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Reducing Spheres and Klein Bottles after Dehn Fillings

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Abstract. Let M be a compact, connected, orientable, irreducible 3-manifold with a torus boundary. It is known that if two Dehn fillings on M along the boundary produce a reducible manifold and a manifold containing a Klein bottle, then the distance between the filling slopes is at most three. This paper gives a remarkably short proof of this result.

Let M be a compact, connected, orientable 3-manifold with a torus boundary T. A *slope* on T is the isotopy class of an essential simple loop. We assume that π and γ are two slopes on T such that $M(\pi)$ is a reducible manifold and $M(\gamma)$ contains a Klein bottle. $\Delta(\pi,\gamma)$ denotes their minimal geometric intersection number. It is proved in [6] that the optimum upper bound of $\Delta(\pi,\gamma)$ is 3, by using the representations of types which come from the intersection of graphs. This paper gives a short proof of this result based on a recent theorem of Jin, Lee, Oh and Teragaito [5].

Theorem 1 *Let M be a hyperbolic 3-manifold. If $M(\pi)$ is reducible and $M(\gamma)$ contains a Klein bottle, then $\Delta(\pi,\gamma) \leq 3$.*

Assume for contradiction that $\Delta(\pi,\gamma) \geq 4$. Let $\widehat{Q}$ be a reducing sphere in $M(\pi)$ which intersects the filling solid torus V_π in a family of meridian disks. We choose $\widehat{Q}$ so that $q = |\widehat{Q} \cap V_\pi|$ is minimal over all reducing spheres in $M(\pi)$. And choose $\widehat{K}$ among all Klein bottles in $M(\gamma)$ so that $k = |\widehat{K} \cap V_\gamma|$ is minimal. Let $Q = \widehat{Q} \cap M$ and $K = \widehat{K} \cap M$. By an isotopy of Q, we may assume that Q and K intersect transversely, and $Q \cap K$ has the minimal number of components. Then as described in [6], we obtain graphs G_Q in $\widehat{Q}$ and G_K in $\widehat{K}$. We use the definitions and terminology of [6].

Lemma 2 *Suppose $k \geq 3$. Then G_Q has at most $\frac{k+2}{2}$ (resp. $\frac{(k+1)}{2}$) mutually parallel edges connecting parallel vertices if k is even (resp. odd).*

Proof In [5] a graph G_P on a projective plane has this property by Lemma 5.1(4). This can be applied to the graph G_Q in exactly the same way. ∎

Lemma 3

(1) *G_K cannot contain Scharlemann cycles on distinct label pairs.*

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(2) *G_K cannot contain an S-cycle.*

Proof (1) This is [2, Theorem 2.4].

(2) If G_K contains an S-cycle, then $M(\pi)$ contains a projective plane [3]. But this contradicts [5, Theorem 1.2]. ∎

Proof of Theorem 1 We assume hereafter that $q \geq 3$ by [1, Lemma 2.3].

Claim There is no vertex x of G_Q such that more than $\frac{3}{2}k$ edges connect x to antiparallel vertices.

Suppose not, then there exist more than $\frac{3}{2}k$ positive x-edges in G_K. Consider the subgraph of G_K consisting of all vertices and all x-edges of G_K. An Euler characteristic count gives that this subgraph contains more than $\frac{1}{2}k$ disk faces whose boundaries are x-edge cycles of G_K. Each disk face contains a Scharlemann cycle by [4, Proposition 5.1]. And all these Scharlemann cycles are, say, 12-Scharlemann cycles by Lemma 3 (1).

Construct a graph Γ in $\widehat{K}$ as follows. Choose a dual vertex in the interior of each face of G_K bounded by a 12-Scharlemann cycle, and let the vertices of Γ be the vertices of G_K together with these dual vertices. The edges of Γ are defined by joining each dual vertex to the vertices of the corresponding Scharlemann cycle in the obvious way. Let n be the number of 12-Scharlemann cycles. Then $n > \frac{1}{2}k$. Γ has $k + n$ vertices and at least $3n$ edges because each Scharlemann cycle has order at least 3 by Lemma 3 (2). Again an Euler characteristic count guarantees that Γ has a disk face E. But E determines a 1-edge cycle bounding a disk face in E which, as long as $q \geq 3$, contains a Scharlemann cycle, contradicting the definition of Γ. This completes the proof of Claim.

Therefore each vertex x of G_Q has at least $(\Delta - \frac{3}{2})k$ labels where edges connecting parallel vertices are incident. Let G_Q^+ be the subgraph of G_Q consisting of all vertices and edges connecting parallel vertices of G_Q. Then every vertex of G_Q^+ has valency at least $(\Delta - \frac{3}{2})k$. If $k \geq 3$, by Lemma 2, any vertex has valency at least 3 in its reduced graph $\overline{G}_Q^+$. So, we can choose a block $\overline{\Lambda}$ of an extremal component of $\overline{G}_Q^+$ with at most one cut vertex. Let Λ be the subgraph of G_Q^+ corresponding to $\overline{\Lambda}$. Notice that $\overline{\Lambda}$ has an interior vertex by [7, Lemma 3.2].

Let v, e and f be the numbers of vertices, edges and faces of $\overline{\Lambda}$ which is regarded as a graph in a disk. Denote by v_i, v_∂ and v_c the numbers of interior vertices, boundary vertices and cut vertices. Hence $v = v_i + v_\partial$ and $v_c = 0$ or 1. Since each face of $\overline{\Lambda}$ is a disk with at least 3 sides, we have $2e \geq 3f + v_\partial$. Combined with $1 = \chi(\text{disk}) = v - e + f$, we get $e \leq 3v - v_\partial - 3 = 3v_i + 2v_\partial - 3$.

Suppose that every interior vertex of $\overline{\Lambda}$ has valency at least 6 and that every boundary vertex except a cut vertex has valency at least 4. Then we have $2e \geq 6v_i + 4(v_\partial - v_c)$. These two inequalities give us that $2v_c \geq 3$, a contradiction.

Therefore either some interior vertex has valency at most 5, or some boundary vertex has valency at most 3 and is not a cut vertex. Thus we have, in Λ, either $4k \leq \Delta k \leq 5(\frac{k}{2} + 1)$ or $\frac{5}{2}k \leq (\Delta - \frac{3}{2})k \leq 3(\frac{k}{2} + 1)$ by Lemma 2. Either case implies that $k \leq 3$ and $k \neq 3$ (use the stronger inequalities of Lemma 2 for odd k).

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For the remaining cases $k = 1$ and 2, we refer to [6, Section 4]. ■

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