vanishes. But then we may write $c = \tilde{c}u$, where $\tilde{c}$ is a cycle of degree $d+2$ and hence null-homologous by the first part of our argument. Thus the map $HC_d^-(\mathcal{A}) \to HH_d(\mathcal{A})$ is an injection. To see that it is a surjection, one inductively

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constructs a lift of a d-cycle c_d in $\mathrm{CC}_\bullet(\mathcal{A})$ to a d-cycle in $\mathrm{CC}_\bullet(\mathcal{A})[[u]]$ using the vanishing of $HH_i(\mathcal{A})$ for $i > d$. □

Remark 5.24. Inspecting the above argument, one finds in addition that there is a short exact sequence $0 \to HH_{d-2}(\mathcal{A}) \to HC^-_{d-1}(\mathcal{A}) \to HH_{d-1}(\mathcal{A}) \to 0$, but we shall not need this.

5.2.2 *Calabi–Yau schemes and anticanonical divisors.* Recall that a scheme X is said to be Cohen–Macaulay of dimension d if $\omega_X[-d]$ is a coherent sheaf (that is, lives in the heart of the t-structure on $\mathrm{IndCoh}(X)$) and is said to be Gorenstein of dimension d if $\omega_X[-d]$ is a line bundle. In the Gorenstein case, we introduce the notation $K_X = \omega_X[-d]$ for the canonical line bundle.

Lemma 5.11. *Let X be Cohen–Macaulay of dimension d. Then $HH_d(\mathrm{Coh}(X)) \cong H^0(X, \omega_X[-d])$ and, for $i > d$, we have $HH_i(\mathrm{Coh}(X)) = 0$. Furthermore, we have an isomorphism $HC^-_d(\mathrm{Coh}(X)) \simeq HH_d(\mathrm{Coh}(X))$.*

Proof. By Lemma 5.8, $HH_i(\mathrm{Coh}(X)) \simeq \mathrm{Ext}^{-i}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X) \simeq \mathrm{Ext}^{d-i}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X[-d])$, which clearly vanishes for $d - i < 0$ since $\omega_X[-d]$ is a sheaf. Moreover, when $i = d$, we have an isomorphism $HH_d(\mathrm{Coh}(X)) \simeq \mathrm{Ext}^0(\Delta_*\mathcal{O}_X, \Delta_*\omega_X[-d]) \simeq H^0(X, \omega_X[-d])$, since pushforward along a closed immersion is fully faithful in degree 0 between sheaves. □

Proposition 5.12. *Let X be a Gorenstein scheme of dimension d. Then giving a left Calabi–Yau structure on $\mathrm{Coh}(X)$ is equivalent to giving a trivialization $\mathcal{O}_X \simeq \omega_X[-d]$.*

Proof. By definition, a left Calabi–Yau structure of dimension k on $\mathrm{Coh}(X)$ is given by a class $\theta \in HC^-_k(\mathrm{Coh}(X))$ such that the corresponding Hochschild class, viewed as a map $\Delta_*\mathcal{O}_X \to \Delta_*\omega_X[-k]$, is an equivalence. Since X is Gorenstein of dimension d, there is only one possibility for k, namely $k = d$, and the induced map $\Delta_*\mathcal{O}_X \to \Delta_*\omega_X[-d]$ is an equivalence if and only the underlying map $\mathcal{O}_X \to \omega_X[-d]$ is an equivalence. Thus a left Calabi–Yau structure on $\mathrm{Coh}(X)$ gives a trivialization $\mathcal{O}_X \simeq \omega_X[-d]$.

Conversely, given a trivialization $\mathcal{O}_X \simeq \omega_X[-d]$, apply Δ_* to obtain a class $\mathrm{Ext}^{-d}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X)$. By Lemma 5.8, we have an isomorphism $HH_d(\mathrm{Coh}(X)) \simeq \mathrm{Ext}^{-d}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X)$, and by Lemma 5.11, we have an isomorphism $HC^-_d(\mathrm{Coh}(X)) \simeq HH_d(\mathrm{Coh}(X))$. Altogether, a trivialization $\mathcal{O}_X \simeq \omega_X[-d]$ gives a left Calabi–Yau structure on $\mathrm{Coh}(X)$. □

Theorem 5.13. *Let Y be a Gorenstein scheme of dimension d, so that $K_Y = \omega_Y[-d]$ is the canonical line bundle. Fix a section $s \in K_Y^{-1}$ with zero scheme $i : X \hookrightarrow Y$ of dimension $d-1$. Then $\mathrm{Coh}(X)$ carries a canonical left Calabi–Yau structure of dimension $d-1$ and the pushforward functor $i_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ carries a compatible canonical left Calabi–Yau structure of dimension d.*

Proof. The choice of section $s \in K_Y^{-1}$ provides a cofiber sequence

$$\mathcal{O}_Y \to i_*\mathcal{O}_X \to K_Y[1]$$

in $\mathrm{IndCoh}(Y)$ and in particular a null-homotopy of the composed map $\mathcal{O}_Y \to K_Y[1]$. Applying $i^!$ to the above cofiber sequence and using the natural equivalences $i^! K_Y[1] \simeq i^!\omega_Y[1-d] \simeq \omega_X[1-d] \simeq K_X$ we obtain a cofiber sequence

$$i^!\mathcal{O}_Y \to i^! i_*\mathcal{O}_X \to K_X$$

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in IndCoh(X). Pre-composing $r : i^! i_* \mathcal{O}_X \to \omega_X$ with the unit $u : \mathcal{O}_X \to i^! i_* \mathcal{O}_X$ of the adjunction, we obtain a morphism

$$\theta : \mathcal{O}_X \to K_X.$$

Since $K_X = \omega_X[-d+1]$, Lemma 5.8 gives a class in $HC^-_{d-1}(\mathrm{Coh}(X)) \simeq \mathrm{HH}_{d-1}(\mathrm{Coh}(X))$. We claim that this class gives a left Calabi–Yau structure of dimension $d-1$ on Coh(X), which by Proposition 5.12 is equivalent to showing that the map $\mathcal{O}_X \to K_X$ is an isomorphism.

To this end, note that since $i : X \hookrightarrow Y$ is by definition the inclusion of the zero-scheme of s, and the map $K_Y \to \mathcal{O}_Y$ is the dual of $s : \mathcal{O}_Y \to K_Y^{-1}$, the above cofiber sequence splits. Furthermore, the component of $\mathcal{O}_X \to i^! i_* \mathcal{O}_X \simeq i^! \mathcal{O}_Y \oplus K_X$ projecting to $i^! \mathcal{O}_Y$ must be homotopic to zero since it is adjoint to a morphism $i_* \mathcal{O}_X \to \mathcal{O}_Y$, the latter being automatically zero since $X \subset Y$ is a proper closed subset. Finally, applying the functor i_* and using the counit of the adjunction, we obtain a factorization $i_* \mathcal{O}_X \to i_* K_X \to i_* i^! i_* \mathcal{O}_X \to i_* \mathcal{O}_X$ of the identity morphism of $i_* \mathcal{O}_X$. Thus $i_* \theta : i_* \mathcal{O}_X \to i_* K_X$ and hence $\theta : \mathcal{O}_X \to K_X$ itself are invertible.

To prove the second part of the theorem, it is enough by Lemma 5.10 to provide a null-homotopy of the composition from top-left to bottom-right in the following diagram, such that the induced vertical arrows are equivalences.

$$\begin{array}{ccccc}
\Delta_* \mathcal{O}_Y & \longrightarrow & (i \times i)_* \Delta_* \mathcal{O}_X \simeq \Delta_* i_* \mathcal{O}_X & \longrightarrow & \Delta_* K_Y[1] \\
\vdots & & \downarrow & & \vdots \\
\Delta_* \mathcal{O}_Y & \longrightarrow & (i \times i)_* \Delta_* K_X \simeq \Delta_* i_* K_X & \longrightarrow & \Delta_* K_Y[1]
\end{array}$$

In fact, however, we have already implicitly proved this before applying Δ_*. Indeed, our constructions above provide a null-homotopy of $\mathcal{O}_Y \to i_* \mathcal{O}_X \to K_Y[1]$ as well as a factorization of $i_* \mathcal{O}_X \to K_Y[1]$ into $i_* \mathcal{O}_X \to i_* K_X \to K_Y[1]$. Furthermore, we have just shown above that the middle vertical arrow is an equivalence, and the outer vertical arrows are nothing but identities. □

5.3 Representation theory

5.3.1 *A_n-quiver.* Let k be a field and let T denote the path algebra over k of the A_n-quiver equipped with the standard orientation. We label the vertices of the quiver by $1, 2, \ldots, n$ so that we have, for every $1 \leqslant i < n$, an arrow $\rho_{i,i+1}$ from i to $i+1$. We denote the idempotent in T corresponding to the vertex i by e_i and further set $e_0 = e_{n+1} = 0$. Our convention is to write concatenation of paths from left to right, so that for example $e_i \rho_{i,i+1} e_{i+1} = \rho_{i,i+1}$. For $0 \leqslant i \leqslant n$, we define $L_i \in \mathrm{Perf}_T$ to be

$$\cdots \to 0 \to e_{i+1} T \to e_i T \to 0 \to \cdots$$

where the right T-module $e_i T$ is situated in degree 0 and the arrow $e_{i+1} T \to e_i T$ is given by left multiplication with $\rho_{i,i+1}$.

Remark 5.25. Note that the objects $L_1, \ldots, L_n$ are cofibrant replacements of the simple T-modules.

We denote by $\amalg_{n+1} \underline{k}$ the free dg category on the set $\{0, 1, 2, \ldots, n\}$.

Theorem 5.14. *The dg functor*

$$f : \coprod_{n+1} \underline{k} \longrightarrow \mathrm{Perf}_T,\ i \mapsto L_i$$

carries a natural left Calabi–Yau structure of dimension 1.

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For the proof we use the notation $\mathcal{A} = \amalg_{n+1} \underline{k}$ and $\mathcal{B} = \mathrm{Perf}_T$.

Proposition 5.15. *There is an S^1-equivariant equivalence of complexes*

$$\mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A}) \simeq k[1]$$

where the right-hand side is equipped with the trivial circle action.

Proof. The category $\mathcal{B}$ admits a semiorthogonal decomposition which makes it easy to compute its mixed complex: additivity of Hochschild homology implies that the functor $\mathrm{Perf}_T \to \amalg_n \mathrm{Perf}_k$ given by evaluation at the n vertices of the quiver induces a quasi-isomorphism $\mathrm{CC}_\bullet(\mathcal{B}) \simeq \mathrm{CC}_\bullet(\amalg_n \mathrm{Perf}_k) \simeq k^n$ of mixed complexes, where k is concentrated in degree 0 with trivial $k[\epsilon]$-module structure. Using this description, the map $\mathrm{CC}_\bullet(\mathcal{A}) \to \mathrm{CC}_\bullet(\mathcal{B})$ can be canonically identified with the $k[\epsilon]$-linear map $k^{n+1} \to k^n$ given by the matrix

$$\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & & 0 \\ & & & \ddots & \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In particular, we obtain an exact triangle

$$k \longrightarrow \mathrm{CC}_\bullet(\mathcal{A}) \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}) \longrightarrow k[1]$$

of mixed complexes which implies the claim. □

Remark 5.26. Proposition 5.15 implies that, up to rescaling, there is a unique class in $H^{-1}(\mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A}))$ which further has a canonical negative cyclic lift.

Proof of Theorem 5.14. The diagonal $\mathcal{A}$-bimodule $\mathcal{A}$ is given by

$$\mathcal{A}(i,j) \cong \begin{cases} k & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and its left dual $\mathcal{A}^!$ is

$$\mathcal{A}^!(i,j) \cong \begin{cases} k^* & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

as follows directly from unraveling the definition of the left dual before (2.4). We choose the identification

$$\xi : \mathcal{A}^! \longrightarrow \mathcal{A}$$

induced by $1^* \mapsto 1$ in each diagonal component. Let $F = f^{\mathrm{op}} \otimes f$ and consider the Quillen adjunction

$$F_! : \mathrm{Mod}_{\mathcal{A}^e} \longleftrightarrow \mathrm{Mod}_{\mathcal{B}^e} : F^*.$$

We form the diagram

$$\begin{array}{ccc} \mathcal{B}^! & \xrightarrow{c^!} & (F_!\mathcal{A})^! \\ & & \simeq \big\downarrow F_!\xi \\ & & F_!\mathcal{A} \xrightarrow{\ c\ } \mathcal{B} \end{array} \tag{5.27}$$

in the ∞-category $\mathcal{D}(\mathrm{Mod}_{\mathcal{B}^e})$. To construct a left Calabi–Yau structure on f it suffices to provide a zero homotopy of the composite which exhibits $\mathcal{B}^!$ as the fiber of c. Indeed, such a zero homotopy gives, by definition, a nonzero class in $H^{-1}(\mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A}))$ which, by Remark 5.26, has a canonical negative cyclic lift.

The envelope $J = j^{\mathrm{op}} \otimes j$ of the Yoneda embedding $j : T \to \mathcal{B}$ induces an equivalence of ∞-categories

$$J^* : \mathcal{D}(\mathrm{Mod}_{\mathcal{B}^e}) \longrightarrow \mathcal{D}(\mathrm{Mod}_{T^{\mathrm{e}}}).$$

It therefore suffices to construct a zero homotopy of the image of the diagram (5.27) in $\mathcal{D}(\mathrm{Mod}_T)$ under J^*. This diagram can be explicitly computed as follows. The functor $j^* f_!$ is given by

$$-\otimes_{\mathcal{A}} M : \mathrm{Mod}_{\mathcal{A}} \longrightarrow \mathrm{Mod}_T$$

where $M \in \mathrm{Mod}_T^{\mathcal{A}}$ can be described as

$$M = \oplus_i L_i.$$

The composite $J^* F_!$ is then given by

$$-\otimes_{\mathcal{A}^e} M^\vee \otimes_k M : \mathrm{Mod}_{\mathcal{A}^e} \longrightarrow \mathrm{Mod}_{T^{\mathrm{e}}}$$

with

$$M^\vee = \oplus_i L_i^\vee \in \mathrm{Mod}_{\mathcal{A}}^T$$

where $(-)\vee$ denotes the T-linear dual. The bimodule $J^* F_! \mathcal{A}$ admits the formula

$$M^\vee \otimes_{\mathcal{A}} M \cong \oplus_i L_i^\vee \otimes L_i$$

and the restricted counit map

$$J^*(c) : \oplus_i L_i^\vee \otimes L_i \longrightarrow T$$

is given by the sum over the evaluation maps. The standard cofibrant replacement of T as an T-bimodule is given by the complex

$$\oplus_i\, Te_i \otimes e_{i+1} T \longrightarrow \oplus_i Te_i \otimes e_i T$$

with $e_i \otimes e_{i+1} \mapsto \rho \otimes e_{i+1} - e_i \otimes \rho$ where ρ denotes the arrow in the quiver from i to $i+1$. Here and below, the summation index i runs from 0 to n where we declare $e_0 = e_{n+1} = 0$. In terms of this cofibrant replacement, the map $J^*(c)$ takes the explicit form

$$\begin{array}{ccc}
\oplus_i Te_i \otimes e_{i+1} T & \xrightarrow{\mathrm{id}} & \oplus_i Te_i \otimes e_{i+1} T \\
\downarrow & & \downarrow \\
\oplus_i (Te_i \otimes e_i T \oplus Te_{i+1} \otimes e_{i+1} T) & \xrightarrow{\alpha} & \oplus_i Te_i \otimes e_i T \\
\downarrow & & \downarrow \\
\underbrace{\oplus_i Te_{i+1} \otimes e_i T}_{J^* F_! \mathcal{A}} & \longrightarrow & \underbrace{0}_{J^* \mathcal{B}}
\end{array}$$

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where the left (respectively right) column describes the object $J^*F_!\mathcal{A}$ (respectively $J^*\mathcal{B}$). The image of diagram (5.27) in $\mathcal{D}(\mathrm{Mod}_{T^e})$, with $J^*F_!(\xi)$ left implicit, is induced by the strict short exact sequence of complexes of T-bimodules

$$\begin{array}{ccccc}
0 & \longrightarrow & \oplus_i Te_i \otimes e_{i+1}T & \xrightarrow{\mathrm{id}} & \oplus_i Te_i \otimes e_{i+1}T \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_i Te_i \otimes e_i T & \xrightarrow{\alpha^\vee} & \oplus_i (Te_i \otimes e_i T \oplus Te_{i+1} \otimes e_{i+1}T) & \xrightarrow{\alpha} & \oplus_i Te_i \otimes e_i T \\
\downarrow & & \downarrow & & \downarrow \\
\underbrace{\oplus_i Te_{i+1} \otimes e_i T}_{J^*\mathcal{B}^!} & \xrightarrow{\mathrm{id}} & \underbrace{\oplus_i Te_{i+1} \otimes e_i T}_{J^*F_!\mathcal{A}} & \longrightarrow & \underbrace{0}_{J^*\mathcal{B}}
\end{array}$$

where the columns correspond to the objects $J^*\mathcal{B}^!$, $J^*F_!\mathcal{A}$, and $J^*\mathcal{B}$, respectively. We may therefore conclude the argument by choosing the trivial zero homotopy to obtain the required class in $H^{-1}(\mathrm{CC}_\bullet(\mathcal{B},\mathcal{A}))$. $\square$

5.4 Symplectic geometry

The functors from Theorem 5.14 naturally fit into the algorithmic framework for Fukaya categories as developed in [Sei08]: choose points $z_1,\dots,z_n \in \mathbb{C}$ contained in the unit disk D, together with smooth paths α_i in D from 1 to each z_i whose interior does not intersect, called *vanishing paths*. We assume that the ordering is chosen so that the tangent directions to the paths $\alpha_1,\dots,\alpha_n$ at 1 are ordered counter-clockwise. Further, for each i, choose a loop γ_i in D based at 1 which runs counter-clockwise about z_i and whose interior does not intersect any of the paths α_i. The loops α_i freely generate the fundamental group of the disk D punctured at $\{z_i\}$. We specify the representation

$$\mu:\ \pi_1(D\backslash\{z_i\},1) \longrightarrow \mathrm{Aut}(\{0,1,\dots,n\}),\ \gamma_i \mapsto (0,i).$$

Using the Riemann existence theorem, we obtain a polynomial $p(z)$ so that the corresponding ramified cover

$$p:\mathbb{C}\to\mathbb{C} \tag{5.28}$$

has branch points $\{z_i\}$ and the monodromy representation of p for a suitable identification $p^{-1}(1)\cong\{x_0,\dots,x_n\}$ is μ.

For any symplectic Lefschetz fibration $q: X\to\mathbb{C}$ with a chosen basis of vanishing paths, Seidel [Sei08] has constructed an A_∞-category $\mathrm{Fuk}(q)$ which interpolates between the Fukaya categories of the total space X and the regular fiber $q^{-1}(1)$ in the following sense: there exist canonical functors

$$\mathrm{Fuk}(X) \xrightarrow{i} \mathrm{Fuk}(q) \xrightarrow{g} \mathrm{Fuk}(q^{-1}(1))$$

where i is fully faithful and g can be described explicitly in terms of a certain directed category construction. The category $\mathrm{Fuk}(q)$ is generated by objects which correspond to Lagrangian vanishing thimbles (equipped with extra data) and the functor g is given by associating to such a thimble its boundary.

In the context of (5.28), this translates to the following. The Fukaya category of the fiber $p^{-1}(1)=\{x_0,x_1,\dots,x_n\}$ is the perfect envelope of the free dg category on the set $x_0,x_1,\dots,x_n$ so that we have

$$\mathrm{Fuk}(p^{-1}(1)) \simeq \amalg_{n+1}\mathrm{Perf}_k.$$

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To each vanishing path α_i there is an associated vanishing thimble constructed as follows. For every i, the two lifts of the path α_i along p with starting point x_0 and x_i, respectively, meet at their endpoint which is the unique ramification point lying over the branch point z_i. The union of these two lifted paths forms a smooth path in $\mathbb{C}$ with boundary $\{x_0, x_i\}$, called the vanishing thimble associated with α_i. The category $\mathrm{Fuk}(p)$ is then defined as the directed subcategory of $\mathrm{Fuk}(p^{-1}(1))$ on the objects

$$x_0[1] \oplus x_1, x_0[1] \oplus x_2, \ldots, x_0[1] \oplus x_n$$

which correspond to the (graded) boundaries of the (graded) vanishing thimbles. We have $\mathrm{Fuk}(p) \simeq \mathrm{Perf}_S$. The functor

$$g : \mathrm{Fuk}(p) \longrightarrow \mathrm{Fuk}(p^{-1}(1))$$

is Morita dual to the functor in Theorem 5.14.

6. Calabi–Yau cospans

Let X and Y be oriented manifolds with boundaries $\partial X \cong S \amalg T$ and $\partial Y \cong T \amalg U$ together with choices of collared neighborhoods of T. Then the pushout $X \amalg_S Y$ is canonically an oriented manifold with boundary. The resulting composition law is the basic operation of oriented cobordism. In this section, we establish a noncommutative analog of this construction for functors of differential graded categories equipped with Calabi–Yau structures.

6.1 A noncommutative cotangent sequence

We expect that there is a cotangent formalism which puts the following discussion into a formal framework but we do not develop it here. Given a dg category $\mathcal{A}$, we propose to interpret the diagonal $\mathcal{A}$-bimodule $\mathcal{A}$ as a shifted noncommutative cotangent complex $L_{\mathcal{A}}[1]$. Further, given a functor $f : \mathcal{A} \to \mathcal{B}$, the exact sequence

$$\begin{array}{ccc} \mathrm{fib}(c) & \longrightarrow & F_!(\mathcal{A}) \\ \downarrow & & \downarrow{\scriptstyle c} \\ 0 & \longrightarrow & \mathcal{B} \end{array}$$

of $\mathcal{B}$-bimodules should be interpreted as a relative cotangent sequence.

$$\begin{array}{ccc} L_{\mathcal{B}/\mathcal{A}} & \longrightarrow & L_{\mathcal{A}}[1] \otimes_{\mathcal{A}} \mathcal{B} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & L_{\mathcal{B}}[1] \end{array}$$

The following result fits naturally into this context and can be interpreted as the exactness of the sequence

$$\begin{array}{ccc} L_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}' & \longrightarrow & L_{\mathcal{A}'} \otimes_{\mathcal{A}} \mathcal{B}' \oplus L_{\mathcal{B}} \otimes_{\mathcal{A}} \mathcal{B}' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & L_{\mathcal{B}'} \end{array}$$

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for a given pushout square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle g}\downarrow & \overset{h}{\searrow} & \downarrow{\scriptstyle i} \\ \mathcal{A}' & \xrightarrow[j]{} & \mathcal{B}' \end{array}$$

of dg categories. This is in complete analogy to the behavior of cotangent complexes with respect to pushouts in the context of commutative differential graded algebras. We emphasize, however, that the above discussion is of purely motivational character: while there do exist noncommutative versions of the cotangent complex for dg categories in the literature (cf. [Tab09] or [Lur11b]), they do not reproduce the above constructions.

Theorem 6.1. *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle g}\downarrow & \overset{h}{\searrow} & \downarrow{\scriptstyle i} \\ \mathcal{A}' & \xrightarrow[j]{} & \mathcal{B}' \end{array} \tag{6.1}$$

be a pushout square in the ∞*-category* $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}(k))$*. Then the corresponding square*

$$\begin{array}{ccc} H_!(\mathcal{A}) & \longrightarrow & I_!(\mathcal{B}) \\ \downarrow & & \downarrow \\ J_!(\mathcal{A}') & \longrightarrow & \mathcal{B}' \end{array} \tag{6.2}$$

in the ∞*-category of* $\mathcal{B}'$*-bimodules is a pushout square.*

Proof. We use the model structure on $\mathcal{C}at_{\mathrm{dg}}(k)$ to reduce the general case to a special pushout square for which we can prove the statement by an explicit calculation involving bar constructions. We may assume that the square (6.1) is a pushout in the category $\mathcal{C}at_{\mathrm{dg}}(k)$ with $\mathcal{A}$ cofibrant and f, g cofibrations so that the square is a homotopy pushout. Further, by the small object argument, we may assume that the functor g is a relative I-cell complex, where I is the set of generating cofibrations in $\mathcal{C}at_{\mathrm{dg}}(k)$. Given a composite of pushout squares

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{A}' & \longrightarrow & \mathcal{B}' \\ \downarrow & & \downarrow \\ \mathcal{A}'' & \longrightarrow & \mathcal{B}'' \end{array}$$

then if (6.2) is a pushout for the top and bottom squares, it is a pushout for the exterior square. This observation generalizes to transfinite compositions of pushout squares so that we may assume that the functor f in (6.1) is a pushout along a generating cofibration. In this situation, we have a composition of pushout squares of the form

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{B}' \end{array} \tag{6.3}$$

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where $\mathcal{X} \to \mathcal{Y}$ is one of the following morphisms:

(1) $\emptyset \to \mathcal{P}$ where $\mathcal{P}$ denotes the k-linear dg category with one object and endomorphism ring k;
(2) $\mathcal{S}^{n-1} \to \mathcal{D}^n$, $n \in \mathbb{Z}$, where:

- $\mathcal{S}^{n-1}$ denotes the k-linear dg category with two objects 1 and 2, freely generated by a morphism $s : 1 \to 2$ in degree $-(n-1)$ satisfying $d(s) = 0$;
- $\mathcal{D}^n$ denotes the k-linear dg category with two objects 1 and 2, freely generated by a morphism $r : 1 \to 2$ of degree $-n$ and a morphism $s : 1 \to 2$ of degree $-(n-1)$ satisfying $d(r) = s$;
- the functor is given by the apparent embedding of mapping complexes.

Note that if (6.2) is pushout (hence bicartesian) for the left-hand and the exterior squares in (6.3), then (6.2) is a pushout for the right-hand square in (6.3) so that we can finally assume that (6.1) is of the form

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{B}' \end{array}$$

where $\mathcal{X} \to \mathcal{Y}$ is one of (1) or (2) above. In case (1), it is immediate to verify that (6.2) is a pushout so that we are left with the following square.

$$\begin{array}{ccc} \mathcal{S}^{n-1} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{D}^{n-1} & \longrightarrow & \mathcal{B}' \end{array}$$

In this case, the morphism complex between objects x, y in $\mathcal{B}'$ can be described explicitly as

$$\mathcal{B}'(x, y) = \bigoplus_{n \geqslant 0} \mathcal{B}(x, x_1) \otimes kr \otimes \mathcal{B}(x_2, x_1) \otimes kr \otimes \cdots \otimes \mathcal{B}(x_2, y)$$

where n copies of kr appear in the nth summand, $x_i = f(i)$, and the differential is given by the Leibniz rule where, upon replacing r by $d(r) = f(s)$, we also compose with the neighboring morphisms in $\mathcal{B}$ so that the level is decreased from n to $n-1$. The square (6.2) being pushout is equivalent to the exactness of the sequence

$$\begin{array}{ccc} H_!(\mathcal{S}^{n-1}) & \longrightarrow & J_!(\mathcal{D}^n) \oplus I_!(\mathcal{B}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}' \end{array}$$

which can be rewritten as follows.

$$\begin{array}{ccc} \mathcal{B} \otimes^L_{\mathcal{S}^{n-1}} \mathcal{B} & \longrightarrow & \mathcal{B} \otimes^L_{\mathcal{D}^n} \mathcal{B} \oplus \mathcal{B} \otimes_{\mathcal{B}'} \mathcal{B} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B} \end{array}$$

All terms of this sequence can be computed explicitly in terms of two-sided bar resolutions of the various diagonal bimodules. To show that the resulting sequence is exact it suffices to show

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that, for every pair of objects (x, y) of $\mathcal{B}'$, the k-linear complex given by the totalization of the evaluation of the sequence at (x, y) is contractible. This can be shown by filtering the totalization by the number of copies of r's and providing explicit contracting homotopies of the associated graded complexes. $\square$

6.2 Composition of Calabi–Yau cospans

Given *cospans* of differential graded categories

$$e \amalg f : \mathcal{A} \amalg \mathcal{B} \longrightarrow \mathcal{X}$$

and

$$g \amalg h : \mathcal{B} \amalg \mathcal{C} \longrightarrow \mathcal{Y},$$

we form a coherent diagram

$$\begin{array}{ccccc} & & & & \mathcal{A} \\ & & & & \downarrow{\scriptstyle e} \\ & & \mathcal{B} & \xrightarrow{f} & \mathcal{X} \\ & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{C} & \xrightarrow[h]{} & \mathcal{Y} & \xrightarrow[j]{} & \mathcal{X} \amalg_{\mathcal{B}} \mathcal{Y} \end{array} \tag{6.4}$$

in $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}(k))$ where the bottom right square is a pushout. We call the functor

$$ie \amalg jh : \mathcal{A} \amalg \mathcal{C} \longrightarrow \mathcal{X} \amalg_{\mathcal{B}} \mathcal{Y}$$

the *composite of the cospans* $e \amalg f$ *and* $g \amalg h$. Applying $\mathrm{CC}_\bullet(-)$ to (6.4), we obtain a coherent S^1-equivariant diagram of complexes

$$\begin{array}{ccccc} F & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B})[-1] & \xrightarrow{\delta_A} & \mathrm{CC}_\bullet(\mathcal{A}) \\ \downarrow & & {\scriptstyle -\delta_B}\downarrow & & \downarrow \\ \mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C})[-1] & \xrightarrow{\delta_{\mathcal{B}}} & \mathrm{CC}_\bullet(\mathcal{B}) & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{X}) \\ {\scriptstyle -\delta_{\mathcal{C}}}\downarrow & & \downarrow & & \downarrow \\ \mathrm{CC}_\bullet(\mathcal{C}) & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{Y}) & \longrightarrow & \mathrm{CC}_\bullet(\mathcal{X} \amalg_{\mathcal{B}} \mathcal{Y}) \end{array} \tag{6.5}$$

in which all squares, with the possible exception of the bottom right square, are bicartesian. We give a more detailed description of (6.5): by definition, we have cofiber sequences

$$\mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B})[-1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{A} \amalg \mathcal{B}) \longrightarrow \mathrm{CC}_\bullet(\mathcal{X})$$

and

$$\mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C})[-1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B} \amalg \mathcal{C}) \longrightarrow \mathrm{CC}_\bullet(\mathcal{Y}).$$

Inverting the natural equivalences $\mathrm{CC}_\bullet(\mathcal{A}) \oplus \mathrm{CC}_\bullet(\mathcal{B}) \simeq \mathrm{CC}_\bullet(\mathcal{A} \amalg \mathcal{B})$ and $\mathrm{CC}_\bullet(\mathcal{B}) \oplus \mathrm{CC}_\bullet(\mathcal{C}) \simeq \mathrm{CC}_\bullet(\mathcal{B} \amalg \mathcal{C})$ and projecting to $\mathrm{CC}_\bullet(\mathcal{B})$, we obtain maps

$$\delta_{\mathcal{B}} : \mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C})[-1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B})$$

and

$$\delta'_{\mathcal{B}} : \mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B})[-1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}).$$

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Defining F to be the fiber (or $F[1]$ to be the cofiber) of the difference $\delta_{\mathcal{B}} - \delta'_{\mathcal{B}}$, we obtain a cofiber sequence

$$\mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C})[-1] \oplus \mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B})[-1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}) \longrightarrow F[1]$$

and an equivalence $F[1] \simeq \mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B}) \times_{\mathrm{CC}_\bullet(\mathcal{B})[1]} \mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C})$. By construction, the compositions $\mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C})[-1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}) \longrightarrow \mathrm{CC}_\bullet(\mathcal{Y})$ and $\mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B})[-1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}) \longrightarrow \mathrm{CC}_\bullet(\mathcal{X})$ are endowed with null-homotopies, and so the composition

$$\mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C})[-1] \oplus \mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B})[-1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}) \longrightarrow \mathrm{CC}_\bullet(\mathcal{X} \amalg_{\mathcal{B}} \mathcal{Y})$$

is endowed with a null-homotopy. We therefore obtain an induced S^1-equivariant morphism

$$\chi : \ \mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B}) \times_{\mathrm{CC}_\bullet(\mathcal{B})[1]} \mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C}) \simeq F[1] \longrightarrow \mathrm{CC}_\bullet(\mathcal{X} \amalg_{\mathcal{B}} \mathcal{Y})$$

which corresponds to the outer rectangle in (6.5). The following result allows us to transport left Calabi–Yau structures along compositions of cospans.

Theorem 6.2. *Let*

$$e \amalg f : \mathcal{A} \amalg \mathcal{B} \longrightarrow \mathcal{X}$$

and

$$g \amalg h : \mathcal{B} \amalg \mathcal{C} \longrightarrow \mathcal{Y},$$

be functors of smooth dg categories. Let

$$\sigma \in \mathrm{CC}_\bullet(\mathcal{X}, \mathcal{A} \amalg \mathcal{B})^{S^1} \times_{\mathrm{CC}_\bullet(\mathcal{B})^{S^1}[1]} \mathrm{CC}_\bullet(\mathcal{Y}, \mathcal{B} \amalg \mathcal{C})^{S^1}$$

so that the projections $\pi_1(\sigma)$ and $\pi_2(\sigma)$ define left Calabi–Yau structures on $e \amalg f$ and $g \amalg h$, respectively. Then $\chi(\sigma)$ defines a left Calabi–Yau structure on the composite $ie \amalg jh$.

Proof. The Calabi–Yau structure on $e \amalg f$ induces an identification of exact sequences

$$\begin{array}{ccccc}
\mathcal{X}^! & \longrightarrow & (E \amalg F)_!(\mathcal{A} \amalg \mathcal{B})^! & \longrightarrow & \mathrm{cof}_{\mathcal{X}} \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{fib}_{\mathcal{X}} & \longrightarrow & (E \amalg F)_!(\mathcal{A} \amalg \mathcal{B}) & \longrightarrow & \mathcal{X}
\end{array}$$

of $\mathcal{X}$-bimodules. We can reinterpret this as an equivalence between the following bicartesian squares.

$$\begin{array}{ccc}
\mathrm{fib}_{\mathcal{X}} & \longrightarrow & E_!\mathcal{A} \\
\downarrow & & \downarrow \\
F_!\mathcal{B} & \longrightarrow & \mathcal{X}
\end{array}$$

and

$$\begin{array}{ccc}
(\mathcal{X})^! & \longrightarrow & (E_!\mathcal{A})^! \\
\downarrow & & \downarrow \\
(F_!\mathcal{B})^! & \longrightarrow & \mathrm{cof}_{\mathcal{X}}
\end{array}$$

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An analogous statement holds for the Calabi–Yau structure on $g \amalg h$. Both Calabi–Yau structures combined yield an equivalence of the diagrams

$$\begin{array}{ccccc} & & I_!(\mathrm{fib}_{\mathcal{X}}) & \longrightarrow & I_!(E_!\mathcal{A}) \\ & & \downarrow & & \downarrow \\ J_!\,\mathrm{fib}_{\mathcal{Y}} & \longrightarrow & M_!\mathcal{B} & \longrightarrow & I_!\mathcal{X} \\ \downarrow & & \downarrow & & \\ J_!(H_!\mathcal{C}) & \longrightarrow & J_!\mathcal{Y} & & \end{array} \tag{6.6}$$

and

$$\begin{array}{ccccc} & & I_!(\mathcal{X}^!) & \longrightarrow & I_!(E_!\mathcal{A}^!) \\ & & \downarrow & & \downarrow \\ J_!(\mathcal{Y}^!) & \longrightarrow & M_!(\mathcal{B}^!) & \longrightarrow & I_!\,\mathrm{cof}_{\mathcal{X}} \\ \downarrow & & \downarrow & & \\ J_!(H_!\mathcal{C}^!) & \longrightarrow & J_!\,\mathrm{cof}_{\mathcal{Y}} & & \end{array} \tag{6.7}$$

of $\mathcal{Z}$-bimodules, where $\mathcal{Z} = \mathcal{X} \amalg_{\mathcal{B}} \mathcal{Y}$ and $M_! \simeq I_!F_! \simeq J_!G_!$. By Theorem 6.1, we have a pushout square.

$$\begin{array}{ccc} M_!\mathcal{B} & \longrightarrow & I_!\mathcal{X} \\ \downarrow & & \downarrow \\ J_!\mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

It follows that we may complete (6.6) and (6.7) by forming pullbacks and pushouts to obtain, an equivalence between the following diagrams.

$$\begin{array}{ccccc} \mathrm{fib}_{\mathcal{Z}} & \longrightarrow & I_!(\mathrm{fib}_{\mathcal{X}}) & \longrightarrow & I_!(E_!\mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ J_!\,\mathrm{fib}_{\mathcal{Y}} & \longrightarrow & M_!\mathcal{B} & \longrightarrow & I_!\mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ J_!(H_!\mathcal{C}) & \longrightarrow & J_!\mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{Z}^! & \longrightarrow & I_!(\mathcal{X}^!) & \longrightarrow & I_!(E_!\mathcal{A}^!) \\ \downarrow & & \downarrow & & \downarrow \\ J_!(\mathcal{Y}^!) & \longrightarrow & M_!(\mathcal{B}^!) & \longrightarrow & I_!\,\mathrm{cof}_{\mathcal{X}} \\ \downarrow & & \downarrow & & \downarrow \\ J_!(H_!\mathcal{C}^!) & \longrightarrow & J_!\,\mathrm{cof}_{\mathcal{Y}} & \longrightarrow & \mathrm{cof}_{\mathcal{Z}} \end{array}$$

Here, we use Lemma 4.2 to obtain that, for a dg functor $r : \mathcal{S} \to \mathcal{T}$ with $\mathcal{S}$ smooth, we have $R_!(\mathcal{S}^!) \simeq (R_!\mathcal{S})^!$. We restrict this equivalence to the exterior bicartesian square to obtain the desired equivalence showing the nondegeneracy of the boundary structure on $ie \amalg jh$. □

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7. Applications

7.1 Localization

Let $f : \mathcal{A} \to \mathcal{B}$ be a functor of dg categories which carries a relative Calabi–Yau structure. Then the choice of negative cocycle on $\mathcal{A}$ also defines a relative Calabi–Yau structure on the zero functor $\mathcal{A} \to 0$. The pushout square

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}/\mathcal{A} \end{array}$$

in $\mathrm{L_{mo}}(\mathcal{C}at_{\mathrm{dg}}(k))$ can be interpreted as a composition of the Calabi–Yau cospans

$$0 \amalg \mathcal{A} \longrightarrow \mathcal{B}$$

and

$$\mathcal{A} \amalg 0 \longrightarrow 0.$$

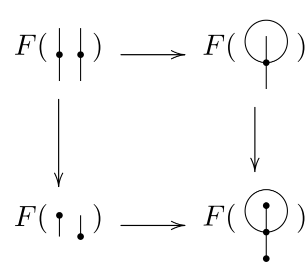
Therefore, the n-dimensional left Calabi–Yau structure on f induces a canonical morphism

$$k[n] \longrightarrow \mathrm{CC}_\bullet(\mathcal{B}, \mathcal{A})^{S^1}. \tag{7.1}$$

The following is therefore an immediate corollary of Theorem 6.2.

Corollary 7.1. *Let $f : \mathcal{A} \to \mathcal{B}$ be a functor of dg categories which carries a left Calabi–Yau structure. Then the negative cyclic cocycle (7.1) defines a left Calabi–Yau structure on the cofiber $\mathcal{B}/\mathcal{A}$.*

Example 7.2. We have a pushout square of topological $\mathbb{Z}$-graded Fukaya categories (cf. § 7.2 below)



in $\mathrm{L_{mo}}(\mathcal{C}at_{\mathrm{dg}}(k))$ which is equivalent to the pushout square

$$\begin{array}{ccc} \mathrm{Coh}(\{0\} \amalg \{\infty\}) & \longrightarrow & \mathrm{Coh}(\mathbb{P}^1) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Coh}(\mathbb{A}^1 \backslash \{0\}) \end{array}$$

where we write $\mathrm{Coh}(-)$ for the derived dg category $D^b(\mathrm{coh}(-))$. The cofiber

$$\mathrm{Coh}(\mathbb{A}^1 \backslash \{0\})$$

carries a natural induced left Calabi–Yau structure which, in this example, can be explicitly verified.

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Example 7.3. Let X be a compact oriented manifold with boundary ∂X. Then we have a pushout square

$$\begin{array}{ccc} \mathcal{L}(\partial X) & \longrightarrow & \mathcal{L}(X) \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & \mathcal{L}(X') \end{array}$$

where X' is the closed oriented manifold given by the complement of the union of those connected components of X which have nonempty boundary. The absolute left Calabi–Yau structure on $\mathcal{L}(X')$ corresponding to the orientation of X' agrees with the one implied by Corollary 7.1.

7.2 Topological Fukaya categories

According to Kontsevich [Kon09], the Fukaya category of a Stein manifold, equipped with suitable extra data, can be described as the global sections of a cosheaf of constructible dg categories on a Lagrangian spine. As stressed in [Kon09], to obtain a version of the Fukaya category of finite type, it is crucial that the construction involves a cosheaf and not a sheaf: by a formal argument, a finite colimit of finite type dg categories is of finite type while finite limits do *not* inherit the finite type property.

Various constructions have been given for the two-dimensional case of a noncompact Riemann surface. We recall the context of [Dyc17] where the topological Fukaya category of a stable marked surface (S, M) is defined (based on the ideas of [DK18]). To obtain an intrinsically defined $\mathbb{Z}$-graded version of the Fukaya category additional structure on the surface must be given: we assume that the punctured surface $S\backslash M$ comes equipped with a framing. The topological Fukaya category of (S, M) can then be constructed combinatorially via the formalism of paracyclic 2-Segal objects.

7.2.1 *State sums for coparacyclic objects.* The paracyclic category Λ_∞ has objects $\langle n\rangle$, $n \geqslant 0$, labeled by the natural numbers. A morphism from $\langle m\rangle$ to $\langle n\rangle$ is a map $\varphi : \mathbb{Z} \to \mathbb{Z}$ preserving $\leqslant$ and satisfying $\varphi(z + m + 1) = \varphi(z) + n + 1$.

Let (S, M) be a stable marked surface with framing on $S\backslash M$ and let $\Gamma \subset S\backslash M$ be a spanning graph. As explained in detail in [DK15], the framing defines combinatorial data on the graph Γ which can be formulated as a functor

$$\delta : I(\Gamma)^{\mathrm{op}} \longrightarrow \Lambda_\infty$$

where $I(\Gamma)$ denotes the incidence category of the graph (cf. [DK15, Definition IV.2]). Given a coparacyclic object $X : \mathrm{N}(\Lambda_\infty) \longrightarrow \mathcal{C}$ with values in an ∞-category $\mathcal{C}$ with colimits, we can then define the *state sum of X on* Γ

$$X(\Gamma) = \operatorname*{colim}_{\mathrm{N}(I(\Gamma)^{\mathrm{op}})} X \circ \delta.$$

The object X is called 2-*Segal* if X maps edge contractions to equivalences in $\mathcal{C}$. As shown in [DK15], any coparacyclic 2-Segal object defines by means of the state sum an invariant $X(S, M)$ of the framed surface (S, M) equipped with an action of the framed mapping class group.

7.2.2 *Left Calabi–Yau structures on topological Fukaya categories of framed surfaces.* We apply the above state sum construction to the coparacyclic 2-Segal dg category

$$F : \mathrm{N}(\Lambda_\infty) \longrightarrow \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}(k)), \quad \langle n\rangle \mapsto \mathrm{MF}^{\mathbb{Z}}(k[z], z^{n+1})$$

where we refer to [Dyc17] for details. The resulting dg category $F(S,M)$ is called the *topological Fukaya category of* (S,M). For every component of the punctured boundary $\partial S \backslash M$, we obtain a functor $\underline{k} \longrightarrow F(S,M)$. Summing over these functors, we obtain the *boundary functor*

$$\coprod_{\pi_0(\partial S\backslash M)} \underline{k} \longrightarrow F(S,M).$$

Example 7.4. Let $S \subset \mathbb{C}$ be the closed unit disk equipped with the standard framing. Let $M \subset \partial S$ be the subset of $(n+1)$st roots of unity. Then the boundary functor can be identified with the functor f from Theorem 5.14. In particular, it carries a left Calabi–Yau structure of dimension 1.

Example 7.5. Let S be the 2-sphere with M consisting of two points. Suppose that $S\backslash M$ is equipped with a framing with winding number $n \in \mathbb{Z}$. Then the topological Fukaya category $F(S,M)$ is Morita equivalent to the differential graded algebra $k[t,t^{-1}]$ with $|t| = 2n$ and zero differential.

Theorem 7.2. *Let (S,M) be a stable marked surface with framing on $S\backslash M$. Then the boundary functor*

$$\coprod_{\pi_0(\partial S\backslash M)} \underline{k} \longrightarrow F(S,M)$$

carries a left Calabi–Yau structure of dimension 1.

Proof. We choose a spanning graph $\Gamma \subset S\backslash M$ with one vertex. Then the topological Fukaya category $F(S,M)$ can be described as a pushout

$$\begin{array}{ccc} & & \amalg_X \underline{k} \\ & & \downarrow \\ \amalg_Y \underline{k} & \longrightarrow & \mathrm{Perf}_T \\ {\scriptstyle g}\downarrow & & \downarrow \\ \amalg_Z \mathrm{Perf}_k & \longrightarrow & F(S,M) \end{array}$$

where X is the set of external half-edges of Γ, Z the set of loops, and Y the set of half-edges appearing in loops. The functor

$$f : \coprod_X \underline{k} \amalg \coprod_Y \underline{k} \longrightarrow \mathrm{Perf}_T$$

is the functor from Theorem 5.14. The component of g corresponding to a loop l in Γ is given by the map

$$\amalg_Y \underline{k} \longrightarrow \mathrm{Perf}_k \tag{7.6}$$

which maps the objects corresponding to the halfedges comprising l to k and $k[2p]$, respectively, and all other objects to 0. The functor f carries a left Calabi–Yau structure by Theorem 5.14. A similar computation shows that g carries a left Calabi–Yau structure which is compatible with the one on f. Therefore, Theorem 6.2 implies the result. □

Remark 7.7. Let (S,M) be as in Theorem 7.2 and assume in addition that S has no boundary. Then the topological Fukaya category $F(S,M)$ carries an absolute left Calabi–Yau structure. This was already stated without proof in [Kon09].

Remark 7.8. Note that Theorem 7.2 does not address the question whether the left Calabi–Yau structure is canonical.

Example 7.9. Let S be a torus with M consisting of one marked point and with the standard framing. Then we have a Morita equivalence

$$F(S, M) \simeq \mathrm{Coh}(X)$$

where X is a nodal cubic curve in $\mathbb{P}^2$. Since $\partial S = \emptyset$, Theorem 7.2 provides an absolute left Calabi–Yau structure on $\mathrm{Coh}(X)$, recovering a special instance of § 5.2.

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