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On the existence of a nodal solution for p-Laplacian equations depending on the gradient

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In the present paper we deal with a quasi-linear elliptic equation depending on a sublinear nonlinearity involving the gradient. We prove the existence of a nontrivial nodal solution employing the theory of invariant sets of descending flow together with sub-supersolution techniques, gradient regularity arguments, strong comparison principle for the p-Laplace operator. The same conclusion is obtained for an eigenvalue problem under a different set of assumptions.

Keywords: p-Laplace operator; gradient dependence; nodal solution; gradient flow

2020 *Mathematics Subject Classification:* 35J60; 35J92

1. Introduction

In the present paper we study the following quasilinear problem

$$\begin{cases} -\Delta_p u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function fulfilling suitable growth conditions at zero and at infinity.

Problem (P) appears in connection with the study of non-Newtonian fluids, where p is related to the characteristics of the medium (dilatant for $p > 2$, pseudoplastic for $p < 2$). The forcing term f is a convection type term, i.e. it depends on the gradient of the unknown function. The dependence on the gradient in the nonlinearity does not allow to apply in a straightforward way variational methods to find solutions of (P). However, the existence of constant sign solutions for (P) has been obtained by means of topological degree, method of sub-supersolutions, fixed point theory and approximation techniques (see for instance [**1**, **7**, **12**, **20**, **21**] and the references therein). When the source term does not depend on the gradient, the study of sign changing solutions for semilinear and quasilinear elliptic problems has been addressed in a number of papers. If $p = 2$ and $f = f(u)$ is superlinear and subcritical, the existence of a nodal solution (together with a positive and a negative one) has been proved for instance in [**6**] employing Morse theory and the strong maximum principle, or in [11] using topological degree technique, assuming also that $f'(0) < \lambda_1$ (being λ_1 the first eigenvalue of the negative Laplacian). In [11], the sublinear case is also addressed and under the condition $f'(0) > \lambda_2$ (being λ_2 the second eigenvalue of the negative Laplacian) the existence of the biggest negative solution, the smallest positive solution and of a nontrivial solution in between (thus nodal) is achieved. The theory of invariant sets of descending flow defined by a pseudogradient vector field is deeply investigated in [19] to localize nodal solutions of a superlinear semilinear problem in different invariant sets. For $p \neq 2$ several contributions extend the quoted results to a nonlinear setting: the existence of nodal solutions is proved in [3–5, 22, 24] where the theory of invariant sets of descending flow is exploited under various assumptions on f both in the superlinear and in the sublinear case, or in [8] by variational and sub–supersolution techniques.

The existence of nodal solutions when f depends on the gradient is still an almost unexplored issue. Motivated by [13], where the existence of a positive and a negative solution (actually extremal) for problem (P) has been proved, the following questions arise naturally.

Question 1.1.

1. Does problem (P) admit a nodal solution for $p > 1$?
2. Beside the smallest positive and the biggest negative solution does there exist a non-trivial nodal solution for problem (P) in between?

The first question has been partially solved in [18] and [15] where the existence of a nodal solution in the presence of a nonlinearity depending on the gradient has been addressed in the case $p = 2$. In [18], the authors assume f to be a superlinear function, locally Lipschitz with respect to both the second and the third variable in a neighbourhood of zero, with some further assumptions on the gradient variable. Using the Nehari method, they obtain a sign changing solution as the limit of a sequence of 'approximated' nodal functions. In [15] the sublinear case is studied and the existence of a nodal solution is ensured by suitable growth conditions at zero in the real (second) variable via the theory of invariant sets of descending flow.

In the present paper, we completely solve the first question above for any $1 < p < \infty$. In the first part of the paper we extend the results of [15] pointing out some difficulties which arise from the quasilinear setting which prevent a straightforward generalization of the conclusions of [15]. Combining the gradient flow theory with some essential tools as a gradient regularity result and strong comparison principle for the p-Laplacian we prove the existence of a nodal solution for a parametrized problem with variational structure. An iteration procedure allows to create a sequence of sign changing solutions converging to a non-trivial nodal solution.

Together with the conclusion of [13] (see also [14]), where the existence of the smallest positive solution and of the biggest negative solution for (P) was proved via sub-super solution methods and fixed point arguments, we obtain a multiplicity theorem under very natural and verifiable assumptions. It still remains an open question whether the nodal solution lies in between (see remark 5.1). In the last part of the paper we give another contribution to the first question, suggesting a different set of hypotheses to prove the existence of sign changing solutions for an eigenvalue problem.

We believe that this work represents the first step in the search of nodal solutions for quasilinear problems depending on the gradient.

Denote by $\|\cdot\|$, the standard norm in $W_0^{1,p}(\Omega)$, i.e. $\|u\| = (\int_\Omega |\nabla u|^p \, \mathrm{d}x)^{1/p}$, and by $\|\cdot\|_q$, $\|\cdot\|_\infty$ the classical norms in $L^q(\Omega)$ and in $L^\infty(\Omega)$ respectively, i.e. $\|u\|_q = (\int_\Omega |u|^q \, \mathrm{d}x)^{1/q}$ and $\|u\|_\infty = \mathrm{supess}_\Omega |u|$. Let λ_1 be the first eigenvalue of the negative p-Laplacian operator on $W_0^{1,p}(\Omega)$, with first positive eigenfunction φ_1 satisfying $\|\varphi_1\| = 1$. The following variational characterization holds

$$\lambda_1 = \inf\left\{ \frac{\|u\|^p}{\|u\|_p^p} : \ u \in W_0^{1,p}(\Omega), u \neq 0 \right\}.$$

It is well known that the cone of nonnegative functions

$$C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u \geqslant 0 \text{ in } \Omega\}$$

has a nonempty interior in the Banach space $C_0^1(\overline{\Omega})$ given by

$$\mathrm{int}(C_0^1(\overline{\Omega})_+) = \left\{ u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega \right\},$$

where ν stands for the outward normal unit vector to $\partial\Omega$.

Our first set of assumptions is:

(f_1) there exist positive constants k_0, θ_0, θ_1 with $\theta_0 + \theta_1 \lambda_1^{1/p'} < \lambda_1$ such that

$$|f(x, s, \xi)| \leqslant k_0 + \theta_0 |s|^{p-1} + \theta_1 |\xi|^{p-1}$$

for all $x \in \Omega$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$;

(f_2) for every $M > 0$ there exists a constant $\eta_M > \lambda_1$ such that

$$\liminf_{s \to 0} \frac{f(x, s, \xi)}{|s|^{p-2} s} \geqslant \eta_M$$

uniformly for all $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leqslant M$;

(f_3) for every $M > 0$ there exists a constant $\zeta_M > 0$ such that

$$\limsup_{s \to 0} \frac{f(x, s, \xi)}{|s|^{p-2} s} \leqslant \zeta_M$$

uniformly for all $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leqslant M$;

(f_4) for every $M > 0$ there exists a constant $m_M > 0$ such that

$$s \to f(x, s, \xi) + m_M |s|^{p-2} s$$

is increasing for all $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leqslant M$.

Under such assumptions we prove the following.

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ a continuous function satisfying $(f_1) - (f_4)$. Then, problem (P) has a nodal solution in $C^1_0(\overline{\Omega})$.*

Combining such conclusion with [13, Theorem 1.3, Corollary 1.1] we can state the following multiplicity result:

Corollary 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ a continuous function satisfying $(f_1) - (f_4)$. Then, problem (P) has the smallest positive solution $u_P \in \mathrm{int}(C^1_0(\overline{\Omega})_+)$, the biggest negative solution $u_N \in -\mathrm{int}(C^1_0(\overline{\Omega})_+)$ and a nodal solution $\tilde{u} \in C^1_0(\overline{\Omega})$.*

In the last part of the paper, still exploiting the same iterative approach, we deduce the existence of a nodal solution for the quasilinear elliptic eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Let us introduce the following assumptions:

$(\tilde{f}_1)$

$$\lim_{(s,\xi)\to\infty} \frac{f(x, s, \xi)}{|s|^{p-1} + |\xi|^{p-1}} = 0$$

uniformly for all $x \in \Omega$;

$(\tilde{f}_2)$ for every $M > 0$,

$$\lim_{s\to 0} \frac{f(x, s, \xi)}{|s|^{p-1}} = 0$$

uniformly for all $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leqslant M$;

$(\tilde{f}_3)$ for every $M > 0$,

$$\lim_{s\to \infty} \frac{f(x, s, \xi)}{|s|^{p-1}} = 0$$

uniformly for all $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leqslant M$;

$(\tilde{f}_4)$ for every $M > 0$, there exists $R_M > 0$ such that $sf(x, s, \xi) > 0$ for all $x \in \Omega$, $|s| > R_M$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leqslant M$;

$(\tilde{f}_5)$ there exist $s^- < 0 < s^+$ such that

$$\inf_{(x,\xi)\in\Omega\times\mathbb{R}^N} F(x, s^\pm, \xi) > 0,$$

$$\text{where } F(x, s, \xi) = \int_0^s f(x, t, \xi)dt.$$

Our second result states the following.

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THEOREM 1.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ a continuous function satisfying $(\tilde{f}_1) - (\tilde{f}_5)$. Then, there exists $\tilde{\lambda}$ such that for each $\lambda > \tilde{\lambda}$, problem (P_λ) has a nodal solution in $C^1_0(\overline{\Omega})$.*

The plan of the paper is the following: in Section 2 we introduce some common preliminaries. Sections 3 and 4 are devoted to prove our main theorems. Finally, in Section 5 some open questions and final remarks are discussed.

2. Preliminaries

In this section we collect some common preliminaries which will be useful in our study. We introduce the following quasilinear problem

$$\begin{cases} -\Delta_p u = g(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\tilde{P}$}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$, $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function such that $g(x, 0, \xi) = 0$ for all x, ξ and

($\mathcal{H}$) there exist positive constants k_0, θ_0, θ_1 with $\theta_0 + \theta_1 \lambda_1^{1/p'} < \lambda_1$ such that

$$|g(x, s, \xi)| \leqslant k_0 + \theta_0 |s|^{p-1} + \theta_1 |\xi|^{p-1}$$

for all $x \in \Omega$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$.

For every $w \in C^1_0(\overline{\Omega})$, let us also consider the parametrized Dirichlet problem

$$\begin{cases} -\Delta_p u = g(x, u, \nabla w) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\tilde{P}_w$}$$

Notice that since $w \in C^1_0(\overline{\Omega})$, classical regularity results implies that each solution u of (P_w) is in $L^\infty(\Omega)$, thus in $C^1_0(\overline{\Omega})$ (see [**16**, **17**]).

Moreover, from $g(x, 0, \xi) = 0$ for all x, ξ we observe that the zero function is a solution of both $(\tilde{P})$ and $(\tilde{P}_w)$ for each $w \in C^1_0(\overline{\Omega})$.

Proposition 2.1 below proves an a priori uniform boundedness which will be crucial for our purposes. It makes use of the following gradient regularity result ([**9**, Theorem 4.3]):

LEMMA 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, and let $u \in W^{1,p}_0(\Omega)$, $1 < p < N$, be a weak solution of the problem*

$$\begin{cases} -\Delta_p u = h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $h \in L^q(\Omega)$, $q \geqslant (p^*)'$.

(i) If $q < N$, then

$$\|\nabla u\|_{q^*(p-1)} \leqslant C \|h\|_q^{\frac{1}{p-1}}.$$

(ii) If $q = N$, then

$$\|\nabla u\|_r \leqslant C\|h\|_q^{\frac{1}{p-1}} \text{ for any } r < \infty.$$

(iii) If $q > N$, then

$$\|\nabla u\|_\infty \leqslant C\|h\|_q^{\frac{1}{p-1}}.$$

In what stated above, C is a constant that depends only on p, N, q.

PROPOSITION 2.1. *Assume $(\mathcal{H})$. Then, for every $u_0 \in C_0^1(\overline{\Omega})$, there exists $\alpha \in]0,1[$ and a positive constant M depending on $k_0, \theta_0, \theta_1, \|u_0\|$ such that if u_n is a solution of $(P_{u_{n-1}})$ one has*

$$\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant M \quad \text{for every } n \in \mathbb{N}.$$

Proof. Let us fix $u_0 \in C_0^1(\overline{\Omega})$. We first prove that if u_n is a solution of $(\tilde{P}_{u_{n-1}})$, then $\|u_n\| \leqslant M_0$ for some constant $M_0 = M_0(\|u_0\|)$ independent on n. Indeed, acting with u_n as test function in $(\tilde{P}_{u_{n-1}})$,

$$\begin{aligned}\|u_n\|^p &\leqslant \int_\Omega |g(x, u_n, \nabla u_{n-1})||u_n| \,\mathrm{d}x \\ &\leqslant k_0\|u_n\|_1 + \theta_0\|u_n\|_p^p + \theta_1\|u_{n-1}\|^{p-1}\|u_n\|_p \\ &\leqslant k_0|\Omega|^{1/p}\lambda_1^{-1/p}\|u_n\| + \theta_0\lambda_1^{-1}\|u_n\|^p + \theta_1\lambda_1^{-1/p}\|u_{n-1}\|^{p-1}\|u_n\|\end{aligned}$$

which implies

$$\left(1 - \theta_0\lambda_1^{-1}\right)\|u_n\|^{p-1} \leqslant k_0|\Omega|^{1/p}\lambda_1^{-1/p} + \theta_1\lambda_1^{-1/p}\|u_{n-1}\|^{p-1}.$$

Denote by

$$\gamma := \frac{k_0|\Omega|^{1/p}\lambda_1^{-1/p}}{1 - \theta_0\lambda_1^{-1}}, \qquad \delta := \frac{\theta_1\lambda_1^{-1/p}}{1 - \theta_0\lambda_1^{-1}}.$$

Notice that $\gamma > 0$ and $0 < \delta < 1$. Thus, using the above notation, we can rewrite

$$\|u_n\|^{p-1} \leqslant \gamma + \delta\|u_{n-1}\|^{p-1}.$$

Iterating this inequality, we obtain

$$\begin{aligned}\|u_n\|^{p-1} &\leqslant \gamma\sum_{i=0}^{n-1}\delta^i + \delta^n\|u_0\|^{p-1} \\ &\leqslant \frac{\gamma}{1-\delta} + \|u_0\|^{p-1},\end{aligned}$$

which means that there exists a constant M_0 (depending on $\|u_0\|$) such that

$$\|u_n\| \leqslant M_0 \quad \text{for every } n \in \mathbb{N}.$$

Denote $h_n(x) = g(x, u_n(x), \nabla u_{n-1}(x))$ and fix q with $(p^*)' \leqslant q \leqslant p'$ such that $\frac{N}{q} \notin \mathbb{N}$. Using the fact that $|u_n|^{p-1}, |\nabla u_{n-1}|^{p-1} \in L^q(\Omega)$ (recall that $u_n \in L^\infty(\Omega)$), and

$(\mathcal{H})$, we deduce that $h_n \in L^q(\Omega)$ and its norm $\|h_n\|_q$ can be estimated by a constant which does not depend on n:

$$\|h_n\|_q^q \leqslant k_0' + \theta_0' \|u_n\|_{q(p-1)}^{q(p-1)} + \theta_1' \|\nabla u_{n-1}\|_{q(p-1)}^{q(p-1)} \leqslant$$
$$\leqslant k_0' + \theta_0'' \|u_n\|_p^{\frac{q}{p'}} + \theta_1'' \|\nabla u_{n-1}\|_p^{\frac{q}{p'}}$$

Hence

$$\|h_n\|_q \leqslant M_0' \quad \text{for every } n \in \mathbb{N}.$$

We can assume $q \leqslant N$, otherwise we are done by lemma 2.1 (iii).

From lemma 2.1 $(i) - (ii)$ we deduce that $|\nabla u_n| \in L^{q^*(p-1)}(\Omega)$ and that

$$\|\nabla u_n\|_{q^*(p-1)} \leqslant C \|h_n\|_q^{\frac{1}{p-1}} \leqslant C_1.$$

Since $u_n \in L^\infty(\Omega)$ we have that $|u_n|^{p-1} \in L^{q^*}(\Omega)$. Moreover, by the previous inequality we also have $|\nabla u_{n-1}|^{p-1} \in L^{q^*}(\Omega)$, thus $h_n \in L^{q^*}(\Omega)$ and, as above $\|h_n\|_{q^*} \leqslant M_1'$.

It is easily seen by induction that $(((q^*)^*)^{\cdots})^* = q^{\overset{k}{**\cdots*}} = \frac{Nq}{N-kq}$ provided $k < \frac{N}{q}$. We choose then $k = [\frac{N}{q}]$ (the maximum integer contained in $\frac{N}{q}$). Recall that since $\frac{N}{q} \notin \mathbb{N}$,

$$\frac{N}{q} - 1 < k < \frac{N}{q}$$

Iterating the previous argument k times, since $\frac{Nq}{N-kq} > N$, by lemma 2.1 (iii)

$$\|\nabla u_n\|_\infty \leqslant C \|h_n\|_{\frac{Nq}{N-kq}}^{\frac{1}{p-1}} \leqslant C_k.$$

The uniform boundedness of the gradient of u_n in $L^\infty(\Omega)$, implies the uniform boundedness of u_n's in $L^\infty(\Omega)$ (see [**16**]). Finally, from [**17**, Theorem 1] we obtain the existence of $\alpha \in]0,1[$ and a positive constant M independent on n such that

$$\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant M \quad \text{for every } n \in \mathbb{N}.$$

□

We introduce now the following abstract setting. Let X be a Banach space, $A : X \longrightarrow X$ continuous and compact operator, $J : X \to \mathbb{R}$ a functional of class $C^1(X, \mathbb{R})$, $p > 1$.

Let us introduce the following conditions:

(J_1) There exist $1 < p < 2$, $d_1, d_2 > 0$ such that

$$\langle J'(u), u - A(u) \rangle \geqslant d_1 \|u - A(u)\|^2 (\|u\| + \|A(u)\|)^{p-2}$$

and

$$\|J'(u)\| \leqslant d_2 \|u - A(u)\|^{p-1},$$

for every $u \in X$.

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(J_2) There exist $p \geqslant 2$, $d_3, d_4 > 0$ such that

$$\langle J'(u), u - A(u)\rangle \geqslant d_3\|u - A(u)\|^p$$

and

$$\|J'(u)\| \leqslant d_4\|u - A(u)\|(\|u\| + \|A(u)\|)^{p-2},$$

for every $u \in X$.

Denote by K the set of critical points of J: it is clear that under either (J_1) or (J_2) K coincides with the set of fixed points of A.

The next lemma allows to replace A with a locally Lipschitz operator B which fulfils the same properties as A.

Lemma 2.2. *([5, Lemma 2.1], [4, Lemma 4.1]) Assume either* (J_1) *or* (J_2)*. Let* D *be a closed convex subset of* X*. Then, there exists a locally Lipschitz continuous compact operator* $B : X \to X$ *which is a convex combination of* A *such that*

(i) $A(u) = B(u)$ *for each* $u \in D$*;*

(ii)

$$\frac{1}{2}\|u - B(u)\| \leqslant \|u - A(u)\| \leqslant 2\|u - B(u)\|$$

for all $u \in X$*;*

(iii) if $1 < p < 2$ *then*

$$\langle J'(u), u - B(u)\rangle \geqslant \frac{d_1}{2}\|u - A(u)\|^2(\|u\| + \|A(u)\|)^{p-2},$$

and if $p \geqslant 2$ *then*

$$\langle J'(u), u - B(u)\rangle \geqslant \frac{d_1}{2}\|u - A(u)\|^p$$

for all $u \in X$*.*

Clearly, critical points of J turn out to be fixed points of B.

In our setting $X = W_0^{1,p}(\Omega)$ endowed with the equivalent norm

$$\|u\|_\mu := \left(\int_\Omega (|\nabla u|^p + \mu|u|^p)\,\mathrm{d}x\right)^{1/p},$$

for $\mu > 0$. For fixed $w \in C_0^1(\overline{\Omega})$, put $A = A_w^\mu : W_0^{1,p}(\Omega) \longrightarrow W_0^{1,p}(\Omega)$ the operator defined by

$$A_w^\mu(u) := (-\Delta_p + \mu h_p(\cdot))^{-1}(g(x, u, \nabla w) + \mu h_p(u)),$$

where $h_p(t) = |t|^{p-2}t$ for each $t \in \mathbb{R}$.

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Let us note that

$$A_w^\mu|_{C_0^1(\overline{\Omega})} : C_0^1(\overline{\Omega}) \longrightarrow C_0^1(\overline{\Omega}).$$

Since problem $(\tilde{P}_w)$ has variational form, we can consider the associated energy functional $J_w \in C^1(W_0^{1,p}(\Omega))$, defined by

$$J_w(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \int_\Omega G(x,u,\nabla w)\, \mathrm{d}x, \text{ for } u \in W_0^{1,p}(\Omega),$$

where $G(x,t,\xi) = \int_0^t g(x,s,\xi)\, \mathrm{d}s$. Because of $(\mathcal{H})$, J_w is coercive, thus bounded from below.

The following inequalities (see [**10**]) ensure properties (J_1) and (J_2) above:

Proposition 2.2. *There exist positive constants c_i, $i = 1,\ldots,4$, such that for all $\xi,\eta \in \mathbb{R}^N$*

$$\begin{aligned}
&||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leqslant c_1(|\xi|+|\eta|)^{p-2}|\xi-\eta|,\\
&(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \geqslant c_2(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2,\\
&||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leqslant c_3|\xi-\eta|^{p-1} \quad \textit{if } 1<p\leqslant 2,\\
&(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \geqslant c_4|\xi-\eta|^p \ \textit{ if } p>2.
\end{aligned}$$

Thus, lemma 2.2 applies. We will exploit it in the next sections with different choices of D, μ and g.

3. Nodal solution for a quasilinear elliptic problem

In this section we assume conditions $(f_1)-(f_4)$ and prove theorem 1.1. Using the theory of invariant sets of descending flow, we will construct first a nodal solution of a parametrized problem and then, following an iterative approach, we will exhibit the existence of a nodal solution for (P).

3.1. On a parametrized problem

Throughout the sequel we will take into account the results of Section 2 with $g = f$. Thus, for every $w \in C_0^1(\overline{\Omega})$, the parametrized Dirichlet problem reads as follows

$$\begin{cases} -\Delta_p u = f(x,u,\nabla w) & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_w)$$

Recall that each solution u of (P_w) is in $C_0^1(\overline{\Omega})$. Also, assumptions (f_2) and (f_3) imply that $f(x,0,\xi) = 0$ for all x,ξ, thus the zero function is a solution of (P) and (P_w) for each $w \in C_0^1(\overline{\Omega})$.

In [**13**], we have proved the following

Lemma 3.1. *([**13**, Lemma 2.2]) Assume (f_2). Then, for every $M>0$ and $w \in C_0^1(\overline{\Omega})$ with $\|w\|_{C^1} \leqslant M$, there exists $\delta = \delta(M) > 0$ such that if $0<\varepsilon<\delta$, then $\varepsilon\varphi_1$ and $-\varepsilon\varphi_1$ are subsolution and supersolution of (P_w).*

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Theorem 3.1. *([13, Theorem 2.1]) Assume* $(f_1), (f_2), (f_3)$*. Then, for every* $w \in C^1_0(\overline{\Omega})$*, there exist* $u^w_P \in \mathrm{int}(C^1_0(\overline{\Omega})_+)$ *and* $u^w_N \in -\mathrm{int}(C^1_0(\overline{\Omega})_+)$ *the smallest positive solution and the biggest negative solution of* (P_w) *respectively.*

From the proof of [13, Theorem 2.1] we deduce that

Remark 3.1. $u^w_P \geqslant \varepsilon\varphi_1$ and $u^w_N \leqslant -\varepsilon\varphi_1$ with $\varepsilon = \varepsilon(M)$ uniform with respect to $w \in C^1_0(\overline{\Omega})$ with $\|w\|_{C^1} \leqslant M$.

We conclude this subsection recalling the strong comparison principle for the p-Laplace operator ([2]). For $h_1, h_2 \in L^\infty(\Omega)$, we say that $h_1 \prec h_2$ if for any $\Omega_0 \subseteq \Omega$ compact subset, there exists $\varepsilon > 0$ such that $h_1(x) + \varepsilon < h_2(x)$ for almost every $x \in \Omega_0$. In particular, if h_1 and h_2 are continuous functions such that $h_1(x) < h_2(x)$ for all $x \in \Omega$, then $h_1 \prec h_2$.

Proposition 3.1. *[2, Proposition 2.6] For* $\lambda \geqslant 0$ *and* $f, g \in L^\infty(\Omega)$*, let* v, u *be solutions of the problems*:

$$\begin{cases} -\Delta_p v + \lambda |v|^{p-2} v = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2} u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

If $f \prec g$ *and* $u \in \mathrm{int}(C^1_0(\overline{\Omega})_+)$*, then* $u - v \in \mathrm{int}(C^1_0(\overline{\Omega})_+)$*.*

3.2. A pseudogradient vector field

Throughout the sequel, $u_0 \in C^1_0(\overline{\Omega})$ is a fixed function, $M > 0$ is given by proposition 2.1 and we choose $m = m_M$ in assumption (f_4). Thus, following the notation of Section 2 with $g = f$ and $\mu = m$ one has

$$\|u\|_m := \left(\int_\Omega (|\nabla u|^p + m|u|^p) \, \mathrm{d}x \right)^{1/p},$$

$$A_w(u) := (-\Delta_p + m h_p(\cdot))^{-1} (f(x, u, \nabla w) + m h_p(u)),$$

$$J_w(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x - \int_\Omega F(x, u, \nabla w) \, \mathrm{d}x,$$

where $F(x, t, \xi) = \int_0^t f(x, s, \xi) \, \mathrm{d}s$.

Let us introduce the set Λ^w which will be crucial in our argument. Let $u^w_P \in \mathrm{int}(C^1_0(\overline{\Omega})_+)$ and $u^w_N \in -\mathrm{int}(C^1_0(\overline{\Omega})_+)$ the smallest positive solution and the biggest negative solution of (P_w) (see theorem 3.1). Let us denote by $[u^w_N, u^w_P]$ the set of all C^1_0-functions u such that $u^w_N \leqslant u \leqslant u^w_P$.

Consider then, the following set

$$\Lambda^w = \{u \in C^1_0(\overline{\Omega}) : \ u \in \mathrm{int}_{C^1_0} [u^w_N, u^w_P]\}.$$

Proposition 3.2.

$$A_w(\Lambda^w) \subseteq \Lambda^w \quad \textit{and} \quad A_w(\mathrm{int}(C^1_0(\overline{\Omega})_+)) \subseteq \mathrm{int}(C^1_0(\overline{\Omega})_+).$$

Proof. First, we show that $A_w(\Lambda^w) \subseteq \Lambda^w$. Let $u \in \Lambda^w$ and $v = A_w(u)$:

$$\begin{aligned} -\Delta_p v + m h_p(v) &= f(x, u, \nabla w) + m h_p(u) \\ \text{(by } (f_4) \text{ and continuity of } f) \quad &\prec f(x, u_P^w, \nabla w) + m h_p(u_P^w) \\ &= -\Delta_p u_P^w + m h_p(u_P^w). \end{aligned}$$

By proposition 3.1, we conclude that $u_P^w - v \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. Analogously one obtains that $v - u_N^w \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. For the other inclusion, let $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ and $v = A_w(u)$. Thus,

$$-\Delta_p v + m h_p(v) = f(x, u, \nabla w) + m h_p(u) > 0$$

By the strong maximum principle [**23**], we conclude that $v \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. □

Let $B = B_w$ be as in lemma 2.2 with $D = \Lambda_w$. Thus, since B_w is a convex combination of A_w, one has

$$B_w(\Lambda^w) \subseteq \Lambda^w \quad \text{and} \quad B_w(\mathrm{int}(C_0^1(\overline{\Omega})_+)) \subseteq \mathrm{int}(C_0^1(\overline{\Omega})_+). \tag{3.1}$$

For every $u \in C_0^1(\overline{\Omega}) \setminus K_w$ (where K_w is the set of all fixed points of A_w) consider the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = -\varphi(t) + B_w(\varphi(t)) \\ \varphi(0) = u. \end{cases} \tag{3.2}$$

Since B_w is locally Lipschitz, the above problem admits a unique solution $\varphi^t(u)$ in $C_0^1(\overline{\Omega})$ called *descending flow curve* with maximal interval of existence $[0, \tau(u)[$. Notice that $\tau(u)$ can be either a positive number or $+\infty$.

By lemma 2.2 *(iii)*,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} J_w(\varphi^t(u)) &= \langle J_w'(\varphi^t(u)), \frac{\mathrm{d}}{\mathrm{d}t}\varphi^t(u) \rangle \\ &= -\langle J_w'(\varphi^t(u)), \varphi^t(u) - B_w(\varphi^t(u)) \rangle \\ &< 0 \end{aligned}$$

and the inequality is strict since $u \notin K_w$ so that it is not a fixed point of B_w. Thus, $J_w(\varphi^t(u))$ is strictly decreasing. Recall also that J_w is coercive, hence bounded from below.

Moreover from (3.2), we have

$$\int_0^t e^s \frac{\mathrm{d}}{\mathrm{d}s}\varphi^s(u)\,\mathrm{d}s = -\int_0^t e^s \varphi^s(u)\,\mathrm{d}s + \int_0^t e^s B_w(\varphi^s(u))\,\mathrm{d}s,$$

or

$$\varphi^t(u) = e^{-t}u + e^{-t}\int_0^t e^s B_w(\varphi^s(u))\,\mathrm{d}s. \tag{3.3}$$

By (3.1), one has the following (see [**19**, proof of Lemma 3.2]).

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Lemma 3.2. *If* $u \in \Lambda^w$, *then* $\varphi^t(u) \in \Lambda^w$ *and if* $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, *then* $\varphi^t(u) \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ *for all* $0 < t < \tau(u)$.

For $D \subseteq C_0^1(\overline{\Omega})$, let us denote by

$$\mathcal{C}(D) = D \cup \{u \in C_0^1(\overline{\Omega}) \setminus K_w : \text{ there exists } \bar{t} \geqslant 0 \text{ such that } \varphi^{\bar{t}}(u) \in D\}.$$

We recall that D is called an *invariant* set of descending flow for J_w, if whenever $u \in D \setminus K_w$, the flow $\{\varphi^t(u) : t \in [0, \tau(u)[\} \subset D$. If $D = \mathcal{C}(D)$, then D is said *complete.*

If D is invariant then, by the definition above, $\mathcal{C}(D)$ is invariant. Moreover it is easy to see that $\mathcal{C}(D)$ is complete as $\mathcal{C}(\mathcal{C}(D)) = \mathcal{C}(D)$. Then $\partial\mathcal{C}(D)$ is also an invariant set of descending flow [**19**, Lemma 2.3]. If D is also open then $\mathcal{C}(D)$ is open [**19**, Lemma 2.4(i)].

By lemma 3.2, Λ^w and $\mathrm{int}(C_0^1(\overline{\Omega})_+)$ are invariant sets of descending flow for J_w, so $\mathcal{C}(\Lambda^w)$ and $\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$ are invariant, as well as $\partial\mathcal{C}(\Lambda^w)$ and $\partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$. Moreover, $\mathcal{C}(\Lambda^w)$ is open in $C_0^1(\overline{\Omega})$ and $\mathcal{C}(\Lambda^w) \neq C_0^1(\overline{\Omega})$ (indeed $u_P^w, u_N^w \in K_w \setminus \Lambda^w$, so they cannot lie in $\mathcal{C}(\Lambda^w)$). Also, $\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$ is open in $C_0^1(\overline{\Omega})$ and $\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+)) \neq C_0^1(\overline{\Omega})$ (indeed $0 \in K_w \setminus \mathrm{int}(C_0^1(\overline{\Omega})_+)$). Since $C_0^1(\overline{\Omega})_+ \subseteq \mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$ and $\partial\mathcal{C}(\Lambda^w) \cap C_0^1(\overline{\Omega})_+ \neq \emptyset$, we have $\partial\mathcal{C}(\Lambda^w) \cap \mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+)) \neq \emptyset$. Because $\partial\mathcal{C}(\Lambda^w) \cap (-C_0^1(\overline{\Omega})_+) \neq \emptyset$, we obtain that $\partial\mathcal{C}(\Lambda^w) \cap \partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+)) \neq \emptyset$. Let us fix

$$u^* \in \partial\mathcal{C}(\Lambda^w) \cap \partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+)).$$

Lemma 3.3. *There exists* $u_w \in K_w$ *and an increasing sequence of positive numbers* $(t_n)_n$ *with* $t_n \to \tau(u^*)$ *such that* $\lim_n \|\varphi^{t_n}(u^*) - u_w\|_m = 0$.

Proof. Let $0 < t_1 < t_2 < \tau(u^*)$. Then,

$$\begin{aligned}
\|\varphi^{t_2}(u^*) - \varphi^{t_1}(u^*)\|_m &\leqslant \int_{t_1}^{t_2} \|\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t(u^*)\|_m \,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \|\varphi^t(u^*) - B_w(\varphi^t(u^*))\|_m \\
\text{(by lemma 2.2, } (ii)) &\leqslant 2\int_{t_1}^{t_2} \|\varphi^t(u^*) - A_w(\varphi^t(u^*))\|_m.
\end{aligned}$$

Assume that $p \geqslant 2$.

Applying Hölder inequality, lemma 2.2 *(iii)*, the monotonicity of the flow $t \to J_w(\varphi^t(u^*))$ and the boundedness from below of J_w, we obtain

$$\begin{aligned}
\int_{t_1}^{t_2} \|\varphi^t(u^*) - A_w(\varphi^t(u^*))\|_m &\leqslant \left(\int_{t_1}^{t_2} \|\varphi^t(u^*) - A_w(\varphi^t(u^*))\|_m^p\right)^{1/p} (t_2 - t_1)^{1/p'} \\
&\leqslant c_1 \left(\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} J_w(\varphi^t(u^*))\,\mathrm{d}t\right)^{1/p} (t_2 - t_1)^{1/p'} \\
&= c_1 (J_w(\varphi^{t_2}(u^*)) - J_w(\varphi^{t_1}(u^*)))^{1/p} \, (t_2 - t_1)^{1/p'} \\
&\leqslant c_2 (t_2 - t_1)^{1/p'}.
\end{aligned}$$

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Putting together the above outcomes we get that

$$\|\varphi^{t_2}(u^*) - \varphi^{t_1}(u^*)\|_m \leqslant c(t_2 - t_1)^{1/p'}.$$

Assume now $1 < p < 2$. By coercivity of J_w, we deduce that the set $\{u : J_w(u) \leqslant J_w(\varphi^0(u^*)) = J_w(u^*)\} \subset \overline{B}(0, b)$ for some $b > 0$ where $\overline{B}(0, b)$ denotes the closed ball in $W_0^{1,p}(\Omega)$ centred at zero of radius b. By the monotonicity of the flow, $J_w(\varphi^t(u^*) \leqslant J_w(u^*)$, thus there exists a constant $b_1 > 0$, such that $\|\varphi^t(u^*)\|_m, \|A_w(\varphi^t(u^*))\|_m \leqslant b_1$ for each t.

Then, by Hölder inequality, lemma 2.2 (iii)

$$\begin{aligned}
\int_{t_1}^{t_2} \|\varphi^t(u^*) - A_w(\varphi^t(u^*))\|_m &\leqslant \Big(\int_{t_1}^{t_2} \|\varphi^t(u^*) - A_w(\varphi^t(u^*))\|_m^2 (\|\varphi^t(u^*)\|_m \\
&\quad + \|A_w(\varphi^t(u^*))\|_m)^{p-2}\, \mathrm{d}t\Big)^{1/2} \\
&\quad \cdot \left(\int_{t_1}^{t_2} (\|\varphi^t(u^*)\|_m + \|A_w(\varphi^t(u^*))\|_m)^{2-p}\, \mathrm{d}t\right)^{1/2} \\
&\leqslant c_1 \left(\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} J_w(\varphi^t(u^*))\right)^{1/2} (t_2 - t_1)^{1/2} \\
&= c_1 (J_w(\varphi^{t_2}(u^*)) - J_w(\varphi^{t_1}(u^*)))^{1/2} (t_2 - t_1)^{1/2} \\
&\leqslant c_2 (t_2 - t_1)^{1/2}.
\end{aligned}$$

Thus, in both cases ($p \geqslant 2$ and $1 < p < 2$), if $\tau(u^*) < \infty$, there exists $u_w \in W_0^{1,p}(\Omega)$ such that

$$\lim_{t \to \tau(u^*)} \|\varphi^t(u^*) - u_w\|_m = 0.$$

Since the interval $[0, \tau(u^*)[$ is maximal it has to be $u_w \in K_w$.

If $\tau(u^*) = \infty$, the boundedness from below of J_w allows us to fix an increasing sequence of positive numbers $(t_n)_n$, $t_n \to \infty$ such that

$$\lim_n \frac{\mathrm{d}}{\mathrm{d}t} J_w(\varphi^t(u^*))|_{t=t_n} = 0.$$

If $p \geqslant 2$, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} J_w(\varphi^t(u^*))|_{t=t_n} &= -\langle J_w'(\varphi^{t_n}(u^*)), \varphi^{t_n}(u^*) - B_w(\varphi^{t_n}(u^*))\rangle \\
(\text{by lemma 2.2, } (iii)) &\leqslant -c_1 \|\varphi^{t_n}(u^*) - A_w(\varphi^{t_n}(u^*))\|_m^p,
\end{aligned}$$

which says that

$$\lim_n \|\varphi^{t_n}(u^*) - A_w(\varphi^{t_n}(u^*))\|_m = 0.$$

Now, let us observe that $(\varphi^{t_n}(u^*))_n$ is bounded in $W_0^{1,p}(\Omega)$. Indeed, by the monotonicity of the flow, $J_w(\varphi^{t_n}(u^*) \leqslant J_w(u^*)$ for every $n \in \mathbb{N}$, that means that $\varphi^{t_n}(u^*) \in J_w^{-1}(] - \infty, J_w(u^*)])$ for every $n \in \mathbb{N}$, and the latter is a bounded set

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because of the coercivity of J_w. Moreover, since A_w is a compact operator, it follows (eventually passing to a subsequence) that there exists $u_w \in W_0^{1,p}(\Omega)$ such that

$$\lim_n \|\varphi^{t_n}(u^*) - u_w\|_m = \lim_n \|A_w(\varphi^{t_n}(u^*)) - u_w\|_m = 0.$$

In particular, $u_w \in K_w$. Also, for some $b_2 > 0$, one has $\|\varphi^{t_n}(u^*)\|_m, \|A_w(\varphi^{t_n}(u^*))\|_m \leqslant b_2$.

If $1 < p < 2$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} J_w(\varphi^t(u^*))|_{t=t_n} &= -\langle J_w'(\varphi^{t_n}(u^*)), \varphi^{t_n}(u^*) - B_w(\varphi^{t_n}(u^*))\rangle \\ (\text{by lemma } 2.2, (iii)) &\leqslant -c_1\|\varphi^{t_n}(u^*) - A_w(\varphi^{t_n}(u^*))\|_m^2 (\|\varphi^{t_n}(u^*)\|_m \\ &\quad + \|A_w(\varphi^{t_n}(u^*))\|_m)^{p-2} \\ &\leqslant -c_2\|\varphi^{t_n}(u^*) - A_w(\varphi^{t_n}(u^*))\|_m^2\end{aligned}$$

which says that

$$\lim_n \|\varphi^{t_n}(u^*) - A_(\varphi^{t_n}(u^*))\|_m = 0$$

and we conclude as above. □

In the next lemma we refine the previous result.

Lemma 3.4. *With the notation of lemma 3.3, one has that* $u_w \in C_0^1(\overline{\Omega})$ *and* $\lim_n \|\varphi^{t_n}(u^*) - u_w\|_{C_0^1} = 0$.

Proof. As in the proof of lemma 3.3, we first observe that the set $\{\varphi^t(u^*) \ : \ t \in [0, \tau(u^*)[\}$ is bounded in $W_0^{1,p}(\Omega)$. Recalling (3.3),

$$\varphi^t(u^*) = e^{-t}u^* + e^{-t}\int_0^t e^s B_w(\varphi^s(u^*))\,\mathrm{d}s.$$

Since $B_w : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is a compact operator, the set

$$\left\{e^{-t}\int_0^t e^s B_w(\varphi^s(u^*))\,\mathrm{d}s \ : \ t \in [0, \tau(u^*)[\right\}$$

is bounded in $C^{1,\alpha}(\overline{\Omega})$, thus relatively compact in $C_0^1(\overline{\Omega})$. This clearly implies that $\{\varphi^t(u^*) \ : \ t \in [0, \tau(u^*)[\}$ is relatively compact in $C_0^1(\overline{\Omega})$. The thesis follows from lemma 3.3. □

We are ready to prove the main result of this subsection.

Theorem 3.2. u_w *is a nodal solution of* (P_w).

Proof. Clearly u_w is a solution of (P_w) since it belongs to K_w. Since the initial point u^* belongs to $\partial\mathcal{C}(\Lambda^w) \cap \partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$, and both sets $\partial\mathcal{C}(\Lambda^w)$, $\partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$ are invariant sets of descending flow, then $(\varphi^{t_n}(u^*))_n \subseteq \partial\mathcal{C}(\Lambda^w) \cap \partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$.

Moreover, $\partial\mathcal{C}(\Lambda^w)\cap\partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$ is a closed set, so that $u_w\in\partial\mathcal{C}(\Lambda^w)\cap\partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$. Being $u_w\in\partial\mathcal{C}(\Lambda^w)$, we obtain that $u_w\notin\mathrm{int}_{C_0^1}[u_N^w,u_P^w]$, in particular $u_w\neq 0$. Actually, by remark 3.1, $u_w\notin\mathrm{int}_{C_0^1}[-\varepsilon\varphi_1,\varepsilon\varphi_1]$. On the other hand, since $u_w\in\partial\mathcal{C}(\mathrm{int}(C_0^1(\overline{\Omega})_+))$, we also have $u_w\notin\mathrm{int}(C_0^1(\overline{\Omega})_+)\cup(-\mathrm{int}(C_0^1(\overline{\Omega})_+))$. This ensures that u_w can not have constant sign. Indeed, if $u_w\geqslant 0$, it would be a non negative, non trivial solution of

$$\begin{cases}-\Delta_p u+mh_p(u)=f(x,u,\nabla w)+mh_p(u) & \text{in }\Omega\\ u=0 & \text{on }\partial\Omega.\end{cases}$$

and by assumption (f_4) and the strong maximum principle by Vazquez [**23**], we would deduce $u_w\in\mathrm{int}(C_0^1(\overline{\Omega})_+)$. Thus, u_w is a nodal solution of (P_w). □

3.3. Existence of a nodal solution for (*P*)

In this subsection through an iteration procedure we prove our first main result.

Proof of theorem 1.1. In theorem 3.2 choose $w=u_0$, where u_0 is the function we fixed at the beginning of Section 3.2. Thus, the existence of a nodal solution u_1 of (P_{u_0}) follows. Proceeding in such way, for each $n\in\mathbb{N}$ denote by u_n the nodal solution of $(P_{u_{n-1}})$ given by theorem 3.2. Hence, we construct a sequence of functions $u_n\in C_0^1(\overline{\Omega})$ such that

$$u_n\notin\mathrm{int}_{C_0^1}[-\varepsilon\varphi_1,\varepsilon\varphi_1] \tag{3.4}$$

$$u_n\notin\mathrm{int}(C_0^1(\overline{\Omega})_+)\cup(-\mathrm{int}(C_0^1(\overline{\Omega})_+)) \tag{3.5}$$

By proposition 2.1, $\|u_n\|_{C^{1,\alpha}(\overline{\Omega})}\leqslant M$, and by the compactness of the embedding $C^{1,\alpha}(\overline{\Omega})\hookrightarrow C_0^1(\overline{\Omega})$, $(u_n)_n$ is relatively compact in $C_0^1(\overline{\Omega})$. Unless to pass to a subsequence, let

$$\tilde{u}:=\lim_{n\to\infty}u_n \text{ in } C_0^1(\overline{\Omega}).$$

Let us prove that $\tilde{u}$ is the solution of (P) we are looking for. Since u_n is a solution of $(P_{u_{n-1}})$, for every $\varphi\in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,\mathrm{d}x=\int_\Omega f(x,u_n,\nabla u_{n-1})\varphi\,\mathrm{d}x.$$

Since $u_n\to\tilde{u}$ in $C_0^1(\overline{\Omega})$, we have that $f(x,u_n,\nabla u_{n-1})\to f(x,\tilde{u},\nabla\tilde{u})$ in $L^{p'}(\Omega)$ and passing to the limit in the above equality we get

$$\int_\Omega|\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\nabla\varphi\,\mathrm{d}x=\int_\Omega f(x,\tilde{u},\nabla\tilde{u})\varphi\,\mathrm{d}x,$$

which is our claim. By (3.4) and (3.5), it follows that $\tilde{u}\notin\mathrm{int}_{C_0^1}[-\varepsilon\varphi_1,\varepsilon\varphi_1]$ and $\tilde{u}\notin\mathrm{int}(C_0^1(\overline{\Omega})_+)\cup(-\mathrm{int}(C_0^1(\overline{\Omega})_+))$. Thus, $\tilde{u}\neq 0$, and as in the proof of theorem 3.2, $\tilde{u}$ can not have constant sign.

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4. Nodal solution for a quasilinear eigenvalue problem

The goal of the present section is to prove the existence of a nodal solution for (P_λ) under assumptions $(\tilde{f}_1)-(\tilde{f}_5)$. While in theorem 1.1 the construction of the nodal solution was based on the existence of the extremal solutions for the parametrized problem (P_w), here, exploiting the dependence on the parameter λ, we apply an abstract theorem by [5], which still relies on the theory of invariant sets of descending flow. After deducing the existence of a sign changing solution for the parametrized problem, we conclude as in the previous section.

4.1. On a parametrized problem

Throughout the sequel we will take into account the results of Section 2 with $g=\lambda f$ and $\mu=\lambda m$ for some m to be chosen later. For every $w\in C_0^1(\overline{\Omega})$, let us consider the parametrized Dirichlet problem

$$\begin{cases} -\Delta_p u=\lambda f(x,u,\nabla w) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{$P_{\lambda,w}$}$$

From hypothesis $(\tilde{f}_1)$, it follows that for each $\lambda>0$ we can fix $\varepsilon=\varepsilon(\lambda)>0$ with $\varepsilon(1+\lambda_1^{\frac{1}{p'}})\lambda<\lambda_1$ and $k_0(\lambda)\in\mathbb{R}$ such that

$$\lambda|f(x,s,\xi)|\leqslant k_0(\lambda)+\varepsilon\lambda|s|^{p-1}+\varepsilon\lambda|\xi|^{p-1}$$

for all $x\in\Omega$, $s\in\mathbb{R}$, and $\xi\in\mathbb{R}^N$. This ensures that under the above conditions, the function $g=\lambda f$ fulfils assumption $(\mathcal{H})$ of Section 2 and proposition 2.1 applies, i.e. for every $\lambda>0$ and $u_0\in C_0^1(\overline{\Omega})$, there exists $\alpha\in]0,1[$ and a positive constant M depending on λ and $\|u_0\|$ such that if u_n is a solution of $(P_{\lambda,u_{n-1}})$, $\|u_n\|_{C^{1,\alpha}(\overline{\Omega})}\leqslant M$.

Remark 4.1. Notice that assumption $(\tilde{f}_2)$ implies that $f(x,0,\xi)=0$ for all x,ξ. Thus the zero function is a solution of both $(P_{\lambda,w})$ and (P_λ).

We will need the following abstract result.

Let X be a Banach space, $D^\pm$ closed convex subsets of X, $A:X\longrightarrow X$ continuous and compact operator and $J:X\to\mathbb{R}$ a functional of class $C^1(X,\mathbb{R})$. Introduce the following conditions.

(D_1) $\mathcal{O}=\text{int}(D^+)\cap\text{int}(D^-)\neq\emptyset$.

(D_2) $A(D^\pm)\subseteq\text{int}(D^\pm)$.

(J_3) For any $b\in\mathbb{R}$ there exists a constant $a=a(b)>0$ such that if $u\in\{u\in X: J(u)\leqslant b\}$ then

$$\|u\|+\|A(u)\|\leqslant a(1+\|u-A(u)\|).$$

(J_4) There exists a path $h:[0,1]\longrightarrow X$ such that $h(0)\in\text{int}(D^+)\setminus D^-$ and $h(1)\in\text{int}(D^-)\setminus D^+$ and

$$\max_{0\leqslant t\leqslant 1} J(h(t))<\alpha_0:=\inf_{D^+\cap D^-} J(u).$$

The next theorem is the abstract tool we will exploit to deduce the existence of a nodal solution for the parametrized problem $(P_{\lambda,w})$, which we state here in a convenient form for our purposes.

THEOREM 4.1. *[5, Theorem 2.2] Assume (D_1), (D_2), (J_3), (J_4) and either (J_1) or (J_2) from Section 2. Then J has a critical point in $\partial\mathcal{C}(\mathcal{O}) \setminus (D^+ \cup D^-)$.*

Following the notation of Section 2, $X = W_0^{1,p}(\Omega)$ endowed with the equivalent norm

$$\|u\|_m^\lambda := \left(\int_\Omega (|\nabla u|^p + \lambda m|u|^p)\,\mathrm{d}x\right)^{1/p},$$

where λ and m will be chosen later in a convenient way,

$$A_w^\lambda(u) := (-\Delta_p + \lambda m h_p(\cdot))^{-1}(\lambda f(x,u,\nabla w) + \lambda m h_p(u)),$$

and B_w^λ as in lemma 2.2. Denote also by $J_w^\lambda : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$

$$J_w^\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \lambda\int_\Omega F(x,u,\nabla w)\,\mathrm{d}x,$$

$$P = \{u \in W_0^{1,p}(\Omega) : u \geqslant 0 \text{ a.e. in } \Omega\}$$

and for $\varepsilon > 0$

$$D_\varepsilon^\pm = \{u \in W_0^{1,p}(\Omega) : \mathrm{dist}_m(u, \pm P) \leqslant \varepsilon\}.$$

PROPOSITION 4.1. *For $\varepsilon > 0$ small enough,*

$$A_w^\lambda(D_\varepsilon^\pm) \subseteq \mathrm{int} D_\varepsilon^\pm.$$

Proof. Let us prove $A_w^\lambda(D_\varepsilon^+) \subseteq \mathrm{int} D_\varepsilon^+$. Notice that by assumptions $(\tilde{f}_2) - (\tilde{f}_4)$, for any $M \geqslant \|w\|_{C^1}$ there exists $m_M > 0$ such that

$$sf(x,s,\xi) + m_M s h_p(s) > 0 \text{ for each } x \in \Omega, s \neq 0, |\xi| \leqslant M. \tag{4.1}$$

In the sequel put $m := m_M$.

By $(\tilde{f}_2)$ and $(\tilde{f}_3)$, for each $\varepsilon > 0$ and $q > p$ there exists another constant $c_M > 0$ such that

$$|f(x,s,\xi) + m h_p(s)| \leqslant (\varepsilon + m)|s|^{p-1} + c_M|s|^{q-1}, \tag{4.2}$$

for each $x \in \Omega, s \neq 0, |\xi| \leqslant M$.

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Let $u \in D_\varepsilon^+$ and $v = A_w^\lambda(u)$. Thus,

$$-\Delta_p v + \lambda m h_p(v) = \lambda f(x, u, \nabla w) + \lambda m h_p(u)$$

and so testing the above equation with $-v^-$, where $v^- = \max\{-v,\ 0\}$, we deduce

$$\begin{aligned}\|v^-\|_m^p &= \lambda \int_\Omega (f(x,u,\nabla w) + m h_p(u))(-v^-)\\ \text{[by (4.1), (4.2)]}\quad &\leqslant \lambda(\varepsilon + m)\int_\Omega (u^-)^{p-1}v^- + \lambda c_M \int_\Omega (u^-)^{q-1}v^-\\ &\leqslant \lambda(\varepsilon+m)\|u^-\|_p^{p-1}\|v^-\|_p + \lambda c_M \|u^-\|_q^{q-1}\|v^-\|_q\\ &\leqslant \frac{\lambda(\varepsilon+m)}{(\lambda_1+\lambda m)^{1/p}}\|u^-\|_p^{p-1}\|v^-\|_m + \frac{\lambda c_M}{(\lambda_1+\lambda m)^{1/p}}\|u^-\|_q^{q-1}\|v^-\|_m.\end{aligned}$$

Hence,

$$d_m(v,P)^{p-1} \leqslant \|v^-\|_m^{p-1} \leqslant \frac{\lambda(\varepsilon+m)}{(\lambda_1+\lambda m)^{1/p}}\|u^-\|_p^{p-1} + \frac{\lambda c_M}{(\lambda_1+\lambda m)^{1/p}}\|u^-\|_q^{q-1}.$$

Since $\|u^-\|_p \leqslant \|u-w\|_p$ for all $w \in P$, we get

$$\begin{aligned}d_m(v,P)^{p-1} &\leqslant \frac{\lambda(\varepsilon+m)}{(\lambda_1+\lambda m)^{1/p}}\|u-w\|_p^{p-1} + \frac{\lambda c_M}{(\lambda_1+\lambda m)^{1/p}}\|u-w\|_p^{q-1}\\ &\leqslant \frac{\lambda(\varepsilon+m)}{(\lambda_1+\lambda m)}\|u-w\|_m^{p-1} + \frac{\lambda c_M}{(\lambda_1+\lambda m)^{\frac{q}{p}}}\|u-w\|_m^{q-1}.\end{aligned}$$

Thus,

$$d_m(v,P)^{p-1} \leqslant \frac{\lambda(\varepsilon+m)}{(\lambda_1+\lambda m)}d_m(u,P)^{p-1} + \frac{\lambda c_M}{(\lambda_1+\lambda m)^{\frac{q}{p}}}d_m(u,P)^{q-1}.$$

Thus, since $q > p$, there exists $\varepsilon_0 > 0$ such that

$$d_m(v,P) < d_m(u,P) \quad \text{if } 0 < d_m(u,P) \leqslant \varepsilon_0, \tag{4.3}$$

and the proof is concluded. □

Remark 4.2. From the above construction, $\varepsilon > 0$ depends on λ and $M \geqslant \|w\|_{C^1}$. In the sequel, our choice of M will be uniform with respect to w (see proposition 2.1).

Theorem 4.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ a continuous function satisfying $(\tilde{f}_1) - (\tilde{f}_5)$. Then, there exists $\tilde{\lambda}$ such that for each $\lambda > \tilde{\lambda}$, and $w \in C_0^1(\overline{\Omega})$, problem $(P_{\lambda,w})$ has a nodal solution in $C_0^1(\overline{\Omega})$.*

Proof. Fix $w \in C_0^1(\overline{\Omega})$. Let us show that all the hypotheses of theorem 4.1 are verified with $A = A_w^\lambda$, $J = J_w^\lambda$, $D^+ = D_\varepsilon^+$, $D^- = D_\varepsilon^-$ and λ big enough.

Condition (D_1) is trivial, (D_2) follows by proposition 4.1 and (J_1), (J_2) by proposition 2.2. Moreover, the map A_w^λ is compact (see [**13**, Lemma 2.3]) and assumption (J_3) is implied by the coercivity of J_w^λ. It remains to check the validity of condition (J_4). By (4.3), we have that

$$K \cap (D_\varepsilon^+ \cap D_\varepsilon^-) = \{0\}.$$

Since J_w^λ is decreasing over the flow and $D_\varepsilon^+ \cap D_\varepsilon^-$ is an invariant set, it follows that for every $u \in D_\varepsilon^+ \cap D_\varepsilon^-$

$$J_w^\lambda(u) \geqslant J_w^\lambda(\varphi_t(u)) \geqslant J_w^\lambda(u^*)$$

where $u^* = \lim_{t\to\tau(u)} \varphi_t(u) \in K \cap (D_\varepsilon^+ \cap D_\varepsilon^-) = \{0\}$. Therefore

$$J_w^\lambda(u) \geqslant J_w^\lambda(0) = 0 \quad \text{for every } u \in D_\varepsilon^+ \cap D_\varepsilon^-.$$

Let us show that there exists $\tilde{\lambda}$ such that for all $\lambda \geqslant \tilde{\lambda}$ (J_4) holds; we follow the construction of [4, Lemma 3.2]. Let

$$a := \inf\{x_1 : x = (x_1, \dots, x_N) \in \Omega\} \text{ and } b := \sup\{x_1 : x = (x_1, \dots, x_N) \in \Omega\}.$$

we consider

$$\Omega_t := \{x \in \Omega : \ (1-t)a + tb < x_1 < b\} \text{ for } t \in [0,1].$$

Thus $\Omega_0 = \Omega$ and $\Omega_1 = \emptyset$. We define

$$h^*(t) = s^+ \chi_{\Omega_t} + s^- \chi_{\Omega\setminus\Omega_t}.$$

Let

$$\delta := |\Omega| \inf_{(x,\xi)\in\Omega\times\mathbb{R}^N} F(x, s^\pm, \xi) > 0$$

(see $(\tilde{f}_5)$). We can approximate h^* by a function $h \in C([0,1], W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega}))$ such that

$$\int_\Omega F(x, h(t), \nabla w)\, \mathrm{d}x \geqslant \delta/2 > 0.$$

We choose $\varepsilon > 0$ such that $s^- < -\varepsilon < 0 < \varepsilon < s^+$, so that $h(0) \in \operatorname{int}(\mathcal{D}_\varepsilon^+) \setminus \mathcal{D}_\varepsilon^-$ and $h(1) \in \operatorname{int}(\mathcal{D}_\varepsilon^-) \setminus \mathcal{D}_\varepsilon^+$. Finally

$$J_w^\lambda(h(t) \leqslant \frac{1}{p}\|\nabla h(t)\|_p^p - \frac{1}{2}\lambda\delta \leqslant C - \frac{1}{2}\lambda\delta.$$

Choose $\tilde{\lambda} = 2C/\delta$. Notice that $\tilde{\lambda}$ does not depend on w. The existence of a nodal solution for $(P_{\lambda,w})$ follows at once by theorem 4.1. □

4.2. Nodal solution of (P_λ)

In this subsection we prove theorem 1.2 using an iterative procedure.

Proof of theorem 1.2. Let us choose $\lambda > \tilde{\lambda}$, where $\tilde{\lambda}$ is as in theorem 4.2. Fix u_0 and let $M > 0$, depending on λ and u_0, be as in proposition 2.1. For each $n \in \mathbb{N}^+$

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denote by u_n the nodal solution in $C_0^1(\overline{\Omega})$ of $(P_{\lambda,u_{n-1}})$ given by theorem 4.2. From its proof,

$$u_n \in \partial\mathcal{C}(\mathrm{int}(D_\varepsilon^+) \cap \mathrm{int}(D_\varepsilon^+)) \setminus (\mathrm{int}(D_\varepsilon^+) \cup \mathrm{int}(D_\varepsilon^-)). \tag{4.4}$$

Let us stress that ε depends on M which is independent on n (remark 4.2). Thus the set $\partial\mathcal{C}(\mathrm{int}(D_\varepsilon^+) \cap \mathrm{int}(D_\varepsilon^+)) \setminus (\mathrm{int}(D_\varepsilon^+) \cup \mathrm{int}(D_\varepsilon^-))$ is a closed invariant set and does not depend on n.

By proposition 2.1, $\|u_n\|_{C^{1,\alpha}} \leqslant M$, and by the compactness of the embedding $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$, $(u_n)_n$ is relatively compact in $C_0^1(\overline{\Omega})$. Unless to pass to a subsequence, let

$$\tilde{u} := \lim_{n\to\infty} u_n \text{ in } C_0^1(\overline{\Omega}).$$

As in the proof of theorem 1.1, $\tilde{u}$ is a solution of (P_λ). By the closedness of the set $\partial\mathcal{C}(\mathrm{int}(D_\varepsilon^+) \cap \mathrm{int}(D_\varepsilon^+)) \setminus (\mathrm{int}(D_\varepsilon^+) \cup \mathrm{int}(D_\varepsilon^-))$, it follows from (4.4) that $\tilde{u}$ is a sign changing function as we claimed.

5. Examples and open questions

This section is devoted to some examples of applications of our main theorems and a few open questions.

Example 5.1. Let k_0, θ_0, θ_1 positive numbers with $\theta_0 + \theta_1 \lambda_1^{1/p'} < \lambda_1$, $g : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ a continuous, positive function such that $g(x,\xi) \leqslant k_0 + \theta_1|\xi|^{p-1}$. Define $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that

$$f(x,s,\xi) = \begin{cases} (\lambda_1 + g(x,\xi))|s|^{p-2}s & \text{if } |s| \leqslant 1 \\ [\lambda_1 + g(x,\xi)) + \theta_0(|s|-1)]\,\dfrac{|s|^{p-2}}{s} & \text{if } |s| > 1 \end{cases}$$

Then, theorem 1.1 applies.

Example 5.2. Let $r, q, t \geqslant 1$ such that $\max\{r,q\} < p < t$, $r + q < p + 1$, and $g : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ a continuous function such that $\inf_{\Omega\times\mathbb{R}^N} g(x,\xi) > 0$ and

$$\lim_{|\xi|\to+\infty} \frac{g(x,\xi)}{|\xi|^{q-1}} = \ell \neq 0.$$

Define $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that

$$f(x,s,\xi) = \begin{cases} |s|^{r-2}s\ g(x,\xi) & \text{if } |s| > 1 \\ |s|^{t-2}s\ g(x,\xi) & \text{if } |s| \leqslant 1 \end{cases}$$

Then, theorem 1.2 applies.

Question 5.1. Because of the extremality of u_N and u_P in corollary 1.1, a non trivial solution of (P) in between, would be a nodal solution. It still remains an open question whether this situation occurs. We believe that in order to prove that such solution exists, extra assumptions would be needed. Indeed, in [**11**] for $p = 2$

it has been proved that, when f does not depend on the gradient, is sublinear, and its derivative at zero is greater than λ_2 (being λ_2 the second eigenvalue of the negative Laplacian), the problem admits the biggest negative and the smallest positive solution which turn out to be also minimizers of the energy functional. The existence of a mountain pass critical point in between follows at once. The variational characterization of λ_2 allows finally to prove that such critical value is non zero (see also [**8**] for an extension to p-Laplace equations for $p \neq 2$).

QUESTION 5.2. Is it possible to find a positive and a negative solution of (P_λ)? We underline that, under our assumption, for fixed $w \in C_0^1(\overline{\Omega})$ the energy functional J_w^λ has the mountain pass geometry for big λ, and a positive /negative solution follows for problem $(P_{\lambda,w})$. It should then be proved that the limit of the approximated sequence is not zero.

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