ON THE POSSIBLE FORMS OF DIFFERENTIAL EQUATION WHICH CAN BE FACTORIZED BY THE SCHRÖDINGER-INFELD METHOD

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1. Introduction. The factorization method, initiated by Schrödinger [**4**] and modified and developed by Infeld [**2**], Duff [**1**], and Infeld and Hull [**3**], furnishes an elegant method of solving eigenvalue problems associated with certain ordinary differential equations of the second order. Not only the eigenvalues and eigenfunctions can thus be obtained, but also certain matrix elements associated with the eigenfunctions. Even if the method cannot be applied directly to eigenvalue problems, the factorization of an equation may still be of interest, since recurrence formulae may thus be established, e.g. for Bessel functions [**3**]. The connection of the method with Truesdell's [**5**] method of the "F-equation" has been discussed by Duff [**1**].

It is therefore of interest to give explicit forms to those differential equations which can be factorized. In the Infeld form of the factorization procedure—which is the only one we shall consider—this is equivalent to finding the solution of a certain differential-difference equation.

2. The form of $k_m(x)$. Infeld's form of the factorization procedure is as follows. The differential equation is written in the form

$$y'' + [r(x, m) + \lambda]y = 0, \tag{2.1}$$

where m is a parameter which can vary continuously .The equation (2.1) can then (by definition) be factorized if and only if it can be written in the two equivalent forms

$$\left[\frac{d}{dx} - k(x, m)\right]\left[\frac{d}{dx} + k(x, m)\right]y = \left[L(m) - \lambda\right]y, \tag{2.2}$$

$$\left[\frac{d}{dx} + k(x, m+1)\right]\left[\frac{d}{dx} - k(x, m+1)\right]y = \left[L(m+1) - \lambda\right]y, \tag{2.3}$$

where $L(m)$ is some function of m. The necessary and sufficient condition that (2.2) and (2.3) shall give the same differential equation (2.1) is easily found to be

$$k_{m+1}^2 - k_m^2 + k'_{m+1} + k'_m = L_m - L_{m+1}, \tag{2.4}$$

where we write for brevity $k_m = k(x, m)$, $L_m = L(m)$, and the dashes stand for differentiation with respect to x. The function k_m must therefore satisfy the differential-difference equation

$$\frac{d}{dx}\,(k_{m+1}^2 - k_m^2 + k'_{m+1} + k'_m) = 0. \tag{2.5}$$

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Having found k_m from (2.5), L_m is given (to within an additive constant) from (2.4), and the function $r(x, m)$ in (2.1) is then given by

$$r(x, m) = k'_m - k_m^2 - L_m. \tag{2.6}$$

Thus all differential equations which can be factorized can be found if we can find all solutions of (2.5). We are only interested in solutions[1] which are continuous functions of m and x.

Particular solutions of (2.5) have been found by Infeld and Hull [**3**] and Duff [**1**] under the assumption that k_m is the sum of a finite number of positive and negative powers of m whose coefficients are functions of x. We shall solve (2.5) under a much more general assumption, namely that the differential equation (2.1) can be transformed into one with rational coefficients. To be more precise, our assumption is the following: *It is possible to find a transformation to new variables z, t defined by*

$$\frac{dt}{dx} = f(t), \qquad y = g(t)z, \tag{2.7}$$

where $f(t)$ is independent of m, such that the equation (2.1), *when multiplied by an appropriate function of t, becomes, for all* λ *and m, an equation whose coefficients are rational functions of t.*

While we do not thus find the most general solution of (2.5) (which seems to be quite hard to find), our solution is nevertheless one of considerable generality, since all equations of interest in applications up to the present appear to be transformable in the above way, though the assumption that $f(t)$ is independent of m imposes a certain additional restriction.

Substituting (2.7) in (2.2), which by supposition is equivalent to (2.1), we find that (2.1) becomes

$$f^2 g\ddot{z} + (2f^2\dot{g} + f\dot{f}g)\dot{z} + [f^2\ddot{g} + f\dot{f}\dot{g} + g(f\dot{k}_m - k_m^2 - L_m + \lambda)]z = 0,$$

where the dots denote differentiation with respect to t. According to our assumption, the ratios of the coefficients of $\ddot{z}$, $\dot{z}$, z, λz are rational functions of t. Using the fact that the sum, product, etc. of two rational functions is also a rational function, and that the derivative of a rational function is also rational, it is then easily shown that the functions f^2, $f\dot{k}_m - k_m^2$ are rational functions of t. Similarly, using (2.3), we find that $f\dot{k}_{m+1} + k_{m+1}^2$, and hence also $f\dot{k}_m + k_m^2$, is a rational function. Hence f^2, $f\dot{k}_m$, k_m^2 are rational functions, from which it follows that k_m/f is also rational. Thus we have

$$f(t) = [R(t)]^{\frac{1}{2}}, \quad k_m(t) = R_m(t)[R(t)]^{\frac{1}{2}}, \tag{2.8}$$

where $R(t)$, $R_m(t)$ are rational functions of t, of which $R(t)$ is independent of m.

[1]The whole factorization procedure becomes quite trivial if solutions which are discontinuous in m are allowed. We may, for instance, define $k(x,m)$ *arbitrarily* for $0 < m < 1$. By successive use of (2.5) with different values of m, we may then determine a function $k(x,m)$ which formally satisfies (2.5), but which is, in general, discontinuous for $m = 0, \pm 1, \pm 2, \ldots$. Such solutions are, of course, of no use in eigenvalue problems.

Using (2.8) and the first of (2.7), (2.5) becomes

$$\frac{d}{dt}\left[R_{m+1}^2 - R_m^2 + \dot{R}_{m+1} + \dot{R}_m + \frac{\dot{R}}{2R}(R_{m+1} + R_m)\right] = 0. \tag{2.9}$$

We now consider the solution of (2.9) in the neighbourhood of the singularities of $R_m(t)$, which can only be poles. Let us first see if these poles can depend on m. The poles of $R_m(t)$ are given by the roots of a polynomial equation in t, say

$$t^N + a_1(m)t^{N-1} + \ldots + a_N(m) = 0. \tag{2.9a}$$

Since $R_m(t)$ is a continuous function of m, the degree N of (2.9a) is independent of m. Let the distinct roots of (2.9a) be $t_1(m), \ldots, t_M(m)$, where $M \leqslant N$. We may assume that these roots remain distinct from each other as m varies, except possibly for special values of m.

Now by considering the solution of (2.9) in the neighbourhood of a pole of $R_m(t)$, it is easily seen that at least one of $R_{m+1}(t)$ or $R_{m-1}(t)$ must have a pole of the same order at the same point. Hence each of the functions $t_i(m)$ must be equal either to one of the $t_i(m+1)$ or to one of the $t_i(m-1)$. Suppose that $M'(0 \leqslant M' \leqslant M)$ of the $t_i(m)$ are *not* equal to any of the $t_i(m+1)$, but that each of the remaining $M - M'$ of the $t_i(m)$ are equal to one of the $t_i(m+1)$. Then there are M' of the $t_i(m+1)$ which are not equal to any of the $t_i(m)$; each of them must therefore be equal to one of the $t_i(m+2)$. Also, since $M - M'$ of the $t_i(m)$ are equal to one of the $t_i(m+1)$, each of the remaining $M - M'$ of the $t_i(m+1)$ must be equal to one of the $t_i(m+2)$. Hence *every* $t_i(m+1)$ $(i = 1, \ldots, M)$ is equal to one of the $t_i(m+2)$, and M' must be zero.

Thus when m is changed into $m+1$, the roots of (2.9a) can only undergo a permutation among themselves, multiplicities being preserved. Hence any symmetric function of the roots of (2.9a) is unaltered when m is changed into $m+1$. The same is therefore true of the coefficients $a_1(m), \ldots, a_N(m)$ occurring in (2.9a). Thus these coefficients, and hence also the poles of $R_m(t)$, are periodic functions of m with period 1.

In what follows we shall, for simplicity, assume that m is an integer, as is indeed assumed by the authors previously mentioned. There is no loss of generality in this, since only values of m which differ by an integer occur in (2.5), and since if $k(x, m)$ is any solution of (2.5) then $k(x, m+c)$ is also a solution, where c is an arbitrary constant (similar remarks apply to (2.9) and its solutions $R(x, m)$). Our final results will hold for arbitrary m, provided arbitrary constants are replaced by arbitrary continuous functions of m of period 1 and the function $(-1)^m$ which occurs in some of the solutions is replaced by $e^{im\pi}$. Our solutions are then continuous functions of m, as required. We may now, according to what has been shown above, treat the poles of $R_m(t)$ as being independent of m.

Suppose, then, that in the neighbourhood of any pole of $R_m(t)$ in the finite part of the t-plane, say $t = t_1$, we have the expansions

$$(2.10)\qquad R_m(t) = \sum_{s=0}^{\infty} a_s(m)(t-t_1)^{-p+s}, \quad \frac{\dot{R}}{2R} = \sum_{s=0}^{\infty} b_s(t-t_1)^{-1+s},$$

where $p \geqslant 1$, $a_0(m) \neq 0$ (b_0 may be zero). Let us, as usual, denote by $\triangle$ the difference operator ($\triangle u_m = u_{m+1} - u_m$), and let us put $\triangle' = \triangle + 2$. Substituting (2.10) in (2.9) and equating to zero the coefficients of the p lowest powers of $(t - t_1)$, we obtain

$$(2.11)\qquad \begin{aligned} &\triangle\Big(\sum_{i=0}^{s} a_i\, a_{s-i}\Big) = 0, && s = 0,1,\ldots,p-2,\\ &\triangle\Big(\sum_{i=0}^{p-1} a_i\, a_{p-1-i}\Big) + (b_0 - p)\triangle' a_0 = 0. \end{aligned}$$

If $p = 1$, only the second of (2.11) applies. Since $\triangle(a_0^2) = \triangle a_0 \,.\, \triangle' a_0$, we see that these equations, solved in succession for $a_0, a_1, \ldots, a_{p-1}$, give: either

$$a_s(m) = c_s(-1)^m, \qquad s = 0,1,\ldots,p-1,$$

or

$$\begin{cases} a_s(m) = c_s, & s = 0,1,\ldots,p-2,\\ a_{p-1}(m) = c\,m + c', \end{cases}$$

where c_s, c, c' are arbitrary constants independent of m. Thus the coefficients of the negative powers of $(t - t_1)$ in the expansion of $R_m(t)$ in the neighbourhood of the pole $t = t_1$ are either linear functions of m or proportional to $(-1)^m$. If $t = \infty$ is a pole of $R_m(t)$, a similar investigation shows that the coefficients of the positive powers of t in the expansion of $R_m(t)$ in the neighbourhood of $t = \infty$ are also either linear functions of m or proportional to $(-1)^m$. We see, then, that the function $\triangle^2 \triangle' R_m(t)$ is an analytic function everywhere in the t-plane, which must, therefore, be a constant. Hence

$$(2.12)\qquad \triangle^2 \triangle' R_m(t) = \lambda_m.$$

Regarding (2.12) as a difference equation for $R_m(t)$, the general solution can be written

$$R_m(t) = A_1(t) + mB_1(t) + (-1)^m C_1(t) + \mu_m,$$

where μ_m is a particular integral of (2.12) which is a function of m only. From (2.8), and returning to the variable x, we see therefore that $k_m(x)$ must be of the form

$$(2.13)\qquad k_m(x) = A(x) + m\,B(x) + (-1)^m C(x) + \mu_m D(x).$$

3. The various types. It remains to find the possible forms of the functions $A(x), \ldots, D(x)$ in (2.13). If (2.13) be substituted in (2.5), we see that, regarded as an equation in m, it is a linear relation between the following 11 functions of m:

$$(3.1)\qquad \begin{aligned} &\mu_{m+1}^2 - \mu_m^2, \quad m\,\mu_{m+1}, \quad (-1)^m \mu_{m+1}, \quad \mu_{m+1},\\ &m\,\mu_m, \quad (-1)^m \mu_m, \quad \mu_m, \quad m, \quad m(-1)^m, \quad (-1)^m, \quad 1. \end{aligned}$$

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If these functions are all linearly independent (for integral values of m), then the coefficient of each of them can be equated to zero in (2.5). This gives $D =$ constant, and μ_m is arbitrary. In the contrary case, there must be one or more linear relations between the functions (3.1), and hence μ_m is not arbitrary.

The first function, $\mu_{m+1}^2 - \mu_m^2$, in (3.1) may be linearly independent of the others. In this case, (2.5) gives again $D =$ constant. In the contrary case, $\mu_{m+1}^2 - \mu_m^2$ is expressible as a linear function of the other 10 functions in (3.1), and (2.5) then becomes a linear relation between these ten functions (whose coefficients are functions of x). These ten functions cannot be linearly independent, since otherwise μ_m would be arbitrary. There must therefore exist at least one linear relation between them. If this relation involves μ_m but not μ_{m+1} (or vice versa), it gives

$$(3.2)\qquad \mu_m = \frac{a + m\,b + (-1)^m c + m(-1)^m d}{a' + m\,b' + (-1)^m c'},$$

where $a, \ldots, c'$ are constants independent of m. If, on the other hand, the relation involves both μ_m and μ_{m+1}, it can be written in the form

$$(3.3)\quad \mu_{m+1} = \frac{a_1 + m\,b_1 + (-1)^m c_1}{a_3 + m\,b_3 + (-1)^m c_3}\mu_m + \frac{a_2 + m\,b_2 + (-1)^m c_2 + m(-1)^m d_2}{a_3 + m\,b_3 + (-1)^m c_3}.$$

Substituting for μ_{m+1} from (3.3) in (2.5), after having eliminated $\mu_{m+1}^2 - \mu_m^2$ in the manner indicated, it becomes a linear relation between the functions

$$m^2\mu_m, \quad m^2(-1)^m\mu_m, \quad m\mu_m, \quad (-1)^m\mu_m, \quad \mu_m,$$
$$m^2, \quad m^2(-1)^m, \quad m(-1)^m, \quad (-1)^m, \quad m, \quad 1.$$

These functions again cannot be linearly independent. Hence we must have

$$(3.4)\quad \mu_m = \frac{c_1 m^2 + c_2 m^2(-1)^m + c_3 m + c_4 m(-1)^m + c_5(-1)^m + c_6}{c'_1 m^2 + c'_2 m^2(-1)^m + c'_3 m + c'_4(-1)^m + c'_5}.$$

Since (3.2) is included in (3.4), we see that (3.4) is the most general form possible for μ_m (unless μ_m is arbitrary).

We now write $(m - 1)$ in place of m in (2.5) and subtract this from the original (2.5). Writing first $2m$, and then $2m - 1$, in place of m in the resulting equation, we obtain the two equations

$$(3.5)\qquad \frac{d}{dx}\Big[(\xi_m^2 - \xi_{m-1}^2)D^2 + (f_m\xi_m - f_{m-1}\xi_{m-1}) + (f_m^2 - f_{m+1}^2) + (\xi_m + 2\eta_m + \xi_{m-1})D'\Big] = 0,$$

$$(3.6)\qquad \frac{d}{dx}\Big[(\eta_m^2 - \eta_{m-1}^2)D^2 + (g_m\eta_m - g_{m-1}\eta_{m-1}) + (g_m^2 - g_{m-1}^2) + (\eta_m + 2\xi_m + \eta_{m-1})D'\Big] = 0,$$

where

$$(3.7)\qquad \xi_m = \mu_{2m+1} = \frac{am^2 + bm + c}{\alpha m^2 + \beta m + \gamma}, \qquad \eta_m = \mu_{2m} = \frac{a'm^2 + b'm + c'}{\alpha' m^2 + \beta' m + \gamma'},$$

$$f_m = (A + B - C) + 2mB, \qquad g_m = (A + C) + 2mB.$$

The constants $a, \ldots, \gamma'$ in (3.7) are related to the constants $c_1, \ldots, c'_5$ occurring in (3.4). From (3.7) we note that

$$(3.8)\qquad \mu_m = \tfrac{1}{2}\Big[\xi_{\frac{1}{2}(m-1)} + \eta_{\frac{1}{2}m}\Big] - \tfrac{1}{2}(-1)^m\Big[\xi_{\frac{1}{2}(m-1)} - \eta_{\frac{1}{2}m}\Big].$$

We suppose that ξ_m, η_m in (3.7) are expressed in their lowest terms.

If $\alpha = \beta = \alpha' = \beta' = 0$, then ξ_m, η_m are polynomials in m of degree two (or less). Leaving this case aside for the moment, at least one of ξ_m, η_m, say ξ_m, must have a pole of at least the first order at some point in the complex m-plane, say $m = m_0$. Then $\xi_m{}^2 - \xi_{m-1}{}^2$ has at least poles of the second order at $m = m_0$ and $m = m_0 + 1$. On the other hand, η_m can have at most a pole of the second order at one of these points. It then follows that $\xi_m{}^2 - \xi_{m-1}{}^2$ is linearly independent of all the other functions of m occurring in (3.5).[2] This equation therefore requires that D is constant. Similarly, if η_m has a pole, (3.6) shows that D is constant.

There remains to be considered the case where ξ_m, η_m are at most quadratics in m, say

$$(3.9)\qquad \xi_m = am^2 + bm + c, \qquad \eta_m = a'm^2 + b'm + c'.$$

Equating to zero the coefficients of m^3 in (3.6), (3.5), we then find that either D is constant or $a = a' = 0$. In the latter case we have, from (3.9) and (3.8),

$$\mu_m = \alpha m + \beta + (-1)^m(\gamma m + \delta),$$

where $\alpha, \ldots, \delta$ are constants. From (2.13), $k_m(x)$ can then be written

$$(3.10)\qquad k_m(x) = A(x) + m\,B(x) + (-1)^m C(x) + m(-1)^m D(x).$$

Apart from this case, which we leave aside for the moment, we have shown that we can take D to be constant without loss of generality, so that from (2.13) we can always write

$$(3.11)\qquad k_m(x) = A(x) + m\,B(x) + (-1)^m C(x) + \mu_m.$$

It is now convenient to write (2.5) in the equivalent form

$$(3.12)\qquad \frac{d}{dx}\Big[\sum_{m_0}^{m}(k'_{m+1} + k'_m) + (k_{m+1}^2 - k_{m_0}^2)\Big] = 0$$

obtained by summing (2.5) with respect to m from m_0 to m, where m_0 is any particular value of m.

[2]The relation (3.5) need only, according to our assumptions, be satisfied for integral values of m, but since, when cleared of fractions, it becomes a polynomial relation in m, it must actually be satisfied for all values of m. The same holds for (3.6).

Substituting (3.11) in (3.12) we get

$$2\mu_{m+1}P' + Q' = 0, \tag{3.13}$$

where

$$P = (A + B) + m\,B - (-1)^m C,$$
$$Q = m^2F + 2m(A' + AB + F) - 2m(-1)^m BC - 2(-1)^m C(A + B) - f(m_0,x),$$
$$F = B^2 + B',$$

and $f(m_0, x)$ is a function which need not be given explicitly. Differentiating (3.13), and eliminating μ_{m+1} between (3.13) and the equation thus obtained we have

$$P'Q'' - Q'P'' = 0. \tag{3.14}$$

In (3.14) only known functions of m occur, so that we can equate coefficients of independent functions of m separately to zero to get a number of differential equations for the functions A, B, C. We thus find

$$\begin{aligned} &F'B'' - F''B' = 0,\\ &F'A'' - F''A' + (A' + AB)'B'' - (A' + AB)''B' = 0,\\ &F'C'' - F''C' + 2\,(BC)'\,B'' - 2\,(BC)''B' = 0, \end{aligned} \tag{3.15}$$

obtained by equating to zero the respective coefficients of m^3, m^2, $m^2\,(-1)^m$ in (3.14). These determine B, A, C in succession.

Having determined the possible forms for A, B, C, the requirement that (2.4) must be satisfied identically in x then determines μ_m and L_m, and also restricts the arbitrary constants occurring in these forms.

First case. If B is not a constant, the solutions of (3.15) are:

$$B = a - b\tan(bx + c),$$

$$A = b_1\tan(bx + c) + c_1e^{ax}\sec(bx + c),$$

$$C = b_2\cot(bx + c);$$

or $\quad B = a + \dfrac{1}{x + b}, \qquad A = \dfrac{b_1}{x + b} + \dfrac{c_1e^{ax}}{x + b}, \qquad C = b_2x;$

or $\quad B = \dfrac{1}{x + b}, \qquad A = \dfrac{b_1}{x + b} + c_1x, \qquad C = b_2x.$

Substitution in (2.4) now gives the following solutions, in which we give also the function $r(x,m)$ as given by (2.4):

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$$\text{I}\begin{cases} k(x,m) = -(m+a)b\tan(bx+c) - d/(m+a), \\ L(m) = (m+a)^2b^2 - d^2/(m+a)^2, \\ r(x,m) = -(m+a)(m+a+1)b^2\sec^2(bx+c) - 2bd\tan(bx+c); \end{cases}$$

$$\text{II}\begin{cases} k(x,m) = -(m+a)b\tan(bx+c) + d\sec(bx+c), \quad L(m) = (m+a)^2b^2, \\ r(x,m) = -[(m+a)(m+a+1)b^2 + d^2]\sec^2(bx+c) \\ \qquad\qquad + bd[2(m+a)+1]\sec(bx+c)\tan(bx+c); \end{cases}$$

$$\text{III}\begin{cases} k(x,m) = -(m+a)b\tan(bx+c) + (-1)^m d\cot(bx+c), \\ L_m = (m+a)^2\, b^2 + 2(-1)^m bd(m+a), \\ r(x,m) = -(m+a)(m+a+1)b^2\sec^2(bx+c) \\ \qquad\qquad - d[d + b(-1)^m]\operatorname{cosec}^2(bx+c) + d^2; \end{cases}$$

$$\text{IV}\begin{cases} k(x,m) = (m+a)/(x+b) - c/(m+a), \qquad L(m) = -c^2/(m+a)^2, \\ r(x,m) = -(m+a)(m+a+1)/(x+b)^2 + 2c/(x+b); \end{cases}$$

$$\text{V}\begin{cases} k(x,m) = (m+a)/(x+b) - c(x+b), \qquad L(m) = 4cm, \\ r(x,m) = -(m+a)(m+a+1)/(x+b)^2 - c^2(x+b)^2 \\ \qquad\qquad - c[2(m-a)+1]; \end{cases}$$

$$\text{VI}\begin{cases} k(x,m) = (m+a)/(x+b) - (-1)^m c(x+b), \\ L(m) = 2(-1)^m\, c(m+a), \\ r(x,m) = -(m+a)(m+a+1)/(x+b)^2 - c^2(x+b)^2 - (-1)^m c. \end{cases}$$

Second case. If B is constant, which can without loss of generality be taken equal to zero, since we may absorb a constant in μ_m in (3.11), the equations (3.15) are then satisfied identically, but we can use the equations

$$A'''A' - A''^2 = 0, \quad A'''C' - A''C'' = 0 \tag{3.16}$$

obtained by equating the coefficients of m, $m(-1)^m$ in (3.14) to zero. Solving (3.16) for A and C we get: either

(i) $A = a + b\, e^{cx}$, $\quad C = a' + b'\, e^{cx}$,

or

(ii) $A = a + bx$, $\quad C$ arbitrary.

Substitution in (2.4) now gives the additional solutions:

$$\text{VII}\begin{cases} k(x,m) = a + bm + ce^{-bx}, \qquad L(m) = -(a+bm)^2, \\ r(x,m) = -c[(2mb + 2a + b)e^{-bx} + ce^{-2bx}]; \end{cases}$$

$$\text{VIII}\begin{cases} k(x,m) = (a+bx) + (-1)^m c/(a+bx), \; L(m) = -2bm - 2c(-1)^m, \\ r(x,m) = -(a+bx)^2 - c[c + b(-1)^m]/(a+bx)^2 + (2m+1)b; \end{cases}$$

$$\text{IX}\begin{cases} k(x,m) = f(m), \text{ where } f \text{ is an arbitrary function of } m,\ L(m) = \{f(m)\}^2, \\ r(x,m) = 0; \end{cases}$$

$$\text{X}\begin{cases} k(x,m) = (-1)^m C(x), \text{ where } C \text{ is an arbitrary function of } x,\ L(m) = 0, \\ r(x,m) = (-1)^m C'(x) - [C(x)]^2. \end{cases}$$

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We now return to the case where $k_m(x)$ is given by (3.10). Substituting in (2.5), and equating to zero the coefficients of the various independent functions of m, we have

$$(3.17)\quad BD = c_1,\ BC + AD = c_2,\ B^2 + B' + D^2 = c_3,\ 4AC + D' = c_4,$$
$$A' + AB + CD = c_5,$$

where $c_1, \ldots, c_5$ are arbitrary constants. If $B = 0$ or $c_1 = 0$ we see that $D =$ constant. If $B \neq 0$, $c_1 \neq 0$, it is possible to eliminate D, C, A', D' between the above equations, and obtain two equations between A and B which must be compatible. These equations are

$$4A^2 - 4\frac{c_2}{c_1}BA + \left(\frac{c_4}{c_1} - 1\right)B^2 - \frac{c_1^2}{B_2} + c_3 = 0,$$

$$4\left[B - \frac{c_1^2}{B^3}\right]A^2 + 2\left(\frac{c_2c_3}{c_1} - 2c_5 + \frac{2c_1c_2}{B^2} - \frac{2c_2}{c_1}B^2\right)A + 2\left(\frac{c_2c_5}{c_1}B - \frac{c_2^2}{B}\right)$$
$$+ \left(B^2 + \frac{c_1^2}{B^2} - c_3\right)\left[\left(\frac{c_4}{c_1} - 1\right)B + \frac{c_1^2}{B^3}\right] = 0.$$

They are compatible if

$$c_4 = c_2^2/c_1, \qquad c_5 = c_2\, c_3/2c_1,$$

where c_1, c_2, c_3 may be arbitrary.

The general solution of (3.17) can then be written

$$(3.18)\quad \begin{cases} \displaystyle\int \frac{B^2}{B^4 - c_3B^2 + c_1^2}\, dB = -x + \text{constant}, \\ 4A^2 - 4\dfrac{c_2}{c_1}BA + \left(\dfrac{c_2^2}{c_1^2} - 1\right)B^2 - \dfrac{c_1^2}{B^2} + c_3 = 0, \\ D = c_1/B, \quad C = (c_2/B) - (c_1A/B^2). \end{cases}$$

Thus we have the additional solution of (2.4):

$$\text{XI}\begin{cases} k_m(x) = A(x) + m\,B(x) + (-1)^m C(x) + m(-1)^m D(x), \\ L(m) = -[c_3\, m^2 + (c_2\, c_3/c_1)m + 2(-1)^m\{c_1\, m^2 + c_2\, m + c_2^2/4c_1\}], \\ r(x,m) = m(m+1)B' + (2m+1)A' + (-1)^m[(m+\tfrac{1}{2})D' + C'] \\ \qquad\qquad - (A^2 + C^2), \end{cases}$$

where A,B,C,D are given by (3.18).

The above types I–XI exhaust all the possible solutions under our assumption. They include as special cases all those which have been given previously [**1**, **3**] Of these eleven types IX, X are trivial, though X is perhaps of some interest as it shows that any equation of the form

$$(3.19)\qquad y'' + [r(x) + \lambda]y = 0,$$

can be formally factorized by the Infeld procedure, regarding it[3] as a particular

[3]Such a device is termed "artificial factorization" by Infeld and Hull [**3**].

case of the more general form (2.1) for a particular value of m. For a function $C(x)$ can always be found such that

$$C'(x) - [C(x)]^2 = r(x), \tag{3.20}$$

and then (3.19) coincides with X for any even m.

Of the remaining types, V, VI, and VII lead to differential equations which can be made to coincide for special values of m by proper choice of the arbitrary constants or by absorbing constants that occur in $r(x,m)$ in the parameter λ. They thus represent different ways of factorizing the same differential equation, namely one of the type

$$y'' + \left[a(x+b)^2 + \frac{c}{(x+b)^2} + \lambda\right]y = 0. \tag{3.21}$$

Similarly, types II and III are essentially the same, as can be seen by writing $\frac{1}{2}(bx + c)$ instead of $(bx + c)$ in III.

On the other hand XI is an essentially new type, though it is doubtful if it can be applied to eigenvalue problems. The integral occurring in (3.18) can easily be evaluated, but it is not possible in general to express B as an explicit function of x in terms of standard functions.

It remains to be seen whether all these types can be transformed to differential equations with rational coefficients in accordance with our original assumption. This is not necessarily the case, since we have only deduced *necessary* conditions that the differential equation (2.1) shall be thus transformable. It is easily seen that the following substitutions suffice:

For I and III: $\tan(bx + c) = t, \quad dt/dx = b(1 + t^2).$

For II: $\sin(bx + c) = t, \quad dt/dx = b(1 - t^2)^{\frac{1}{2}}.$

For VII: $e^{-bx} = t, \quad dt/dx = -bt.$

IV, V, VI, VIII, IX are already in the required form. It is evident that X cannot in general be so transformed, nor does it seem possible in general to transform XI. In the particular case in which A, as determined by the quadratic equation in (3.18), is a rational function of B (which occurs if $c_3 = \pm 2c_1$), it is possible to transform XI to a differential equation with rational coefficients by means of the substitution

$$B(x) = t, \quad \frac{dt}{dx} = c_3 - t^2 - \frac{c_1^2}{t^2}.$$

It will be seen that, in all the above cases, it is not necessary to change the dependent variable, as envisaged by (2.7), in order to effect the required transformation.

Apart from X and XI, all the differential equations, when reduced to forms with rational coefficients, are either of hypergeometric or confluent hypergeometric type.

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