CROSSED PRODUCTS AND MAXIMAL ORDERS

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Introduction. Let Γ be a maximal order over a complete discrete rank one valuation ring R in a central simple algebra over the quotient field of R. The purpose of this paper is to determine necessary and sufficient conditions for Γ to be equivalent to a crossed product over a tamely ramified extension of R.

It is a classical result that every central simple algebra over a field k is equivalent to a crossed product over a Galois extension of k. Furthermore, it has been proved by Auslander and Goldman in [2] that every central separable algebra over a local ring is equivalent to a crossed product over an unramified extension.

Let R denote a discrete rank one valuation ring. The set of maximal orders $M'(R)$ over R forms a subset of the set of hereditary orders $H'(R)$ over R (see [3]). An equivalence relation on the set of hereditary orders has been defined in [2]. Namely, if Λ_1 and Λ_2 are in $H'(R)$, then Λ_1 is said to be equivalent to Λ_2 if there exist finitely generated free R-modules E_1 and E_2 and an R-algebra isomorphism

$$\Lambda_1 \otimes_R \mathrm{Hom}_R(E_1, E_1) \cong \Lambda_2 \otimes_R \mathrm{Hom}_R(E_2, E_2).$$

It is established in [2] that an hereditary order which is equivalent to a maximal order is itself a maximal order.

The author has proved in [10] that every crossed product $\Delta(f, S, G)$ over a tamely ramified extension S of a discrete rank one valuation ring R is an hereditary order, and that $\Delta(f, S, G)$ is a maximal order if and only if the order of the conductor group H_f is one (see Section 1 for the definition of H_f). She has also exhibited in this paper an example of a non-maximal hereditary order which is not equivalent to a crossed product over a tamely ramified extension. Now let Γ be a maximal order over a complete discrete rank one valuation ring R in a central simple algebra Σ over the quotient field of R.

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The main theorem of this paper states that a necessary and sufficient condition for Γ to be equivalent to a crossed product over a tamely ramified extension of R is the existence of a splitting field K of Σ for which

(1) the integral closure S of R in K is a tamely ramified extension of R

(2) f is in the image of the natural map $Z^2(G, U(S)) \to Z^2(G, U(K))$ where f is a 2-cocycle with the property that $\Delta(f, K, G)$ is equivalent to Σ.

At the end of the paper we present an example of a maximal order which is not equivalent to a crossed product over a tamely ramified extension.

The following notation shall be in use throughout the paper. If R is a local ring, then $\bar{R}$ shall denote its residue class field. The multiplicative group of units of a ring R shall be denoted by $U(R)$. Unless otherwise stated, R shall always denote a complete discrete rank one valuation ring, S the integral closure of R in a finite Galois extension of the quotient field of R, and G the Galois group of the quotient field extension. Since R is complete, S is also a complete discrete rank one valuation ring. The inertia group and the inertia ring of the extension S of R shall be denoted by G_I and U respectively; and the image of a 2-cocycle f under the natural map $Z^2(G, U(S)) \to Z^2(G, U(\bar{S}))$ shall be denoted by $\bar{f}$. For the definitions of crossed product, hereditary order, and tamely ramified extension we refer the reader to [10]. For convenience we recall that when the extension S of R is a tamely ramified extension of complete discrete rank one valuation rings then the inertia group G_I is cyclic, and the e^{th} roots of unity are present in the inertia ring U, where e is the order of G_I.

1. Cohomology and tame ramification. A crossed product over a tamely ramified extension is a maximal order if and only if its conductor group is trivial (see [10]). Therefore this section is devoted to the study of cohomology and the conductor group in the tamely ramified case.

Definition Let S be a tamely ramified extension of a complete discrete rank one valuation ring R. For each cohomology class $[f]$ we define four subgroups of the cyclic group G_I:

(1) Ω_f is the maximal subgroup of G_I such that the image of $[f]$ under the restriction map $H^2(G, U(S)) \to H^2(\Omega_f, U(S))$ is trivial,

(2) Γ_f is the maximal subgroup of G_I such that the image of $[\bar{f}]$ under the restriction map $H^2(G, U(\bar{S})) \to H^2(\Gamma_f, U(\bar{S}))$ is trivial,

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(3) J_f is the maximal subgroup of G_l with the property that $[f]$ is in the image of the inflation map $H^2(G/J_f, U(S)) \to H^2(G, U(S))$,

(4) H_f is the maximal subgroup of G_l with the property that $[\bar{f}]$ is in the image of the inflation map $H^2(G/H_f, U(\bar{S})) \to H^2(G, U(\bar{S}))$.

The group H_f was named the *conductor group* in [10]. An element f of $Z^2(G, U(S))$ is said to be properly normalized if f is trivial on $\Omega_f \times \Omega_f$. Similarly, an element $\bar{f}$ in $Z^2(G, U(\bar{S}))$ is said to be properly normalized if $\bar{f}$ is trivial on $\Gamma_f \times \Gamma_f$. The purpose of this section is to establish the equalities $\Omega_f = \Gamma_f$ and $J_f = H_f$.

PROPOSITION 1.1. *Let S be a tamely ramified extension of a complete discrete rank one valuation ring R, and f an element of $Z^2(G, U(S))$. Then $\Omega_f = \Gamma_f$, and f is cohomologous to a properly normalized 2-cocycle.*

Proof. Since the image of $[f]$ under the restriction map $H^2(G, U(S)) \to H^2(\Omega_f, U(S))$ is trivial, certainly the image of $[\bar{f}]$ under the map $H^2(G, U(\bar{S})) \to H^2(\Omega_f, U(\bar{S}))$ is trivial. Therefore $\Omega_f \subseteq \Gamma_f$.

Let U denote the inertia ring of S over R. Since S is a tamely ramified extension of R we know that $\bar{U} = \bar{S}$. To show that $\Gamma_f \subseteq \Omega_f$ we shall make use of the fact that Γ_f is a cyclic group to first observe that the map $\Psi : H^2(\Gamma_f, U(S)) \to H^2(\Gamma_f, U(\bar{U}))$ induced by the natural map $S \to \bar{S} = \bar{U}$ is a monomorphism. For let $[f_\Gamma]$ be in the kernel of Ψ, and let u be an element of $U(T)$ such that $[f_\Gamma]$ corresponds to u mod $N(U(S))$ under the canonical identification $H^2(\Gamma_f, U(S)) = U(T)/N(U(S))$ where T is the integral closure of R in the fixed field of Γ_f, and $N(U(S))$ denotes the norm of $U(S)$ in T. Since $\Psi([f_\Gamma])$ is the identity we know that $(\bar{u})(\bar{c}^n) = \bar{1}$ for some element $\bar{c}$ in $U(\bar{U})$ where n is the order of Γ_f using the identification $H^2(\Gamma_f, U(\bar{U})) = U(\bar{U})/(U(\bar{U}))^n$. Therefore the separable polynomial $\bar{P}(X) = X^n - 1/\bar{u}$ in $\bar{U}[X]$ has a root in $\bar{U}$. By Hensel's lemma it follows that $P(X) = X^n - 1/u$ has a solution, say c, in U. Therefore $N(c) = c^n$ and $uc^n = 1$, and hence f_Γ is cohomologus to the trivial 2-cocycle and Ψ is a monomorphism.

Now letting f_Γ denote the restriction of f to $\Gamma_f \times \Gamma_f$ it follows from the definition of Γ_f together with the above observation that f_Γ is cohomologous to the trivial 2-cocycle in $Z^2(\Gamma_f, U(S))$. Therefore there exists a map $\phi : \Gamma_f \to U(S)$ such that $f_\Gamma(\sigma, \tau) = \phi(\sigma)\phi^\sigma(\tau)/\phi(\sigma\tau)$ for σ and τ in Γ_f. Extend ϕ to

G by defining $\phi(\rho)=1$ if ρ is not in Γ_f. Then the element g of $Z^2(G, U(S))$ defined by $g(\sigma, \tau)=f(\sigma, \tau)\phi(\sigma\tau)/\phi(\sigma)\phi^\sigma(\tau)$ is cohomologous to f and has the property that $g(\sigma, \tau)=1$ when σ and τ are in Γ_f. Thus $\Gamma_f \subseteqq \Omega_f$ and this concludes the proof.

In order to establish that $J_f=H_f$ we next prove three preliminary lemmas in which it is always assumed that S is a tamely ramified extension of a complete discrete rank one valuation ring R.

LEMMA 1.2. *For each element f of $Z^2(G, U(S))$ there exists an element g of $Z^2(G, U(S))$ such that g is cohomologous to f, whenever ρ is in H_f it is true that $g(\tau, \rho)=1$, and $\bar{g}(\tau, \rho)=1$ if τ or ρ is in H_f.*

Proof. By Prop. 1.1 we may as well assume that f is a properly normalized 2-cocycle. Then $\bar{f}$ is also properly normalized. From the definition of H_f we know by Prop. 2.3 of [10] that there exists a map $\bar{\phi}: G\to U(\bar{S})$ such that the 2-cocycle $\bar{g}$ in $H^2(G, U(\bar{S}))$ defined by $\bar{g}(\tau, \sigma)=\bar{f}(\tau, \sigma)\bar{\phi}(\tau)\bar{\phi}^\tau(\sigma)/\bar{\phi}(\tau\sigma)$ has the property that $\bar{g}(\tau, \sigma)=1$ if τ or σ is in H_f, and that the restriction of $\bar{\phi}$ to H_f takes values in the multiplicative group of h^{th} roots of unity where h is the order of H_f.

We shall use $\bar{g}$ to produce the definition of the desired 2-cocycle g. Let $G=\cup\tau_j H_f$ be a disjoint left coset decomposition of G relative to the subgroup H_f, and let σ now denote a generator of H_f. If $\bar{\phi}(\sigma)$ is the h^{th} root of unity $\bar{\eta}$, define $\phi(\sigma)$ to be η where η is an h^{th} root of unity in $U(S)$ whose existence is guaranteed by the assumption that the extension S of R is a tamely ramified extension of complete local rings and Hensel's lemma.

For each j define $\phi(\tau_j)$ and $\phi(\tau_j\sigma)$ by choosing representatives of $\bar{\phi}(\tau_j)$ and $\bar{\phi}(\tau_j\sigma)$ in $U(S)$ such that $g(\tau_j, \sigma)=1$ where $g(\tau_j, \sigma)$ is defined by $g(\tau_j, \sigma)=f(\tau_j, \sigma)\phi(\tau_j)\phi^{\tau_j}(\sigma)/\phi(\tau_j\sigma)$. We next define $\phi(\tau_j\sigma^2)$ to be a representative of $\bar{\phi}(\tau_j\sigma^2)$ for which $g(\tau_j\sigma, \sigma)=1$ where $g(\tau_j\sigma, \sigma)=f(\tau_j\sigma, \sigma)\phi(\tau_j\sigma)\phi^{\tau_j\sigma}(\sigma)/\phi(\tau_j\sigma^2)$. Proceeding in this way we finally define $\phi(\tau_j\sigma^{h-1})$ by choosing a representative of $\bar{\phi}(\tau_j\sigma^{h-1})$ for which $g(\tau_j\sigma^{h-2}, \sigma)=1$ where $g(\tau_j\sigma^{h-2}, \sigma)=f(\tau_j\sigma^{h-2}, \sigma)\phi(\tau_j\sigma^{h-2})$ $\phi^{\tau_j\sigma^{h-2}}(\sigma)/\phi(\tau_j\sigma^{h-1})$. Thus we have defined a map $\phi: G\to U(S)$. It remains to verify that the 2-cocycle g cohomologous to f by ϕ satisfies the conclusion of the lemma. In order to do this we first check that the above definitions imply that $g(\tau_j\sigma^{h-1}, \sigma)=1$. Now

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$$\begin{aligned} g(\tau_j\sigma^{h-1}, \sigma) &= f(\tau_j\sigma^{h-1}, \sigma)\phi(\tau_j\sigma^{h-1})\phi^{\tau_j\sigma^{h-1}}(\sigma)/\phi(\tau_j) \\ &= \prod_{i=0}^{h-1} f(\tau_j\sigma^i, \sigma)\phi^{\tau_j\sigma^i}(\sigma)/\prod_{i=0}^{h-2} g(\tau_j\sigma^i, \sigma) \\ &= f(\tau_j, \sigma^h)[\phi^{\tau_j}(\sigma)]^h \\ &= 1 \end{aligned}$$

since the h^{th} root of unity $\phi(\sigma)$ is present in the inertia ring and hence is left fixed by σ.

Therefore $g(\tau, \sigma) = 1$ for all τ in G where σ is a generator of H_f. We verify finally that $g(\tau, \sigma^i) = 1$ for $1 \leq i \leq h$. From the associativity relation on g together with the above, we have that $g(\tau, \sigma^i) = g(\tau\sigma^{i-1}, \sigma)g(\tau, \sigma^{i-1})/g^{\tau}(\sigma^{i-1}, \sigma) = 1$ for $1 \leq i \leq h$ and therefore $g(\tau, \sigma) = 1$ for all τ in G and ρ in H_f.

As in Prop. 2.1 of [10], for each element τ of G we let $n(\tau)$ be the integer defined modulo e by the relation $\tau\sigma\tau^{-1} = \sigma^{n(\tau)}$ where σ is a generator of G_l and e is the order of G_l. With this definition it is easy to check that $\tau\rho\tau^{-1} = \rho^{n(\tau)}$ for each ρ in G_l.

LEMMA 1.3. *Assume the notation of Lemma* 1.2. *Then there exists a* 2-*cocycle* $\hat{g}$ *in* $Z^2(G, U(S))$ *cohomologous to* g *such that* $\hat{g}(\tau, \rho) = \hat{g}(\rho^{n(\tau)}, \tau) = 1$ *for each element* ρ *in* H_f *and* τ *in* G.

Proof. Let ρ be in H_f and τ in G. Denote by K the quotient field of S and by F the fixed field of $\{\rho^{n(\tau)}\}$. We first show that $N_{K/F}(g(\rho^{n(\tau)}, \tau)) = 1$. By the assumption on g and its associativity property we have that $g(\rho^{n(\tau)}, \tau\rho^i) = g(\rho^{n(\tau)}, \tau)$ and $g(\rho^{in(\tau)}, \tau) = g(\rho^{n(\tau)}, \tau\rho^{i-1})g^{\rho^{n(\tau)}}(\rho^{(i-1)n(\tau)}, \tau)$ for all i. These equalities imply that $g(\rho^{jn(\tau)}, \tau) = \prod_{i=0}^{j-1} g^{\rho^{in(\tau)}}(\rho^{n(\tau)}, \tau)$ for $1 \leq j \leq b$, from which it follows that $\prod_{i=0}^{b-1} g^{\rho^{in(\tau)}}(\rho^{n(\tau)}, \tau) = g(\rho^{bn(\tau)}, \tau) = g(1, \tau) = 1$ where b is the order of $\{\rho^{n(\tau)}\}$. Thus $N_{K/F}(g(\rho^{n(\tau)}, \tau)) = 1$.

Since $N_{K/F}(g(\rho^{n(\tau)}, \tau)) = 1$ it follows from Th. 3 p. 171 of [11] and the fact that K is a tamely ramified inertial extension of F that $g(\rho^{n(\tau)}, \tau) = y^{\rho^{n(\tau)}}\xi/y$ for some y in $U(S)$ and b^{th} root of unity ξ. And $\xi = 1$ since $\overline{g(\rho^{n(\tau)}, \tau)} = \bar{1}$. Now we may construct $\hat{g}$. Let $G = \cup H_f\tau_j$ be a disjoint coset decomposition of G relative to H_f. Fix a generator σ of H_f. For each τ in G define $\phi(\tau) = 1/y$ where τ is in $H_f\tau_j$ and y is an element of $U(S)$ for which $g(\sigma^{n(\tau_j)}, \tau_j) = y^{\sigma^{n(\tau_j)}}/y$. Now define $\hat{g}$ by $\hat{g}(\tau, \beta) = g(\tau, \beta)\phi(\tau)\phi^{\tau}(\beta)/\phi(\tau\beta)$. It is easy to verify that $\hat{g}$ has the desired properties.

LEMMA 1.4. *Assume the notation of Lemma 1.3. Then there exists a 2-cocycle q in $Z^2(G, U(S))$ cohomologous to $\hat{g}$ and satisfying $q(\tau, \sigma)=1$ whenever τ or σ is in H_f.*

Proof. Let $G=\cup H_f\tau_j$ be a disjoint right coset decomposition of G relative to the subgroup H_f. Define $\phi : G \to U(S)$ by $\phi(\sigma\tau_j)=1/\hat{g}(\sigma, \tau_j)$ where σ is in H_f. Define $q : G\times G \to U(S)$ by $q(\tau, \rho)=\hat{g}(\tau, \rho)\phi(\tau\sigma)/\phi(\tau)\phi^{\tau}(\rho)$. Let $\tau=\omega\tau_j$ be any element of G where ω is in H_f, and let σ be any element of H_f.

Then from the definition of q we obtain the equality $q(\tau, \sigma)=q(\omega\tau_j, \sigma)=\hat{g}(\omega\tau_j, \sigma)\hat{g}(\omega, \tau_j)/\hat{g}(\omega\sigma^{n(\tau_j)}, \tau_j)$. By the associativity relation satisfied by the 2-cocycle $\hat{g}$ we have that $\hat{g}(\omega\sigma^{n(\tau_j)}, \tau_j)\hat{g}(\omega, \sigma^{n(\tau_j)})=\hat{g}(\omega, \sigma^{n(\tau_j)}\tau_j)\hat{g}^{\omega}(\sigma^{n(\tau_j)}, \tau_j)$; and together with the assumption on $\hat{g}$ this implies that $\hat{g}(\omega\sigma^{n(\tau_j)}, \tau_j)=\hat{g}(\omega, \tau_j\sigma)$. Since $\hat{g}(\omega\tau_j, \sigma)\hat{g}(\omega, \tau_j)=\hat{g}(\omega, \tau_j\sigma)\hat{g}^{\omega}(\tau_j, \sigma)=\hat{g}(\omega, \tau_j\sigma)$ we conclude that $q(\tau, \sigma)=1$.

On the other hand $q(\sigma, \tau)=q(\sigma, \omega\tau_j)=\hat{g}(\sigma, \omega\tau_j)\hat{g}^{\sigma}(\omega, \tau_j)/\hat{g}(\sigma\omega, \tau_j)$. But $\hat{g}(\sigma, \omega\tau_j)\hat{g}^{\sigma}(\omega, \tau_j)=\hat{g}(\sigma\omega, \tau_j)\hat{g}(\sigma, \omega)=\hat{g}(\sigma\omega, \tau_j)$. Therefore $q(\sigma, \tau)=1$, and this concludes the proof.

PROPOSITION 1.5. *Let S be a tamely ramified extension of a complete discrete rank one valuation ring R, and f an element of $Z^2(G, U(S))$. Then $H_f=J_f$.*

Proof. By the definition of J_f there exists a 2-cocycle g in $Z^2(G, U(S))$ such that g is cohomologous to f and $g(\sigma, \tau)=1$ if σ or τ is in J_f. If g is cohomologous to f by $\phi : G\to U(S)$, then $\bar{g}$ is cohomologous to $\bar{f}$ by $\bar{\phi} : G\to U(\bar{S})$. The fact that $\bar{g}(\sigma, \tau)=1$ if σ or τ is in J_f implies that $J_f\subseteq H_f$.

To obtain the inclusion $H_f\subseteq J_f$ we apply the preceding lemmas to f, and so obtain a 2-cocycle q in $Z^2(G, U(S))$ cohomologous to f and satisfying $q(\sigma, \tau)=1$ whenever σ or τ is in H_f. It now follows from the definition of J_f that $H_f\subseteq J_f$.

2. Maximal orders. In order to establish necessary and sufficient conditions for a maximal order to be equivalent to a crossed product over a tamely ramified extension in the complete case, the following lemma shall be useful.

LEMMA 2.1. *Let Σ_1 and Σ_2 be equivalent central simple k-algebras, where k is the quotient field of a discrete rank one valuation ring R. If Γ_1 and Γ_2 are maximal orders in Σ_1 and Σ_2 respectively, then Γ_1 is equivalent to Γ_2.*

Proof. Since Σ_1 and Σ_2 are equivalent, there exist finitely generated k-modules V_1 and V_2 such that

$$\Sigma_1 \otimes_k \mathrm{Hom}_k(V_1, V_1) \cong \Sigma_2 \otimes_k \mathrm{Hom}_k(V_2, V_2).$$

Let Ω_1 and Ω_2 be maximal orders in $\mathrm{Hom}_k(V_1, V_1)$ and $\mathrm{Hom}_k(V_2, V_2)$ respectively. It is a classical result that Ω_1 and Ω_2 are of the form $\Omega_1 = \mathrm{Hom}_R(E_1, E_1)$ and $\Omega_2 = \mathrm{Hom}_R(E_2, E_2)$ where E_1 and E_2 are finitely generated free R-submodules of V_1 and V_2 respectively. Now Ω_1 and Ω_2 are central separable R-algebras, and therefore it follows from Prop. 8.6 of [2] that $\Gamma_1 \otimes_R \Omega_1$ and $\Gamma_2 \otimes_R \Omega_2$ are maximal orders. Since all maximal orders in a central simple algebra over a discrete rank one valuation ring are isomorphic (see Prop. 3.5 of [3]) we conclude that $\Gamma_1 \otimes \Omega_1 \cong \Gamma_2 \otimes \Omega_2$. Therefore Γ_1 is equivalent to Γ_2.

PROPOSITION 2.2. *Let S be a tamely ramified extension of a complete discrete rank one valuation ring R, and f an element of $Z^2(G, U(S))$. Then every maximal order in the central simple k-algebra $\Delta(f, K, G)$ is equivalent to a crossed product over a tamely ramified extension of R.*

Proof. By Lemma 1.4 we know that there exists a 2-cocycle q in $Z^2(G, U(S))$ such that q is cohomologous to f and $q(\tau, \sigma) = 1$ whenever τ or σ is in H_f.

The subgroup H_f is a normal subgroup of G, so that the fixed field L of H_f is a Galois extension of k with Galois group G/H_f. Let T denote the integral closure of R in L and observe that T is a tamely ramified extension of R. To show that q takes values in $U(T)$ we shall make use of the following definition. For each element τ of G let $m(\tau)$ be the integer defined modulo e by the relation $\tau^{-1}\omega\tau = \omega^{m(\tau)}$ where ω is a generator of the inertia group G_I and e is the order of G_I. We proceed to show that $q^\sigma(\tau, \rho) = q(\tau, \rho)$ for all σ in H_f and all τ and ρ in G. By the associativity property of q we have the equalities

$$q(\sigma, \tau\rho) q^\sigma(\tau, \rho) = q(\sigma\tau, \rho) q(\sigma, \tau)$$
$$q(\tau\sigma^{m(\tau)}, \rho) q(\tau, \sigma^{m(\tau)}) = q(\tau, \sigma^{m(\tau)}\rho) q^\tau(\sigma^{m(\tau)}, \rho)$$

from which it follows that $q^\sigma(\tau, \rho) = q(\sigma\tau, \rho)$ and also $q(\tau\sigma^{m(\tau)}, \rho) = q(\tau, \sigma^{m(\tau)}\rho)$. Therefore $q(\sigma\tau, \rho) = q(\tau\sigma^{m(\tau)}, \rho) = q(\tau, \rho\sigma^{m(\tau)m(\rho)})$. And the equality

$$q(\tau, \rho\sigma^{m(\tau)m(\rho)}) q^\tau(\rho, \sigma^{m(\tau)m(\rho)}) = q(\tau\rho, \sigma^{m(\tau)m(\rho)}) q(\tau, \rho)$$

implies that $q^{\tau}(\tau, \rho) = q(\tau, \rho)$. Hence $q(\tau, \rho)$ is in the fixed field of H_f, and so q takes values in $U(T)$.

We may now consider the crossed product $\Delta(g, T, G/H_f)$ where g is the preimage of q under the inflation map $Z^2(G/H_f, U(T)) \to Z^2(G, U(S))$. It follows from the definition of the conductor group H_f and the second Noether isomorphism theorem, that the conductor group H_g is trivial. Therefore we conclude from Theorem 2.5 of [10] that $\Delta(g, T, G/H_f)$ is a maximal order in $\Delta(g, L, G/H_f)$. Now the central simple k-algebra $\Delta(g, L, G/H_f)$ is equivalent to $\Delta(q, K, G)$ (see [1]). If Γ denotes a maximal order in $\Delta(q, K, G)$ it follows from the preceding lemma that Γ is equivalent to the crossed product $\Delta(g, T, G/H_f)$.

Thus we have stablished the following main theorem.

THEOREM 2.3. *Let Γ be a maximal order over a complete discrete rank one valuation ring R in a central simple algebra Σ. For Γ to be equivalent to a crossed product over a tamely ramified extension of R it is necessary and sufficient that there exists a splitting field K of Σ such that*

(1) *the integral closure S of R in K is a tamely ramified extension of R*

(2) *f is in the image of the natural map $Z^2(G, U(S)) \to Z^2(G, U(K))$ where f is a 2-cocycle for which $\Delta(f, K, G)$ is equivalent to Σ.*

COROLLARY 2.4. *Let Σ be a central simple algebra over the quotient field k of a complete discrete rank one valuation ring R. If Σ has a splitting field K such that the integral closure S of R in K is a tamely ramified inertial extension of R, then each maximal order Γ in Σ is equivalent to a crossed product over a tamely ramified extension of R.*

Proof. We shall prove first that if an extension S of R is a tamely ramified inertial extension of complete discrete rank one valuation rings, then the natural map $H^2(G, U(S)) \to H^2(G, U(K))$ is an epimorphism, where K denotes the quotient field of S, and G is the Galois group of K over k. Let f be an element of $Z^2(G, U(K))$ and let $[f]$ correspond to c mod $N(U(K))$ under the canonical identification $H^2(G, U(K)) = U(k))/N(U(K))$ which holds because G is a cyclic group. As usual, N denotes norm. Next write c in the form $c = up^x$ where u is in $U(R)$, x is an integer, and p denotes the prime element of R. Let e be the order of G. Because of the assumption on S and R we know that for a proper choice of the prime element P of S it is true that $P^e = vp$ for some element v in $U(R)$, and $\sigma(P) = \xi P$, where ξ is a primitive e^{th} root of unity in R and σ is a generator of G (see Prop. 3.1 of [10]). Therefore $N(P) = \pm vp$, and so the element $b = (\pm vp)^{-x}$ is also a norm. Hence cb is an element of $U(R)$ which is congruent to c mod $N(U(K))$, and from this it follows that the map $H^2(G, U(S)) \to H^2(G, U(K))$ is an epimorphism.

Now we may prove the corollary. Since Σ is split by K we know that Σ is equivalent to a crossed product $\Delta(f, K, G)$ for some element $[f]$ in $H^2(G, U(K))$, (see [1]). By the first part of the proof we may assume that f is in the image of the natural map $Z^2(G, U(S)) \to Z^2(G, U(K))$. It now follows from the theorem that a maximal order Γ in Σ is equivalent to a crossed product over a tamely ramified extension of R.

EXAMPLE 2.5. We present an example to show that a maximal order over a discrete rank one valuation ring need not be equivalent to a crossed product over a tamely ramified extension.

Consider the ring of polynomials $Z[X]$ with coefficients in the integers Z. Let $R = Z[X]_{(2)}$ be the localization of $Z[X]$ at the minimal prime ideal generated by the element 2. Let $K = k(\sqrt{2})$ where k denotes the quotient field of R. Then the integral closure S of R in K is $S = R[\sqrt{2}]$ and the Galois group G of K over k is of order two. Note that S is not a tamely ramified extension of R since the field characteristic of $\overline{R}$ and the ramification index of S over R are both equal to two. Consider the element $[f]$ of $H^2(G, U(S))$ which corresponds to X mod $N(U(S))$ under the canonical identification $H^2(G, U(S)) = U(R)/N(U(S))$, and the crossed product $\Delta = \Delta(f, S, G)$.

It may be verified by computation that $\Delta\sqrt{2}$ is the unique maximal two-sided ideal of Δ. Since $\Delta\sqrt{2}$ is a free left Δ-module it follows from Theorems 2.2 and 2.3 of [3] that Δ is a maximal order.

Suppose now that $\Delta(f, S, G)$ is equivalent to a crossed product $\Delta(g, T, H)$. We shall prove that T cannot be a tamely ramified extension of R. The definition of equivalence implies that there exist finitely generated free R-modules E_1 and E_2 such that $\Delta(f, S, G) \otimes_R \mathrm{Hom}_R(E_1, E_1) \cong \Delta(g, T, H) \otimes_R \mathrm{Hom}_R(E_2, E_2)$. Let rad $T = (A)$. Then the above isomorphism must map $\sqrt{2}$ into Au where u is a unit in $\Delta(g, T, H) \otimes \mathrm{Hom}(E_2, E_2)$. Therefore $A^2 = 2v$ for some element v in $U(T)$. Hence the ramification index of T over R is two, and so T cannot

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be a tamely ramified extension of R.

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