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A REMARK ON A THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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Let f be a real function defined on $[0, 1]\times\mathbb{R}^3$ and let $\eta \in (0, 1)$. Very recently, C.P. Gupta and V. Lakshimikantham, making use of the Leray-Schauder continuation theorem and Wirtinger-type inequalities, established an existence result for the problem

$$\begin{cases} x''' = f(t, x, x', x'') \\ x(0) = x(\eta) = x(1) = 0 \end{cases}$$

(Theorem 1 and Remark 4 of [Nonlinear Anal. **16** (1991), 949–957]).

The aim of the present paper is simply to point out how, by means of a completely different approach, it is possible to improve that result not only by requiring much more general conditions on f, but also by giving a precise pointwise estimate on x'''.

Let f be a real function defined on $[0, 1]\times\mathbb{R}^3$; $\eta \in (0, 1)$; $k \in [1, +\infty)$; $L^k([0, 1])$ the space of all (equivalence classes of) measurable functions $\psi\colon [0, 1] \to \mathbb{R}$ such that $\|\psi\|_{L^k([0,1])} = \left(\int_0^1 |\psi(t)|^k\, dt\right)^{1/k} < +\infty$; $W^{3,k}([0, 1])$ the space of all $u \in C^2([0, 1])$ such that u'' is absolutely continuous in $[0, 1]$ and $u''' \in L^k([0, 1])$.

Consider the problem

$$\text{(P)} \qquad \begin{cases} x''' = f\left(t, x, x', x''\right) \\ x(0) = x(\eta) = x(1) = 0. \end{cases}$$

A function $u\colon [0, 1] \to \mathbb{R}$ is said to be a generalised solution of (P) if $u \in W^{3,k}([0, 1])$, $u(0) = u(\eta) = u(1) = 0$ and, for almost every $t \in [0, 1]$, one has $u'''(t) = f(t, u(t), u'(t), u''(t))$.

Our interest in problem (P) originated reading [3]. In that paper the authors implicitly established the following existence theorem (see [**3**, Theorem 1 and Remark 4]).

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Theorem A. *Assume that:*

(i_1) *the function $t \to f(t, z_1, z_2, z_3)$ is measurable for every $z_1, z_2, z_3 \in \mathbb{R}$;*

(i_2) *the function $(z_1, z_2, z_3) \to f(t, z_1, z_2, z_3)$ is continuous for almost every $t \in [0, 1]$;*

(i_3) *there exist $p, q, r, S \in L^1([0, 1])$ such that, for almost every $t \in [0, 1]$ and every $z_1, z_2, z_3 \in \mathbb{R}$, one has*

$$|f(t, z_1, z_2, z_3)| \leqslant p(t)\,|z_1| + q(t)\,|z_2| + r(t)\,|z_3| + S(t);$$

(i_4) *there exist $a, b, c \in \mathbb{R}$ and $\alpha \in L^1([0, 1])$ such that, for almost every $t \in [0, 1]$ and every $z_1, z_2, z_3 \in \mathbb{R}$, one has*

$$f(t, z_1, z_2, z_3)z_2 \geqslant az_2^2 + b\,|z_1|\,|z_2| + c\,|z_2|\,|z_3| + \alpha(t)z_2.$$

Then, problem (P) has at least one generalised solution $u \in W^{3,1}([0, 1])$ provided

$$\|r\|_{L^1([0,1])} < 1$$

and

$$\left(1 - \|r\|_{L^1([0,1])}\right)\left(\frac{4}{\pi^2}\,|a| + \frac{4}{\pi^3}\,|b| + \frac{2}{\pi}\,|c|\right) + \frac{1}{\pi}\,\|p\|_{L^1([0,1])} + \|q\|_{L^1([0,1])} + \|r\|_{L^1([0,1])} < 1.$$

The method employed in [3], in proving Theorem A, consists of using the Leray-Schauder continuation theorem and Wirtinger-type inequalities.

The aim of the present paper is simply to point out how, by means of a completely different approach based on the classical Schauder-Tychonoff fixed point theorem, it is possible to establish the following result, which improves Theorem A.

Theorem 1. *Let the assumptions (i_1), (i_2) and (i_3) of Theorem A be satisfied. Then problem (P) has at least one generalised solution $u \in W^{3,1}([0, 1])$ provided*

$$\frac{1}{8}\,\|p\|_{L^1([0,1])} + \frac{1}{2}\,\|q\|_{L^1([0,1])} + \|r\|_{L^1([0,1])} < 1.$$

Moreover, for almost every $t \in [0, 1]$, one has

$$|u'''(t)| \leqslant \sup\{|f(t, z_1, z_2, z_3)| : \ |z_1| \leqslant \gamma/8,\ |z_2| \leqslant \gamma/2,\ |z_3| \leqslant \gamma\},$$

where

$$\gamma = \frac{\|S\|_{L^1([0,1])}}{1 - \left(\frac{1}{8}\,\|p\|_{L^1([0,1])} + \frac{1}{2}\,\|q\|_{L^1([0,1])} + \|r\|_{L^1([0,1])}\right)}. \tag{1}$$

Before proving Theorem 1, we collect, in Proposition 1 below, the inequalities we shall use in the sequel.

Proposition 1. *Let $u \in W^{3,1}([0, 1])$ be such that $u(0) = u(\eta) = u(1) = 0$. Then one has:*

(j₁) $\max_{t\in[0,1]} |u(t)| \leqslant \frac{1}{8} \max_{t\in[0,1]} |''(t)|$;

(j₂) $\max_{t\in[0,1]} |u'(t)| \leqslant \frac{1}{2} \max_{t\in[0,1]} |u''(t)|$;

(j₃) $\max_{t\in[0,1]} |u''(t)| \leqslant \int_0^1 |u'''(t)|\, dt$.

Proof: The assertions (j_1) and (j_2) follow at once from Theorem 8.2 of [**1**]. The assertion (j_3) derives from the fact that, owing to our assumptions, there exists $\xi \in [0, 1]$ such that $u''(\xi) = 0$. □

We now give the following

Proof of Theorem 1: For every $z = (z_1, z_2, z_3) \in \mathbb{R}^3$ we put $\|z\| = \max\{8|z_1|, 2|z_2|, |z_3|\}$. It is easy to check that the extended real function $M: t \mapsto \sup_{\|z\|\leqslant\gamma} |f(t, z)|$, $t \in [0, 1]$, where γ is given by (1), is measurable. Moreover, owing to (i_3), $M \in L^1([0, 1])$.

Consider the set

$$K = \{w \in L^1([0, 1]) \colon |w(t)| \leqslant M(t) \text{ almost everywhere in } [0, 1]\}.$$

Of course, K is nonempty and convex. By the Dunford-Pettis theorem (see, for instance, [2], Theorem 1, p.101) it is also weakly compact. For every $w \in K$ and every $t \in [0, 1]$, we put:

$$\alpha(w) = \frac{\eta}{2(1-\eta)} \int_0^1 (1-\sigma)^2 w(\sigma) d\sigma - \frac{1}{2\eta(1-\eta)} \int_0^\eta (\eta-\sigma)^2 w(\sigma) d\sigma;$$

$$\beta(w) = \frac{1}{2\eta(1-\eta)} \int_0^\eta (\eta-\sigma)^2 w(\sigma) d\sigma - \frac{1}{2(1-\eta)} \int_0^1 (1-\sigma)^2 w(\sigma) d\sigma; \tag{2}$$

$$\Phi(w)(t) = \frac{1}{2} \int_0^t (t-\sigma)^2 w(\sigma) d\sigma + \alpha(w)t + \beta(w)t^2;$$

$$\Psi(w)(t) = f(t, \Phi(w)(t), \Phi(w)'(t), \Phi(w)''(t)),$$

where, as usual, $\Phi(w)'(t) = \frac{d\Phi(w)(t)}{dt}$ and $\Phi(w)''(t) = \frac{d^2\Phi(w)(t)}{dt^2}$. Let us prove that $\Psi(K) \subseteq K$. To this end, fix $w \in K$ and observe that, by (i_3) and (1), one has

$$\|w\|_{L^1([0,1])} \leqslant \|M\|_{L^1([0,1])} = \int_0^1 \sup_{\|z\|\leqslant\gamma} |f(t, z)|\, dt$$

$$\leq \left(\frac{1}{8}\|p\|_{L^1([0,1])} + \frac{1}{2}\|q\|_{L^1([0,1])} + \|r\|_{L^1([0,1])}\right)\gamma + \|S\|_{L^1([0,1])} = \gamma.$$

Taking into account that $\Phi(w) \in W^{3,1}([0, 1])$ and $\Phi(w)(0) = \Phi(w)(\eta) = \Phi(w)(1) = 0$, from Proposition 1 it follows that

$$\max_{t\in[0,1]} |\Phi(w)(t)| \leqslant \frac{1}{8} \max_{t\in[0,1]} |\Phi(w)''(t)| \leqslant \frac{1}{8}\int_0^1 |w(t)|\,dt \leqslant \frac{\gamma}{8};$$

$$\max_{t\in[0,1]} |\Phi(w)'(t)| \leqslant \frac{1}{2} \max_{t\in[0,1]} |\Phi(w)''(t)| \leqslant \frac{1}{2}\int_0^1 |w(t)|\,dt \leqslant \frac{\gamma}{2};$$

$$\max_{t\in[0,1]} |\Phi(w)''(t)| \leqslant \int_0^1 |w(t)|\,dt \leqslant \gamma.$$

Hence, $|\Psi(w)(t)| \leqslant \sup_{\|z\|\leqslant\gamma} |f(t, z)| = M(t)$ for almost every $t \in [0, 1]$. This implies that $\Psi(w) \in K$.

Now, let us prove that the operator Ψ is weakly continuous. Owing to the weak compactness of K, we need only verify that the graph $gr(\Psi)$ of Ψ is weakly closed in $K \times K$. Taking into account Theorem 7, p.313, of [4], it suffices to show that $gr(\Psi)$ is sequentially weakly closed. Let $w \in K$ and let $\{w_n\}$ be a sequence in K weakly converging to w in $L^1([0, 1])$. From (2) it follows that, for every $t \in [0, 1]$, $\lim_{n\to\infty} \Phi(w_n)(t) = \Phi(w)(t)$, $\lim_{n\to\infty} \Phi(w_n)'(t) = \Phi(w)'(t)$ and $\lim_{n\to\infty} \Phi(w_n)''(t) = \Phi(w)''(t)$. Therefore, by (i_2), the sequence $\{\Psi(w_n)\}$ converges almost everywhere in $[0, 1]$ to $\Psi(w)$. Bearing in mind that for almost every $t \in [0, 1]$ and every $n \in \mathbb{N}$ one has

$$|\Psi(w_n)(t)| \leqslant M(t),$$

by the Lebesgue dominated convergence theorem, we obtain that $\lim_{n\to\infty} \Psi(w_n) = \Psi(w)$ in $L^1([0, 1])$. So, $\{\Psi(w_n)\}$ converges weakly to $\Psi(w)$.

At this point, we are allowed to apply the Schauder-Tychonoff fixed point theorem to Ψ. There is, therefore, $w \in K$ such that $w = \Psi(w)$. The function $u(t) = \Phi(w)(t)$, $t \in [0, 1]$, satisfies our conclusion. ∎

REMARK 1. We point out that Theorem 1 of [3] deals with the case $p, q, r \in L^2([0, 1])$, $S \in L^1([0, 1])$. It is easy to check that, in this case, the above-mentioned result and our Theorem 1 are mutually independent.

Taking into account that $\|\psi\|_{L^1([0,1])} \leqslant \|\psi\|_{L^2([0,1])}$ for every $\psi \in L^2([0, 1])$, from Theorem 1, it is possible to derive the following

THEOREM 2. *Let the assumptions (i_1) and (i_2) of Theorem A be satisfied. Moreover, suppose that:*

(i_3') *there exist $p, q, r, S \in L^2([0, 1])$ such that, for almost every $t \in [0, 1]$ and every $z_1, z_2, z_3 \in \mathbf{R}$, one has*

$$|f(t, z_1, z_2, z_3)| \leqslant p(t)\,|z_1| + q(t)\,|z_2| + r(t)\,|z_3| + S(t).$$

Then, problem (P) has at least one generalised solution $u \in W^{3,2}([0,1])$ *provided*

$$\frac{1}{8}\|p\|_{L^2([0,1])} + \frac{1}{2}\|q\|_{L^2([0,1])} + \|r\|_{L^2([0,1])} < 1.$$

Moreover, for almost every $t \in [0,1]$, *one has*

$$|u'''(t)| \leqslant \sup\{|f(t, z_1, z_2, z_3)| : |z_1| \leqslant \gamma'/8, |z_2| \leqslant \gamma'/2, |z_3| \leqslant \gamma'\},$$

where

$$\gamma' = \frac{\|S\|_{L^2([0,1])}}{1 - \left(\frac{1}{8}\|p\|_{L^2([0,1])} + \frac{1}{2}\|q\|_{L^2([0,1])} + \|r\|_{L^2([0,1])}\right)}.$$

REMARK 2. The previous condition on p, q and r is not so restrictive as that requested in (2.17) or in Remark 4 of [**3**].

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