Rate of Change of Momentum.—The object of this note is to suggest how Newton's Laws of Motion may be approached explicitly by means of rate of change of momentum. It will be seen that the term "force" is not made prominent at the expense of the notion of rate of change of momentum. Once the student believes that rate of change of momentum is a vital aspect of stopping or starting the motion of a body, he is not inclined to overestimate his knowledge of "force" apart from the rate of change of momentum which accompanies it. Since students of ordinary endowment learn to observe things before words, we begin with some common examples. As for the term "momentum," it is sufficient to begin by saying that

$$\text{momentum} = \text{mass} \times \text{velocity}.$$

The student experienced in using a balance will not cavil about the term "mass."

(1) In catching a cricket ball, a force is required. There are two ways of catching the ball, one which hurts and one which does not; in the first case the force that acts to stop the ball is great, in the second case this force is small. Now in the first case the momentum mv is *suddenly* reduced to zero by keeping the hands still; in the second case the same momentum is *slowly* reduced to zero by bringing the hands away in the line of travel of the ball. There is the same change of momentum in the two cases but not the same *rate* of change of momentum. The rate of change is greatest when the ball hurts most. If a cannon ball, mass M, has the same velocity v as the cricket ball and both are stopped in the same time, the *time* taken to reduce the momentum Mv to zero is the same as the *time* taken to reduce the momentum mv to zero; but the change of momentum is Mv as against mv, therefore the rate of change of momentum of the cannon ball is greater than the rate of change of momentum of the cricket ball; it is harder to stop the cannon ball—a greater "force" is required.

(2) When one end of a see-saw comes near the ground, the knees are bent in order that momentum may be lost slowly; then the force that stops the see-saw is small. If inadvertently the see-saw is suddenly stopped by the ground, there is an unpleasant bump, for there is a sudden change of momentum; we say the "force" acting on the see-saw is large.

(3) In jumping from the top of a wall to the ground the usual practice of bending the knees when the ground is reached is explained by the fact that an attempt is made to have as small a rate of change of momentum as possible. When this is the case, the jump is made with a sense of less "force"; when this is not the case, that is, when the momentum is quickly destroyed, the "force" is so large as to be unpleasant. It would be dangerous to drop to the ground from the height of an ordinary table, if the knees were kept stiff.

(4) The action of spring or hydraulic buffers at a railway terminus is explained by the principle that if a large amount of momentum has to be destroyed it should be done slowly to prevent damage. The hands in (1) and the knees in (2) and (3) act as buffers.

(5) If a boy drops from a low table on to the platform of a weighing machine, a large force acts on the machine. It may be measured roughly by setting the machine so that there is little or no kick when the jump is made. If the boy now takes off from the platform so as just to reach the table again, will the force acting on the machine in the take-off be greater or less than or the same as that acting on the machine in the drop? The way to settle the question is to compare the two rates of change of momentum; m and v are the same, therefore the changes of momenta are the same, so that the question of *time* is all-important. According to experiments made in a gymnasium less time is occupied in taking off than in finishing the drop, so that the machine in general records a greater force in the first case than in the second. By taking off very slowly and dropping stiffly a gymnast could reverse this result—indeed a larger or smaller force means a greater or less rate of change of momentum.

(6) Quitting a street car by running along and holding the rail produces no discomfort, because the momentum of the passenger is lost slowly. If a person standing on the street grasps the rail of a passing car and mounts all of a sudden, he is getting up a speed equal to that of the car in a very short time, his rate of change of momentum is great, the pull on his arm may be so great as to make him think his arm is being pulled off. The same effect is noticed if the car suddedly bolts forward while he is mounting the stair to

the top, using the hand-rail. An even worse sensation results if a passenger is coming down the stair of a car in motion when it suddenly increases its speed, especially if his right hand is not free to grasp the rail.

(7) A small box set on a shelf fixed to the inside of a door will be upset if the door is suddenly closed, for the rate of change of momentum required for it to keep its place may easily exceed the product of its weight and the coefficient of friction.

(8) When a motor car is moving with constant speed in a circle on a flat road, is there a rate of change of momentum? The magnitude of the momentum mv remains unchanged; if we conclude that there is therefore no change of momentum, and therefore no rate of change of momentum, then it would be the most natural thing in the world for a car travelling uniformly along a straight road to suddenly turn a corner at the same speed. This leads to an examination of change of momentum when the *direction* but not the magnitude of a momentum is changed. The parallelogram of momenta gives the change of momentum in this case.

The Laws of Motion might now be taken as summing up the principles that underlie these and similar examples. When there is no rate of change of momentum, as in the case of a body at rest, or that of a curling stone moving uniformly in a straight path on smooth ice, there is no force acting. When there is a rate of change of momentum, it is measured by

$$(mv' - mv)/t = m \,.\, (v' - v)/t = ma.$$

The unit rate of change of momentum is therefore most simply defined as that of a body of mass 1 lb. moving with a uniform acceleration of 1 ft. per sec. per sec. This unit is the poundal. Equivalent statements are obtained by using the word force as a synonym for rate of change of momentum, giving the usual formula $f = ma$ poundals.

The poundal, if first introduced as a unit of rate of change of momentum and then synonymously as a unit of force, does not come into opposition with any preconceived common unit of force such as the "pound" or "pound-weight" or "weight of 1 lb." But both the poundal and the weight of 1 lb. are units of force, so that there must be a relation between them, as between the inch and the foot. The weight of 1 pound is simply its tendency to fall or, more exactly, its rate of change of momentum when it is falling. The mass is then 1 and the acceleration is 32, so that

$$\begin{aligned}\text{the weight of 1 lb.} &= 32 \text{ units of rate of change of momentum}\\ &= 32 \text{ poundals,}\end{aligned}$$

and, in general,

$$\text{weight of } m \text{ lbs.} = mg \text{ poundals.}$$

An alternative form of the equation $f = ma$ is

$$\frac{F}{W} = \frac{a}{g},$$

where the force F acting on a body and the weight W of the body are in the same units. In this way the term poundal may be avoided. The use of this second form, with all its advantages, introduces into the working of even such a simple problem as that of "Atwood's Machine" some algebraic forms that are rather annoying, so that an attempt is sometimes made to avoid the poundal and yet retain the simplicity of the formula $f = ma$. Write the equation $F/W = a/g$ in the form

$$F = \left(\frac{W}{g}\right) . a = ka.$$

What is k? If a is in ft.-sec. units, g is 32, so that the measure of k is $\frac{1}{32}$ of W. If W is the weight of W lbs., F is equal to the weight of ka lbs.; if W is the weight of W tons, F is equal to the weight of ka tons; k is called the number of *slugs*. If we started by defining momentum as kv where v is measured in ft.-sec. units and k in slugs we should, instead of $f = ma$, have $F = ka$ directly, where F would be in lbs.-weight, say.

One more addition may be made. In the calculation of force we usually explicitly calculate rate of change of momentum, really using "force" as a contracted form, so that all the "forces" such as weight, frictional resistance, reaction, etc., are in units of rate of change of momentum, by hypothesis.

Let a particle, of mass m, move round a circle, centre O and radius r, with uniform speed v; to find the rate of change of momentum at the moment when the particle is at P. Let Q be another position of the particle.

The momentum at P $= mv$, along the tangent,

$= mv$, perpendicular to OP.

The momentum at Q $= mv$, perpendicular to OQ.

Hence OP, OQ may be taken to *represent* the momenta at P and Q; for the momenta are equal in magnitude and OP = OQ, and the directions of the momenta are perpendicular to OP and OQ.

The change of momentum is then *represented* in magnitude and direction by PQ, so that its direction is actually perpendicular to PQ; let us confine ourselves to the magnitude in the meantime. Now the number of inches in the length of the line from E (Edinburgh) to G (Glasgow) on a Map of Scotland is not likely to be the number of miles between Edinburgh and Glasgow, so that we cannot say

length of EG = distance from Edinburgh to Glasgow

nor length of EA = distance from Edinburgh to Aberdeen.

But

$$\frac{\mathrm{EG}}{\mathrm{EA}} = \frac{\text{dist. from Edinburgh to Glasgow}}{\text{dist. from Edinburgh to Aberdeen}}$$

is a true equation.

So the actual change of momentum between P and Q is not equal to PQ.

But

$$\frac{\text{actual change of momentum}}{\text{actual momentum at P}} = \frac{\mathrm{PQ}}{\mathrm{OP}},$$

therefore actual change of momentum

$$= \text{actual momentum at P} \times \frac{\mathrm{PQ}}{\mathrm{OP}}$$

$$= mv \cdot \frac{\mathrm{PQ}}{r} \cdot = \frac{mv}{r} \cdot \mathrm{PQ}.$$

Hence the average rate of change of momentum in the time t taken to go from P to Q

$$= \frac{mv}{r} \cdot \frac{\mathrm{PQ}}{t}$$

$$= \frac{mv}{r} \cdot \frac{\mathrm{arcPQ}}{t} \cdot \frac{\mathrm{PQ}}{\mathrm{arcPQ}}$$

$$= \frac{mv}{r} \cdot v \cdot \frac{\mathrm{PQ}}{\mathrm{arcPQ}} = \frac{mv^2}{r} \cdot \frac{\mathrm{PQ}}{\mathrm{arcPQ}}.$$

When PQ is all but zero, $\dfrac{\text{PQ}}{\text{arc}\,\text{PQ}}$ is all but 1, so that the average rate of change of momentum is all but $\dfrac{mv^2}{r} \times 1$.

Assuming then that there is a definite rate of change of momentum *at the moment* when the particle is at P, we see that this is measured *exactly* by $\dfrac{mv^2}{r} \times 1$ or $\dfrac{mv^2}{r}$.

Now use force in place of rate of change of momentum and we get

$$f = \frac{mv^2}{r} \text{ units of rate of change of momentum,}$$

$$= \frac{mv^2}{r} \text{ poundals.}$$

The average rate of change of momentum is *represented* in direction by PQ, and is therefore actually perpendicular to PQ. Hence the direction of the force *at P* is from P to O.

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