The Veronesean of quadrics and associated loci

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(*Received* 31*st October*, 1935; *received in revised form* 1*st January*, 1936. *Read* 1*st November*, 1935.)

In this paper we consider the correspondence between tangential quadrics of [3] and points of [9]. Godeaux[1] has considered this geometrically, with the object of obtaining a representation for a twisted cubic of three dimensions. We have considered it from a standpoint more algebraic than that of Godeaux, with particular reference to the types of pencils of quadrics that correspond to special lines of [9], and to the interpretation in [9] of the fact that the condition for a net of quadrics to be part of the polar system of a cubic surface is poristic.

§1. We denote a general point of [3] by P_x, its coordinates being (x_0, x_1, x_2, x_3) and its equation $P_x = 0$.

We denote a general plane of [3] by Π_X, its coordinates being (X_0, X_1, X_2, X_3) and its equation $\Pi_X = 0$.

The equation of a locus quadric q is $\Sigma\, a_{ij}\, x_i\, x_j = 0$. The equation of a tangential quadric Q is $\Sigma\, A_{ij}\, X_i\, X_j = 0$, and q is therefore *apolar* to Q if $\Sigma\, a_{ij}\, A_{ij} = 0$.

The suffixes i, j are always interchangeable and have the values 0, 1, 2, 3.

We denote by Q_y a general point of [9] whose coordinates are $(y_{00}, y_{11}, y_{22}, y_{33};\ y_{12}, y_{20}, y_{01};\ y_{03}, y_{13}, y_{23})$. The point Q_A or $(A_{00}, A_{11}, A_{22}, A_{33};\ A_{12}, A_{20}, A_{01};\ A_{03}, A_{13}, A_{23})$ *represents* the quadric Q, while the prime ϖ_a or $\Sigma\, a_{ij}\, y_{ij} = 0$ *represents* the quadric q. Hence if ϖ_a contains Q_A, q and Q are apolar and conversely.

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[1] "La géométrie de la cubique gauche," *Bulletin de la Soc. Roy. des Sciences de Liége* (3) 14 (1927).

Consider the locus

$$|y_{ij}|_r = 0,$$

where the suffix r indicates that the determinant $|y_{ij}|$ is of rank r.

If $r = 3$, Q_y represents a *conic*. The locus of such points is a variety V_8^4 of dimension 8 and order 4. If $r = 2$, Q_y represents a *point pair*. The locus of such points is a variety V_6^{10} of order 10 and dimension 6. If $r = 1$, Q_y represents a *repeated point*. The locus of such points is a variety V_3^8 of order 8 and dimension 3 whose parametric equations may be written

$$y_{ij} = x_i x_j. \tag{1}$$

V_3^8 is therefore triple on V_8^4 and quadruple on V_6^{10}, while V_6^{10} is double on V_8^4. We shall denote (1, 0, 0, 0) by P_0, (0, 1, 0, 0) by P_1, (0, 0, 1, 0) by P_2 and (0, 0, 0, 1) by P_3. We shall denote by Q_{xy} the point representing the point pair $P_x P_y = 0$, whether $x = y$ or not.

§ 2. *Representation of V_3^8 on a solid.* The representation of V_3^8 on the solid is analogous to that of the Veronese surface on the plane, and we give a brief summary of corresponding elements and properties:—

On Solid	On V_3^8
A point P_x	A point Q_{xx}
A curve of degree n	A curve of degree $2n$
A surface of degree n	A surface of degree $4n$
A plane	A Veronese surface
A quadric	An octic del Pezzo surface[1].
Two quadrics have in general a unique common self-polar tetrad.	*There is a unique solid through a general line of* [9] *meeting V_3^8 in 4 points.* Such a solid we call *four-secant.*
A quadric $\lambda P_x^2 + \mu P_y^2 = 0$ is a point pair.	*Chords of V_3^8 lie on V_6^{10}.*
A quadric $\lambda P_x^2 + \mu P_y^2 + \nu P_z^2 = 0$ is a tangential conic.	*Trisecant planes of V_3^8 lie on V_8^4.*

Godeaux (*loc. cit.* § 4) obtains these results geometrically.

[1] Represented by quartics with two fixed double points.

§ 3. *Flat spaces on V_8^4 and V_6^{10}. Tangent spaces.*

Flat spaces on V_8^4 are of two kinds:—

(i) [5]'s σ_5 *representing conics of a plane*;

(ii) [4]'s σ_4 *representing conics touching a line at a point.*

There is *one* σ_5 through a general point of V_8^4, ∞^1 through a point of V_6^{10} and ∞^2 through a point of V_3^8. The equations of a typical σ_5 are

$$\sum_{i=0}^{3} X_i y_{ij} = 0, \qquad (j = 0, 1, 2, 3).$$

Two spaces σ_5 representing conics in planes α, β meet in the *plane* of the conic representing the line $(\alpha\beta)$. The typical space σ_4 represents the system of conics

$$(z_0X_0 + z_1X_1 + z_2X_2 + z_3X_3)(\lambda X_0 + \mu X_1 + \nu X_2 + \rho X_3) + \sigma(x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3)^2 = 0,$$

and at any point of it the six expressions

$$(2z_i z_j y_{ij} - z_i^2 y_{jj} - z_j^2 y_{ii})/(x_i z_j - x_j z_i)^2$$

must be equal. There are ∞^5 spaces σ_4 altogether, ∞^1 through a general point of V_8^4 [Godeaux, *loc. cit.*, § 7].

Flat spaces on V_6^{10} are of two kinds:—

(i) *Tangent solids* σ_3 *to* V_3^8,

(ii) *Planes* σ_2 *of conics on* V_3^8 [Godeaux, *loc. cit.*, § 9].

Through a point Q_{xy} on V_6^{10} there is *one* plane σ_2, namely that of the conic on V_3^8 through Q_{xx} and Q_{yy}. The ∞^1 spaces σ_4 through Q_{xy} are obtained by joining it to the tangent solids σ_3 to V_3^8 at points of this conic.

The tangent prime at any point of V_8^4 meets V_3^8 in two Veronese surfaces, since it contains the space σ_5 through the point.

Consider next tangent [6]'s σ_6 to V_6^{10}. A space σ_6 represents tangential quadrics with a common generator g, and touches V_6^{10} at all points of the plane γ containing the conic of V_3^8 representing g. It meets V_8^4 in a repeated quadric plane cone of vertex γ. One system of generating [4]'s of this cone are spaces σ_4, while the other system

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consists of [4]'s each lying in a space σ_5. The equations of the tangent [6] at Q_{xy}, which joins the tangent solids to V_3^8 at Q_{xx} and Q_{yy} are:—

$$2y_{02}B_{03}B_{32}+2y_{23}B_{20}B_{03}+2y_{30}B_{32}B_{20}+y_{00}B_{23}^2+y_{22}B_{30}^2+y_{33}B_{02}^2=0,$$
$$2y_{01}B_{03}B_{31}+2y_{13}B_{10}B_{03}+2y_{30}B_{31}B_{10}+y_{00}B_{13}^2+y_{11}B_{30}^2+y_{33}B_{01}^2=0,$$
$$2y_{01}B_{02}B_{21}+2y_{12}B_{10}B_{02}+2y_{20}B_{21}B_{10}+y_{00}B_{12}^2+y_{11}B_{20}^2+y_{22}B_{01}^2=0,$$

where $B_{ij} = x_i\,\xi_j - x_j\,\xi_i$ and therefore

$$B_{01}\,B_{23} + B_{02}\,B_{31} + B_{03}\,B_{12} = 0.$$

There are ∞^1 spaces σ_6 through a general point of S_9, and ∞^2 through a point of V_6^{10}.

A space σ_6 meets V_6^{10} in a *rational quartic plane cone*, for the point pairs of an ∞^6 system of tangential quadrics with a common generator g consist of a point of the generator and an arbitrary point, are therefore ∞^4, and can be put in (1, 1) correspondence with the points of a [4]. Any net of quadrics with a common generator has *four* point pairs, and hence follows the fact that the locus is quartic. The vertex of the cone is γ. It is possible by a process of elimination to prove that the only solids meeting V_6^{10} in space curves are those which lie in spaces σ_6. In any space σ_6 there will be ∞^4 exceptional solids which are the intersection of two spaces σ_6 and meet V_6^{10} in a quadric surface (*cf.* Godeaux, *loc. cit.* § 21).

§ 4. *Lines of* [9].

Consider the various possible positions of a line in [9] with regard to V_8^4. A general line meets V_8^4 in 4 distinct points, but lines in special positions may meet V_6^{10} or V_3^8 or touch or osculate V_8^4. We give below a table shewing the types of (tangential) quadrics corresponding to the various types of special position possible for a line of [9]. We thus obtain by varying the position of a line in [9] *all* the standard tangential quadrics dual to those given by Bromwich.[1]

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[1] Bromwich, *Quadratic Forms and their classification by means of Invariant Factors*, Cambridge Tract No. 3.

No.	*Segre characteristic and position of line in* [9].	*Base envelope of corresponding pencil in* [3].
1.	[1111]; a general line.	A general quartic developable of genus 1.
2.	[(11) 11]; a line meeting V_6^{10}.	Two quadric cones with two common tangent planes each touching all quadrics of the pencil at a fixed point.
3.	[(11)(11)]; a chord of V_6^{10} not on V_8^4.	Four pencils of planes whose axes form a skew quadrilateral.
4.	[(111) 1]; a line meeting V_3^8.	A quadric cone twice; the quadrics of the pencil have ring-contact.
5.	[211]; a line touching V_8^4, not meeting V_6^{10}.	A quartic developable with a double tangent plane, on which all quadrics of the pencil have a common point.
6.	[(21) 1]; a line touching V_8^4 at a point of V_6^{10} (three-point contact).	Two quadric cones with a common generator and a common tangent plane along it.
7.	[2 (11)]; a tangent to V_8^4 which meets V_6^{10}.	A quadric cone and two pencils of planes whose axes meet and touch the cone.
8.	[(211)]; a tangent to V_8^4 at a point of V_3^8 (*i.e.* having four-point contact).	Two pencils of planes through intersecting axes, each counted twice. Quadrics of the system have a common point on each axis.
9.	[22]; a bitangent to V_8^4.	A pencil of planes through a line l and a cubic developable, two of whose generating planes contain l.
10.	[(22)]; a tangent to V_6^{10}.	Three pencils of planes through skew lines l_1, l_2 and a transversal l_3, the last pencil being counted twice.
11.	[4]; a line having four-point contact with V_8^4.	A cubic developable and a pencil of planes through a generator of it.
12.	[31]; a line which osculates V_8^4.	A quartic developable with a stationary generating plane.
13.	[(31)]; a line through a point of V_6^{10} having four-point contact with V_8^4.	Two pencils of planes and a quadric cone through the intersection of their axes, touching the joining plane of those axes.
14.	A line on V_8^4 in a space σ_4 [No Segre Characteristic].	A pencil of planes counted twice and a quadric cone.
15.	[111]; a line on V_8^4 in a space σ_5.	Four pencils of planes through four concurrent lines.

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§5. *Planes of* [9].

The general plane meets V_8^4 in a non-singular quartic. Hesse in his paper on the bitangents of a non-singular plane quartic[1] considered the condition for the general member of a net of (locus) quadrics to be a cone. If the net be $\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 = 0$, this condition is quartic in the λ's. In the dual case we could take the λ-plane to be the plane representing the net and the quartic its intersection with V_8^4.

Any special relation of quadrics of a net must be reflected in a corresponding speciality of the quartic in which the representative plane meets V_8^4. This curve may be degenerate in various ways, including the extreme case of a line repeated four times, which occurs when the plane touches V_6^{10} along a line or has 4-point contact with V_8^4 along a line. The latter occurs for the net

$$2a\,(X_1 X_2 - X_0 X_3) + bX_2^2 + 2cX_1 X_3 + 2dX_2 X_3 + eX_3^2 = 0,$$

(where a, b, c, d are connected by two linear relations).

§6. *Solids of* [9].

In general a solid meets V_8^4 in a symmetroid. The most interesting special case is when this becomes a Kummer Surface. The Kummer Surface whose equation is in the standard form

$$(\delta_0 \lambda_0 r_0)^{1/2} + (\delta_1 \lambda_1 r_1)^{1/2} + (\delta_2 \lambda_2 r_2)^{1/2} = 0,$$

where $\quad \delta_0 + \delta_1 + \delta_2 = 0, \quad r_0 \equiv a_2 \lambda_1 - a_1 \lambda_2 - b_0 \lambda_3,$

$$r_1 \equiv -a_2 \lambda_0 + a_0 \lambda_2 - b_1 \lambda_3, \quad r_2 \equiv a_1 \lambda_0 - a_0 \lambda_1 - b_2 \lambda_3,$$

can be regarded as the condition for a quadric of the web

$$(1) \quad \begin{aligned} &2\lambda_0(\delta_0 X_1 X_2 - a_2 X_1 X_3 + a_1 X_2 X_3) + 2\lambda_1(\delta_1 X_0 X_2 + a_2 X_0 X_3 - a_0 X_2 X_3) \\ &+ 2\lambda_2(\delta_2 X_0 X_1 - a_1 X_0 X_3 + a_0 X_1 X_3) + 2\lambda_3(b_0 X_0 + b_1 X_1 + b_2 X_2) X_3 = 0 \end{aligned}$$

to be a conic. The quadrics of this web have six common tangent planes. Of the 16 nodes, 10 lie on V_6^{10} and 6 are due to contact with V_8^4 by the solid representing the web. Actually these last six nodes all lie on the trope $\lambda_3 = 0$, and all the other tropes contain 4 nodes on V_6^{10} and two contact nodes. The quadrics for which $\lambda_3 = 0$ are

[1] O. Hesse, *Werke*, p. 376 = *Journal für Math.*, 49 (1855), p. 279.

inscribed in the cubic developable determined by the six common tangent planes. Conversely it can be shewn that *any* web with six common tangent planes may be regarded as of the form (1).

Godeaux (*loc. cit.*, § 19) starts with a web of quadrics through six points, and proves geometrically that the representative solid of the web meets V_8^4 in a Kummer Surface. He uses the trope which does not meet V_6^{10} to *represent* the twisted cubic through the six points.

§ 7. *The Polar Porism.*

(i) Consider the tangential pencil

$$(\lambda A_{00}+\mu B_{00})X_0^2+(\lambda A_{11}+\mu B_{11})X_1^2+(\lambda A_{22}+\mu B_{22})X_2^2+(\lambda A_{33}+\mu B_{33})X_3^2=0$$

which we call Ω, and let the line representing it be l. *Any* quadric which forms with Ω a *polar net* (*i.e.* part of the polar system of a class cubic surface) is represented by a point Q_c of a plane through l lying on a certain quadric solid cone. The plane $Q_c l$ may be regarded as a perfectly general plane representing a polar net and will be denoted by a. The 4-secant solid through l we denote by Σ.

For if a represent a polar net, we may suppose that there is a class cubic surface with respect to which

Π_1 is the plane whose polar quadric, A, is $A_{00}\,X_0^2 + \ldots\ldots = 0$,

Π_2 is the plane whose polar quadric, B, is $B_{00}\,X_0^2 + \ldots\ldots = 0$,

and Π_3 is the plane whose polar quadric, C, is represented by Q_c.

If we write down the conditions for the pole of Π_1 with respect to B to be the same as that of Π_2 with respect to A, and two similar sets, and eliminate the ratios of the coordinates of Π_1, Π_2, Π_3, then *we find that the coordinates of Q_c satisfy a twelfth-order skew-symmetrical determinantal equation reducible*[1] *to*

$$(\lambda\, y_{01}\, y_{23} + \mu y_{02}\, y_{31} + \nu y_{03}\, y_{12})^2 = 0, \tag{1}$$

where $\lambda \equiv (A_{00}\,B_{11} - A_{11}\,B_{00})\,(A_{22}\,B_{33} - A_{33}\,B_{22}) \equiv [01]\,[23],$

$$\mu \equiv [02]\,[31], \qquad \nu \equiv [03]\,[12]$$

and therefore

$$\lambda + \mu + \nu \equiv 0. \tag{2}$$

The quadric (1) *is a solid cone V_l, vertex Σ.* Any point of V_3^8 is on it because of (2). There are ∞^1 such quadrics with a given vertex, forming a linear pencil whose base is the quartic cone joining Σ to V_3^8.

[1] See Salmon; *Geometry of Three Dimensions* (4th ed. 1882), p. 209.

(ii) There are two systems of [6]'s on V_l. Those of the first system (3) meet V_3^8 in rational normal sextics through Q_{00}, Q_{11}, Q_{22}, Q_{33}, while those of the second, (4), meet V_3^8 in conics and the four isolated points Q_{00}, Q_{11}, Q_{22}, Q_{33}. The equations are

$$\left.\begin{aligned} by_{03} - cy_{02} + d\lambda y_{23} &= 0, \\ cy_{01} - ay_{03} + d\mu y_{31} &= 0, \\ ay_{02} - by_{01} + d\nu y_{12} &= 0, \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} b'\nu y_{12} - c'\mu y_{13} + d'y_{01} &= 0, \\ c'\lambda y_{23} - a'\nu y_{12} + d'y_{02} &= 0, \\ a'\mu y_{31} - b'\lambda y_{32} + d'y_{03} &= 0. \end{aligned}\right\} \tag{4}$$

There are four exceptional members of the second system (those for which three of a', b', c', d', are zero) which meet V_3^8 in a Veronese surface. If we project from Σ on to a [5], *both* systems of [6]'s project into planes meeting the projection of V_3^8 in conics.

If a *point pair* $P_x P_y = 0$ be represented by Q_c, and p join the points P_x, P_y, then p is the locus of poles of some plane Π_3 with respect to the quadrics of Ω, since the poles of Π_1 and Π_2 with respect to C are in this case on p.

If p is the locus of poles of a plane with respect to quadrics of Ω, the same is true when p is regarded as the join of any other point pair $P'_x P'_y$ of p, so if V_l contains Q_{xy} it also contains the whole σ_2-plane through Q_{xy} and therefore the [6] joining it to Σ. This [6] is clearly of the second system (4) above.

The lines p form a tetrahedral complex with fundamental tetrahedron $P_0 P_1 P_2 P_3$; their Plücker coordinates $(l_1, m_1, n_1, \lambda_1, \mu_1, \nu_1)$ with respect to this tetrahedron satisfy :—

$$\frac{l_1 \lambda_1}{\lambda} = \frac{m_1 \mu_1}{\mu} = \frac{n_1 \nu_1}{\nu}. \tag{5}$$

But the Plücker coordinates of l with respect to the tetrahedron $Q_{00} Q_{11} Q_{22} Q_{33}$ also satisfy (5) and so any line l' of the tetrahedral complex in Σ thus defined by (5) gives the same quadric (4) as l does. Hence:—

(*a*) *If a quadric represented by Q_c forms a polar net with a pencil Ω represented by l, through which the four-secant solid Σ passes, then the quadric represented by Q_c also forms a polar net with the pencil represented by any other line l' which cuts on the faces of $Q_{00} Q_{11} Q_{22} Q_{33}$ a related range.*

(*b*) *The range in which l cuts the faces of* $Q_{00}Q_{11}Q_{22}Q_{33}$ *is related to that in which the locus of poles of a plane with respect to quadrics of* Ω *cuts the faces of* $P_0P_1P_2P_3$.

(iii) Any solid lying in a 5-secant [4] of V_3^8 represents the polar system of a class cubic surface. For if the [4] meets V_3^8 in $Q_{00}Q_{11}Q_{22}Q_{33}$ and a point Q_{44}, representing the repeated point pair $X_4^2 = 0$, and if we arrange that the relation between X_0, X_1, X_2, X_3 and X_4 is

$$X_0 + X_1 + X_2 + X_3 + X_4 \equiv 0,$$

then any solid of the [4] represents a web capable of the form

$$a_0 X_0^2 + a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_4^2 = 0,$$

where there is a single linear relation between the α's, namely

$$p_0 a_0 + p_1 a_1 + p_2 a_2 + p_3 a_3 + p_4 a_4 = 0.$$

Put $p_k a_k = \lambda_k$ $(k = 0, 1, 2, 3, 4)$ and the equation of the web can be written

$$(A) \qquad \lambda_0\left(\frac{X_0^2}{a_0}\right) + \lambda_1\left(\frac{X_1^2}{a_1}\right) + \lambda_2\left(\frac{X_2^2}{a_2}\right) + \lambda_3\left(\frac{X_3^2}{a_3}\right) + \lambda_4\left(\frac{X_4^2}{a_4}\right) = 0,$$

where $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$.

The web (A) is the polar system of the class cubic surface

$$\frac{X_0^3}{a_0} + \frac{X_1^3}{a_1} + \frac{X_2^3}{a_2} + \frac{X_3^3}{a_3} + \frac{X_4^3}{a_4} = 0.$$

Any plane in a 5-secant [4] of V_3^8 lies in ∞^1 such solids of that 5-secant [4], and therefore represents part of the polar system of a class cubic surface. It follows that *all 5-secant* [4]*'s of* V_3^8 *through l lie on* V_l. For suppose one did not. Then we should be able to find in it a plane through l representing part of the polar system of a class cubic surface, which did not lie on V_l, contrary to what was proved above.

(iv) It remains to consider the question of whether a plane through l known to lie on V_l necessarily represents a polar net. To that end we shall shew that in any [6], S_6, of V_l meeting V_3^8 in a sextic [*i.e.* of system (3)], there is just one 5-secant [4] of V_3^8 through l and an arbitrary point. The sextic Γ_6 in which the [6] with equations (3) meets V_3^8 represents the twisted cubic common to the quadrics

$$\left.\begin{aligned} bx_0x_3 - cx_0x_2 + d\lambda x_2x_3 = 0,\\ cx_0x_1 - ax_0x_3 + d\mu x_3x_1 = 0.\end{aligned}\right\}$$

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The parametric equations of the sextic are found to be

$$y_{00} = \theta^2 (c\theta - d\lambda)^2 (c\theta + d\mu)^2;\ y_{11} = a^2\theta^2 (c\theta - d\lambda)^2;\ y_{22} = b^2\theta^2 (c\theta + d\mu)^2;$$
$$y_{33} = (c\theta - d\lambda)^2 (c\theta + d\mu)^2;\quad y_{01} = a\theta^2 (c\theta - d\lambda)^2 (c\theta + d\mu);$$
$$y_{02} = b\theta^2 (c\theta - d\lambda)(c\theta + d\mu)^2;\ y_{03} = \theta (c\theta - d\lambda)^2 (c\theta + d\mu)^2,$$

where $\theta = x_0/x_3$, and we assume that $d \neq 0$, which we can do without loss of generality, so that $y_{00}, y_{11}, y_{22}, y_{33}, y_{01}, y_{02}, y_{03}$ are independent coordinates for S_6.

No 5-secant [4] through l can meet Σ in a plane unless it contains Σ entirely. For otherwise, let Σ_1 be the [5] joining the [4] in question to Σ. There will be a system of ∞^3 locus quadrics apolar to all those represented by points of Σ_1, and this system will have nine non-coplanar common points, which is impossible.

Suppose the 5-secant [4] U through l to be given as the intersection of the [5]'s UQ_{00} and UQ_{33}, primes of S_6. Then the equations of U may be written

$$\left.\begin{aligned} [23]\,y_{11} + [31]\,y_{22} + [12]\,y_{33} + P_{01}\,y_{01} + P_{02}\,y_{02} + P_{03}\,y_{03} &= 0,\\ [12]\,y_{00} + [20]\,y_{11} + [01]\,y_{22} + \quad + P'_{01}y_{01} + P'_{02}y_{02} + P'_{03}y_{03} &= 0,\end{aligned}\right\}$$

seeing that UQ_{00} meets Σ in the plane lQ_{00} whose equations in S_6 are

$$[23]\,y_{11} + [31]\,y_{22} + [12]\,y_{33} = y_{01} = y_{02} = y_{03} = 0.$$

The parameter of Q_{00} is ∞ and that of Q_{33} is 0. If we substitute the parameters into the equations of UQ_{00} and UQ_{33} and then divide the latter by θ, we get two quintic polynomials in θ which both give the parameters of the intersections of U and V_3^8, and are therefore proportional. We therefore have the identity

$$a^2 [23]\, \theta^2 (c\theta - d\lambda)^2 + b^2 [31]\, \theta^2 (c\theta + d\mu)^2 + [12]\, (c\theta - d\lambda)^2 (c\theta + d\mu)^2$$
$$+ \theta (c\theta - d\lambda)(c\theta + d\mu)\,[aP_{01}\theta (c\theta - d\lambda) + bP_{02}\theta (c\theta + d\mu) + P_{03} (c\theta - d\lambda)(c\theta + d\mu)]$$
$$\equiv K\{[12]\, \theta (c\theta + d\lambda)^2 (c\theta - d\mu)^2 + a^2[20]\,(c\theta - d\lambda)^2 + b^2 [01]\, \theta(c\theta + d\mu)^2$$
$$+ (c\theta - d\lambda)(c\theta + d\mu)[aP'_{01}\theta(c\theta - d\lambda) + bP'_{02}\theta(c\theta + d\mu) + P'_{03}(c\theta - d\lambda)(c\theta + d\mu)]\}.$$

Putting $\theta = 0$ in both sides gives

$$KP'_{03} \equiv [12]. \qquad (1)$$

Putting $c\theta = d\,[01][23]$ or $c\theta = -\,d\,[02][31]$ gives

$$K = d\,[23]\,[31]/c.$$

If we now divide the identity by $\theta(c\theta - d\lambda)(c\theta + d\mu)$ we reduce it to

$$aP_{01}\theta(c\theta - d\lambda) + bP_{02}\theta(c\theta + d\mu) + P_{03}(c\theta - d\lambda)(c\theta + d\mu)$$
$$-(d/c)[23][31]\{(c\theta - d\lambda)(c\theta + d\mu) + aP'_{01}(c\theta - d\lambda) + bP'_{02}(c\theta + d\mu)\}$$
$$+(1/c)\{a^2[23](c\theta - d\lambda) + b^2[31](c\theta + d\mu)\} \equiv 0.$$

As this is quadratic in θ, it will be sufficient to express the fact that it has three roots 0, $\frac{d\lambda}{c}$, $-\frac{d\mu}{c}$. Hence we obtain by putting $\theta = 0$,

$$-P_{03}d[01][02] + (d^2/c)[01][02][03][31] + (ad/c)[01][23]P'_{01}$$
$$-(bd/c)[02][31]P'_{02} - (a^2/c)[01][23]^2 + (b^2/c)[02][31]^2 = 0. \qquad (2)$$

By putting $c\theta = d\lambda$ we obtain

$$d[01][23]P_{02} - d[23][31]P'_{02} + b[31] = 0, \qquad (3)$$

and by putting $c\theta = -d\mu$ we obtain

$$d[02][31]P_{01} + d[23][31]P'_{01} - a[23] = 0. \qquad (4)$$

(1) (2) (3) (4) are 4 linear equations between the non-homogeneous coefficients P_{01}, P_{02}, P_{03}, P'_{01}, P'_{02}, P'_{03}. The conditions for the [4] U to go through an arbitrary point of S_6 not on l would give 2 further linear relations, and therefore there is in general a unique solution, *i.e. through a plane* α *containing a line* l *and on* V_l *there is one* 5-*secant* [4] *of* V_3^8 *in each space* S_6 *containing* α.

Since there are ∞^1 spaces S_6 on V_l through α, there are therefore ∞^1 5-secant [4]'s of V_3^8 through α [*N.B.* 2 spaces S_6 do *not* intersect in a 5-secant [4] of V_3^8].

We have now proved:

(1) *A necessary condition for a plane* α *to represent a polar net is that it lie on the quadric* V_l *defined as above for any line* l *in* α. Hence the *general* plane of [9] does not represent a polar net.

(2) *If a plane* α *lies on* V_l *and* l *is in* α, *then in each space* S_6 *on* V_l *through* α *there is one* 5-*secant* [4] *of* V_3^8 *through* α. *Hence there are* ∞^1 5-*secant* [4]'s *of* V_3^8 *through* α *altogether.* In each of the 5-secant [4]'s there are ∞^1 solids through α, and therefore [§ 7 (iii)] we can find ∞^2 cubic surfaces to whose polar systems the net represented by α belongs. We have thus found a geometrical reason in [9] for the fact that the condition for a net to be polar is poristic.

(v) It is possible also to shew in this way that self-conjugate pentagrams, for a net of quadrics which is polar, form a g_5^1 on a twisted cubic.[1] For we can prove that the [6], T, which joins two 5-secant [4]'s through α contains them all, and that it cannot degenerate into a [5]. T meets V_3^8 in a curve which must be a sextic, also T lies in the prime P which touches V_l all along α. All spaces S_6 through α lie in P and any two meet in $[S\alpha]$, which is not a 5-secant [4] of V_3^8. Hence the sextics in which these spaces S_6 meet V_3^8 represent twisted cubics of *the same* system on the locus quadric Q representing P. It can be shewn that T, which lies in P, meets V_3^8 in a sextic representing a twisted cubic of *the opposite* system on Q, which therefore meets every one of the cubics represented by intersections of spaces S_6 with V_3^8 in 5 points. The [4]'s joining such sets of 5 points generate the quadric plane cone in which T meets V_l. The fact that sets of points representing self-conjugate pentagrams lie on [4]'s which generate a quadric plane cone proves that they form a g_5^1 on the twisted cubic represented by the intersection of T and V_3^8.

Conversely, if we consider the g_5^1 on the twisted cubic which is the locus of $(\theta^3, \theta^2, \theta, 1)$ given by

$$(a_0+\lambda a'_0)\theta^5+(a_1+\lambda a'_1)\theta^4+(a_2+\lambda a'_2)\theta^3+(a_3+\lambda a'_3)\theta^2+(a_4+\lambda a'_4)\theta+(a_5+\lambda a'_5)=0,$$

we find that the net of quadrics having the sets of this g_5^1 for self-conjugate pentagrams is represented by the plane

$$\left.\begin{aligned} a_0y_0+a_1y_1+a_2y_2+a_3y_3+a_4y_4+a_5y_5&=0,\\ a_0y_1+a_1y_2+a_2y_3+a_3y_4+a_4y_5+a_5y_6&=0,\\ a'_0y_0+a'_1y_1+a'_2y_2+a'_3y_3+a'_4y_4+a'_5y_5&=0,\\ a'_0y_1+a'_1y_2+a'_2y_3+a'_3y_4+a'_4y_5+a'_5y_6&=0,\end{aligned}\right\}$$

where $y_0, \ldots, y_6$ are coordinates in the space of the sextic representing the locus of $(\theta^3, \theta^2, \theta, 1)$ so chosen that the parametric equations to this sextic are

$$y_0:y_1:y_2:y_3:y_4:y_5:y_6=\theta^6:\theta^5:\theta^4:\theta^3:\theta^2:\theta:1.$$

[1] See Edge "A special net of quadrics," *Proc. Edin. Math. Soc.* (2) 4 (1936) 185.