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Uniqueness of Almost Everywhere Convergent Vilenkin Series

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Abstract. D. J. Grubb [3] has shown that uniqueness holds, under a mild growth condition, for Vilenkin series which converge almost everywhere to zero. We show that, under even less restrictive growth conditions, one can replace the limit function 0 by an arbitrary $f \in L^q$, when $q > 1$.

1 Introduction

Let $\mathbf{N} := \{0, 1, 2, \dots\}$, and $\mathcal{P} := \{p_0, p_1, \dots\}$ be any sequence of integers which satisfies $p_n \geq 2$. For each $n \in \mathbf{N}$ set $P_n := p_0 p_1 \cdots p_{n-1}$, where the empty product is by definition 1. The *multiplicative Vilenkin group* associated with $\mathcal{P}$ is the set $G := \{(x_0, x_1, \dots) : x_k \in \mathbf{N} \text{ and } 0 \leq x_k < p_k\}$ together with the operation

$$x \dot{+} y := (x_0 \oplus y_0, x_1 \oplus y_1, \dots),$$

where $x = (x_0, x_1, \dots)$, $y = (y_0, y_1, \dots)$ and, for each k, $x_k \oplus y_k$ represents the sum of x_k and y_k modulo p_k. The dual group of G is the system $(w_n, n \in \mathbf{N})$, defined for $x = (x_0, x_1, \dots)$, by

$$w_n(x) := \prod_{k=0}^{\infty} \exp\left(\frac{2\pi i n_k x_k}{p_k}\right), \tag{1}$$

where the coefficients n_k are integers which satisfy $0 \leq n_k < p_k$ and $n = \sum_{k=0}^{\infty} n_k P_k$ (see Vilenkin [4] for details). When $p_k := 2$ for all k, the group G is called the *dyadic group* and the characters w_n are called the *Walsh system*. When $p_k = O(1)$, the system $\{w_n\}$ is called a (multiplicative) Vilenkin system of *bounded type*.

It is well known that G is a compact group for each collection of radices $\mathcal{P}$, and that the corresponding *Vilenkin system* $\{w_n\}$ is a complete orthonormal system on G. Moreover, the group G can be identified with the interval $[0, 1)$ by taking an $x = (x_0, x_1, \dots) \in G$ to the number

$$\bar{x} := \sum_{k=0}^{\infty} x_k P_{k+1}^{-1}.$$

Under this identification, Haar measure on G is taken to Lebesgue measure on $[0, 1)$.

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A *Vilenkin series* is a series of the form $S := \sum_{k=0}^{\infty} a_k w_k$, where a_k is some sequence of complex numbers. For each $x \in G$ and $n \in \mathbf{N}$, the *partial sums* of a Vilenkin series S are defined by

$$S_n(x) := \sum_{k=0}^{n-1} a_k w_k(x).$$

The partial sums S_{P_n} form a martingale in $L^1(G)$ which allows one to use martingale convergence theorems on Vilenkin series.

A Vilenkin-Fourier series Sf is a Vilenkin series whose coefficients can be computed by the following formula for some integrable function f:

$$a_k = \widehat{f}(k) := \int_G f\, \overline{w_k}\, dm, \quad k \in \mathbf{N}.$$

We shall prove the following theorem.

Theorem 1 *Suppose that S is a Vilenkin series, $\{n_\nu\}$ is a subsequence of positive integers, $f \in L^q(G)$ for some $q > 1$, and E is a countable subset of G. If*

$$\lim_{\nu\to\infty} S_{P_{n_\nu}}(x) = f(x) \tag{2}$$

for almost every $x \in G$,

$$\limsup_{\nu\to\infty} |S_{P_{n_\nu}}(x)| < \infty, \quad x \notin E, \tag{3}$$

and if

$$\lim_{n\to\infty} P_n^{-1} S_{P_n}(x) = 0, \quad x \in E, \tag{4}$$

then S is the Vilenkin-Fourier series of f.

Theorem 1 is known when f is a finite-valued, integrable function and (2) holds off a countable set rather than almost everywhere (see D. J. Grubb [3], and Bokaev and Skvortsov [1]). Grubb [3] has also shown that if $S_{P_n} \to 0$ almost everywhere on G and (3) holds everywhere on G, then S is the zero series. Clearly, Theorem 1 contains all these results when $f \in L^q$, $q > 1$, *i.e.,* shows that uniqueness holds under mild growth conditions for simultaneously almost everywhere convergence *and* nonzero limits.

The proofs of Grubb and Bokaev and Skvortsov rely heavily on differentiation theory and do not seem to generalize to the case when the almost everywhere limit is nonzero, *i.e.,* cannot be used to prove Theorem 1. Our proof is more primitive (a proof by contradiction), but reveals the essential nature of the growth condition (4) (see the lemma in Section 2).

2 Preliminaries

For each nonnegative integer n, define *intervals of rank n* on G by $I_0(0) := G$, and

$$I_n(j) := \Big\{ x = (x_0, x_1, \dots) \in G : \sum_{k=0}^{n-1} x_k P_{k+1}^{-1} = \frac{j}{P_n} \Big\}$$

for $j = 0, 1, \dots, P_n - 1$, $n = 1, 2, \dots$. Recall that $\{I_n(0)\}_{n=0}^{\infty}$ is a nested sequence of subgroups of G which forms a neighborhood base at the origin, and for each n, $\{I_n(j)\}_{j=0}^{P_n-1}$ is a collection of pairwise disjoint compact sets in G whose union is G (see [4]). In particular, given $x \in G$ and $n \in \mathbf{N}$, there is a unique $0 \le j < P_n$ such that $x \in I_n(j)$. We shall denote this interval by $I_n(x)$.

Denote the Haar measure of a subset E of G by $m(E)$ and Lebesgue measure of a subset E of $[0, 1)$ by $|E|$. Notice that under the identification of G with $[0, 1)$, the interval $I_n(j)$ corresponds to the interval $[\,jP_n^{-1}, (j+1)P_n^{-1})$. In particular, $m(I_n(j)) = P_n^{-1}$ for $0 \le j < P_n$ and $n \in \mathbf{N}$.

It is well known that the partial sums of the Vilenkin-Fourier series of an integrable f satisfy

$$(S_{P_n} f)(x) = P_n \int_{I_n(x)} f\, dm \tag{5}$$

for $n \in \mathbf{N}$. Hence by Lebesgue's differentiation theorem, $S_{P_n} f \to f$ almost everywhere, in $L^q(G)$ norm, $q \ge 1$, and since the indefinite integral is absolutely continuous, we also have

$$\lim_{n\to\infty} P_n^{-1}(S_{P_n} f)(x) = 0 \quad \text{for all } x \in G. \tag{6}$$

To prove Theorem 1, we shall construct a sequence of intervals on which S_{P_n} is nonzero. The following result shows that under hypothesis (4), this construction of intervals can proceed indefinitely, and can do so to avoid any unwanted point x^*.

Lemma *Let $x^* \in G$ and S be a Vilenkin series which satisfies (4) at $x = x^*$. If $S_{P_{n_0}}$ is nonzero on some $I_0 := I_{n_0}(y_0)$, then there is an interval $J \subseteq I_0$ of rank m_0 such that $x^* \notin J$ and $S_{P_{m_0}}$ is nonzero on J.*

Proof If $x^* \notin I_0$, there is nothing to prove. If $x^* \in I_0$, then $I_0 = I_{n_0}(x^*)$. By hypothesis $S_{P_{n_0}}$ is nonzero on I_0. Since each Vilenkin function w_k whose index k satisfies $k < P_{n_0}$ is constant on I_0, it follows that $S_{P_{n_0}}(x) =: \alpha_0$ for all $x \in I_0$, where α_0 is some fixed nonzero constant. In particular, if $\beta_0 := \alpha_0/P_{n_0}$, then

$$S_{P_{n_0}}(x) = P_{n_0}\beta_0 \quad \text{for all } x \in I_0. \tag{7}$$

For each nonnegative integer k, let $I_k := I_{n_0+k}(x^*)$ and let j_k be the index which satisfies $I_{k+1} = I_{n_0+k+1}(j_k)$. Set $\mathcal{W}_k := \{\ell : I_{n_0+k+1}(\ell) \subset I_k,\ \ell \ne j_k\}$. Then $\mathcal{W}_k$ contains exactly $p_{n_0+k} - 1$ integers, $x^* \notin I_{n_0+k}(\ell)$ for all $\ell \in \mathcal{W}_k$, and

$$I_{n_0+k} = I_{n_0+k+1} \cup \bigcup_{\ell\in\mathcal{W}_k} I_{n_0+k+1}(\ell). \tag{8}$$

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For each $\ell \in \mathcal{W}_k$, fix a point x_ℓ in $I_{n_0+k+1}(\ell)$. Suppose for a moment that ν is an integer which satisfies $P_{n_0+k} \le \nu < P_{n_0+k+1}$. By (8) and (1), the set $\{w_\nu(x) : x = x^*$ or $x = x_\ell$ for some $\ell \in \mathcal{W}_k\}$ contains every p_{n_0+k}th root of unity. Since, for any p, the sum of pth roots of unity is zero, it follows that $w_\nu(x^*) = -\sum_{\ell\in\mathcal{W}_k} w_\nu(x_\ell)$ for each $\nu \in [P_{n_0+k}, P_{n_0+k+1})$. In particular

$$S_{P_{n_0+k+1}}(x^*) - S_{P_{n_0+k}}(x^*) = -\sum_{\ell\in\mathcal{W}_k} (S_{P_{n_0+k+1}} - S_{P_{n_0+k}})(x_\ell). \tag{9}$$

Suppose that the lemma is false. Since $x^* \notin I_{n_0+k+1}(\ell)$, it follows that

$$S_{P_{n_0+k+1}}(x) = 0, \quad x \in I_{n_0+k+1}(\ell) \tag{10}$$

for all $\ell \in \mathcal{W}_k$. We shall use this to prove that

$$S_{P_{n_0+k}}(x^*) = P_{n_0+k}\beta_0 \tag{11}$$

for $k = 0, 1, \ldots$. Notice that this will lead to a contradiction. Indeed, by (11),

$$\lim_{k\to\infty} \frac{S_{P_{n_0+k}}(x^*)}{P_{n_0+k}} = \beta_0 \ne 0$$

contrary to hypothesis. It remains to prove (11).

We shall prove (11) by induction on k. By (7), (11) holds for $k = 0$. Suppose that (11) holds for some $k \ge 1$. Fix $\ell \in \mathcal{W}_k$ and $x \in I_{n_0+k+1}(\ell)$. By (10),

$$0 = S_{P_{n_0+k+1}}(x) = (S_{P_{n_0+k+1}}(x) - S_{P_{n_0+k}}(x)) + S_{P_{n_0+k}}(x).$$

Since $S_{P_{n_0+k}}$ is constant on $I_{n_0+k}(x^*)$, it follows from this identity and (11) that

$$S_{P_{n_0+k+1}}(x) - S_{P_{n_0+k}}(x) = -S_{P_{n_0+k}}(x^*) = -P_{n_0+k}\beta_0.$$

Substituting $x = x_\ell$ into this last identity, and summing over $\ell \in \mathcal{W}_k$, we have by (9) that

$$S_{P_{n_0+k+1}}(x^*) - S_{P_{n_0+k}}(x^*) = -\sum_{\ell\in\mathcal{W}_k} (-P_{n_0+k}\beta_0) = (p_{n_0+k} - 1)P_{n_0+k}\beta_0. \tag{12}$$

Combining (11) and (12), we finally obtain

$$S_{P_{n_0+k+1}}(x^*) = (p_{n_0+k} - 1)P_{n_0+k}\beta_0 + S_{P_{n_0+k}}(x^*) = p_{n_0+k}P_{n_0+k}\beta_0 = P_{n_0+k+1}\beta_0.$$

Thus (11) holds for all $k \ge 0$. ∎

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3 A Proof of Theorem 1

For simplicity, we assume that $n_\nu = n$. Obvious modifications change this proof into one which holds for subsequences. Suppose to the contrary that S is not the Vilenkin-Fourier series of f. Then the series $S - Sf$ is not the zero series, *i.e.,* we can find an integer n_0 such that $S_{P_{n_0}} - S_{P_{n_0}} f$ is nonzero on some interval I_{n_0} of rank n_0.

Let $E := \{x_1, x_2, \dots\}$. Then (4) holds for $x = x_1$, and it follows from (6) that (4) holds for the series $S - Sf$ at $x = x_1$. Hence by the lemma, we can choose an interval I_{m_0} of rank $m_0 > n_0$ such that $x_1 \notin I_{m_0}$ and $S_{P_{m_0}} - S_{P_{m_0}} f$ is nonzero on I_{m_0}. We claim that there is an interval I_{n_1}, of rank n_1, such that $I_{n_1} \subseteq I_{m_0}$ and

$$|S_{P_{n_1}}(x)| > 1 + |(S_{P_{n_1}} f)(x)| \quad \text{for all } x \in I_{n_1}. \tag{13}$$

Suppose the claim is false. Then given any interval $J \subseteq I_{m_0}$, of rank $j \geq m_0$, there is at least one point $x \in J$ such that $|S_{P_j}(x)| \leq 1 + |(S_{P_j} f)(x)|$. But S_{P_j} is constant on intervals of rank j, hence

$$|S_{P_j}| \leq 1 + |S_{P_j} f| \tag{14}$$

on J for all $j \geq m_0$ and all $J \subseteq I_{m_0}$. Hence (14) holds everywhere on I_{m_0} for all $j \geq m_0$.

Since $S_{P_j} f \to f$ in $L^q(G)$ norm, (14) implies that $\xi_j := S_{P_j} - S_{P_j} f$ is bounded in $L^q(I_{m_0})$ norm. Since by (2), $\xi_j \to f - f = 0$ almost everywhere on I_{m_0}, as $j \to \infty$, it follows from a generalized Bounded Convergence Theorem (see [5]) that $\int_{I_{m_0}} \xi_j \, dm \to 0$ as $j \to \infty$. But by orthogonality and the fact that ξ_{m_0} is constant on I_{m_0}, we have

$$0 = \lim_{j \to \infty} \int_{I_{m_0}} \xi_j \, dm = \int_{I_{m_0}} \xi_{m_0} \, dm = m(I_{m_0}) \xi_{m_0}(y_0)$$

for any $y_0 \in I_{m_0}$. Therefore, $\xi_{m_0} := S_{P_{m_0}} - S_{P_{m_0}} f$ is zero on I_{m_0}, contrary to the choice of m_0. This contradiction proves (13).

Inequality (13) contains two consequences which hold everywhere on I_{n_1}:

$$|S_{P_{n_1}}| > 1 + |S_{P_{n_1}} f| \geq 1 + 0 = 1,$$

and

$$|S_{P_{n_1}} - S_{P_{n_1}} f| \geq |S_{P_{n_1}}| - |S_{P_{n_1}} f| > 1 > 0.$$

Thus we have found an interval $I_{n_1} \subset I_{n_0}$ such that $x_1 \notin I_{n_1}$, $|S_{P_{n_1}}| > 1$, and $S_{P_{n_1}} - S_{P_{n_1}} f \neq 0$ on I_{n_1}. Continuing this construction, we generate nested intervals I_{n_k} such that

$$x_k \notin I_{n_k} \tag{15}$$

and

$$|S_{P_{n_k}}| > k \quad \text{on } I_{n_k}. \tag{16}$$

Since the intervals I_{n_k} are compact, there is a point x_0 which belongs to all I_{n_k}. By (16), then, $\limsup_{k \to \infty} |S_{P_{n_k}}(x_0)| = \infty$. In view of (4), this forces $x_0 \in E$, *i.e.,* $x_0 = x_k$ for some k. Thus $x_k \in I_{n_k}$, which contradicts (15). ∎

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