ON SCHAUDER BASES FOR SPACES OF CONTINUOUS FUNCTIONS[1)]

H.W. Ellis and D.G. Kuehner[2)]

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1. Introduction. In a finite dimensional vector space V a set x_i, $i = 1, 2, \ldots, n$ of vectors of V is said to be a basis, base, or coordinate system for V if the vectors x_i are linearly independent and if each vector in V is a linear combination of the elements x_i with real coefficients. If a topology for V is defined in terms of a norm $\|\cdot\|$ then $\{x_i\}$ is a basis for V if and only if to each $x \in V$ corresponds a unique set of constants a_i such that

$$\| x - \sum_1^n a_i x_i \| = 0.$$

In infinite dimensional normed vector spaces the above concepts of basis have different generalizations. The first or algebraic definition gives a Hamel basis which is a maximal linearly independent set [1, p. 2]. We shall be interested in the other or topological definition.

DEFINITION. A set of elements x_i, $i = 1, 2, \ldots$ in a real Banach space B is a countable or Schauder base or basis if to each x in B corresponds a unique set of real constants $\{a_i\}$ such that

$$\lim_{n\to\infty} \| x - \sum_1^n a_i x_i \| = 0.$$

A Schauder basis will be called an unconditional basis if, for each permutation p of the positive integers,

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2) Holder of a National Research Council Studentship.

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$$\lim_{n\to\infty} \| x - \sum_{i=1}^{n} a_{p(i)} x_{p(i)} \| = 0.$$

If a Banach space B has a Schauder basis $\{x_i\}$, the set of finite linear combinations of the basis elements with rational coefficients is dense in B. Since this set is countable, B must be separable. An important unsolved problem of analysis, first formulated by Schauder, is the basis problem. <u>Does every separable Banach space have a Schauder basis</u>? By actual construction it has been shown that most of the familiar separable Banach spaces do have Schauder bases. That the trigonometric system $\{\frac{1}{2}, \sin nx, \cos nx\}$, $n = 1, 2, \ldots$, is a Schauder basis for $L^2(-\pi, \pi)$ follows from known theorems in the theory of Fourier series [6, pp. 74-5]. This system is also a countable basis for $L^p(-\pi, \pi)$, $p > 1$ [6, 7.3 (i), p. 153] but not for $L^1(-\pi, \pi)$ [6, p. 155]. The Haar functions form a Schauder basis for $L^p(-\pi, \pi)$, $p \geq 1$ and generalizations of the Haar functions form a Schauder basis for a wide class of Banach function spaces [2]. The present state of the basis problem is given in [1, chapter IV].

If Schauder basis is replaced by unconditional basis the answer to the corresponding basis problem is known. In [1] it is shown that if X is an arbitrary compact Hausdorff space then C(X), the Banach space of continuous functions on X, can have an unconditional basis only if C*(X), the conjugate space, is separable. Since C*(X) is separable only in trivial cases this shows that in general the spaces C(X) do not have unconditional bases [1, corollary 1, p. 77]. If the answer to the basis problem is negative it is possible, but unlikely, that some of these spaces also fail to have Schauder bases. We do not believe that it has been shown that every separable C(X) has a Schauder basis. In the appendix we show that there is no loss of generality if X is assumed to be compact metric. In this case it is shown in [2] that there exists a system of generalized Haar functions for which the partial sums of the series for an arbitrary $f \in C(X)$ converge uniformly to f. However, since the generalized Haar functions are not in general in C(X) they cannot form a basis for C(X) and it does not appear possible to obtain a basis by some simple modification of these functions. In 1927 Schauder constructed a countable basis for the case $X = [0, 1]$ [5, see § 2]. It appears to be well known that a Schauder basis can be constructed for X an n-dimensional rectangle in Euclidean n-space although we do not know of any proofs in the literature.

The principal part of this exposition is the construction of two different Schauder bases for C(X), X a closed square. The

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first has properties more nearly analogous to the one dimensional case. In the second method the basis is defined in terms of Cartesian products of one dimensional basis elements and the results extend readily to n dimensional rectangles. We also consider briefly the existence of countable bases in certain spaces of continuous functions vanishing at infinity and in C(X) where X is compact as a Cartesian product of compact subsets of the line. In an appendix we consider the implications in the general case of the separability of C(X).

2. Schauder's basis in C(X), $X = [0, 1]$. Let $r_1 = 0$, $r_2 = 1$, $r_3, \ldots$ be dense in X. Define $x_1 = x_1(P) = 1 - P$; $x_2 = P$, $0 \leq P \leq 1$. If r_n falls between the adjacent points r, r' of the set r_i, $i = 1, 2, \ldots, n-1$, define

$$
\begin{aligned}
x_n(P) &= 0, && 0 \leq P \leq r, \quad r' \leq P \leq 1, \\
&= (P-r)/(r_n-r), && r \leq P \leq r_n, \\
&= (P-r')/(r-r_n), && r_n \leq P \leq r' .
\end{aligned} \tag{2.1}
$$

Given f(P) in C(X), set

$$
\begin{aligned}
&a_1 = f(r_1), \\
&s_n(P) = \sum_1^n a_i\, x_i(P), \quad a_n = f(r_n) - s_{n-1}(r_n), \quad n = 1, 2, \ldots
\end{aligned} \tag{2.2}
$$

We note that

(i) $x_n(P) \in C(X)$, $\|x_n\| = \sup\{|x_n(P)| : 0 \leq P \leq 1\} = 1$;

(ii) $x_n(P) = 0$, $P = r_1, r_2, \ldots, r_{n-1}$;

(iii) $s_n(P)$, $n = 2, 3, \ldots$, is in C(X), coincides with f(P) for $P = r_i$, $i = 1, 2, \ldots, n$ and is piecewise linear between adjacent points of the set $\{r_i\}$, $i = 1, \ldots, n$.

From (iii), the uniform continuity of f on X and the fact that the set $\{r_i\}$ is dense in X it follows that $\|f - s_n\| \to 0$ as $n \to \infty$. To prove uniqueness suppose that $f(P) = \sum a_i\, x_i(P)$ $= \sum b_i\, x_i(P)$. If e_i is the continuous linear functional on C(X) defined by $e_i(x) = x(r_i)$, $i = 1, 2, \ldots$,

$$0 = e_1\left[\sum (b_i - a_i)x_i\right] = (b_1 - a_1)e_1(x_1) = b_1 - a_1.$$

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Proceeding inductively, using (ii), it follows that $a_i = b_i$, $i = 1, 2, \ldots$.

3. A countable basis for C(S), S a closed square, first method. We call a function f(P) defined on S a pyramid function if there exists a pyramid with base $A \subset X$, with vertex above A, altitude unity, and such that f(P) vanishes outside of A and defines the lateral surface of the pyramid for P in A. In this section the basis functions will be pyramid functions. They are defined in stages corresponding to progressively finer partitions of S, the partitions being alternately determined by lines parallel to the sides and to the diagonals of S.

The basis functions x_i, $i = 1, 2, 3, 4$ are pyramid functions with vertices at the corners P_i of S and with triangular bases determined by the two sides adjacent to P_i and the corresponding diagonal. We define a_i by (2.2) with P_i replacing r_i, $i = 1, 2, \ldots$. The second stage consists of the single pyramid function x_5 on S as base with vertex at P_5, the center of S. It is easily seen that $s_5(P)$ is in C(S), coincides with f(P) at the center and four corners of S and determines a surface with plane triangular faces above the four triangles into which the diagonals partition S.

If D_1 is one of the isosceles right triangles into which the diagonals of S partition S and P_6 is the midpoint of the hypotenuse of D_1, $x_6(P)$ is the pyramid function with base D_1 and vertex at P_6. The refinement of the partition of S corresponding to x_6 is obtained by bisecting D_1 by the line P_5P_6. Then $s_6(P)$ defines a continuous surface with plane triangular faces above the triangles of this partition and coinciding with f(P) for $P = P_i$, $i = 1, 2, \ldots, 6$. The other three functions of stage three are similarly related to the three remaining triangles into which the diagonals partition S.

The remaining even stages correspond to successive partitions of each of the squares in the preceding even stage into four equal squares. The corresponding basis elements are then pyramid functions on square bases with vertices above the centers of the bases. Each remaining odd stage corresponds to a partitioning of S by all the diagonals of all the squares of the preceding even stage into a number of squares in S and a number of right isosceles triangles each with hypotenuse in a side of S. The basis elements are then right pyramid functions on square bases in the first case and pyramid functions with triangular bases and vertices above the midpoint of the hypotenuse in the second. The vertex of the i-th basis function is denoted by P_i.

Clearly (i) and (ii) of §1 are satisfied with r_i replaced by P_i. In place of (iii) we have

(iii)' $s_n(P) \in C(S)$, $n = 1, 2, \ldots$ and, for $n \geq 4$, $s_n(P)$ coincides with $f(P)$ for $P = P_i$, $i = 1, 2, \ldots, n$ and defines a surface with plane triangular faces above non-overlapping triangles in S with the points P_i, $i = 1, 2, \ldots, n$ as vertices of these triangles.

The dimensions of these triangles approach zero uniformly as $n \to \infty$. The proof that the functions $\{x_n\}$ form a countable basis for C(S) is essentially the same as that outlined above for one dimension.

The alternate stages in the construction have been used to give (i), (ii) and (iii)' and yet make sure that there is no line L such that, for some N, $x_n(P) = 0$, $P \in L$, $n > N$ for then $\lim_n s_n(P)$ would be piecewise linear on L and obviously not in general equal to f(P). If we had used only the functions in the even stages above, the lines $x = i/2^n$, $y = i/2^n$, $i = 1, 2, \ldots, 2^n$, $n = 1, 2, \ldots$ would all have been of this type. For the construction we have given the set of points P_i is dense not only in S but also on each of these lines.

If the square S is not closed, C(S) is not separable and thus does not have a countable basis [§6, theorem A]. Similarly no countable basis exists for C(X), X the whole plane. The functions $x_n(P)$, $n > 4$, form a countable basis for $C_o(S)$, the space of continuous functions vanishing on the boundary of the square S and in this case S need not be closed. We note that if h is a homeomorphism of the open square S^o onto the plane X, the functions $x_n'(P')$, $n = 5, 6, \ldots$ defined by $x_n'(hP) = x_n(P)$ form a countable basis for $C_o(X)$, the space of continuous functions on the plane vanishing at infinity. Alternatively a countable basis of pyramid functions is obtained for $C_o(X)$ by ordering the unit squares with integer vertices, taking a countable basis of pyramid functions for each square, ordering these elements by the diagonal method and finally combining those of the elements from the odd stages that are discontinuous (as functions on X) into continuous functions.

4. <u>Second method</u>. Let X = [a, b], Y = [c, d], suppose that $\{r_i\}$, $\{r_i'\}$, $r_1 = r_1' = 0$, $r_2 = r_2' = 1$, are dense in X and Y

respectively and let $\phi_i(x)$, $\phi_i'(y)$ be Schauder one dimensional bases for $C(X)$ and $C(Y)$ corresponding to these dense sets. Define

$$\phi_{ij}(x,y) = \phi_i(x)\,\phi_j'(y)$$

and order the functions ϕ_{ij} as follows

$$(4.1)\qquad \phi_{11},\ \phi_{21},\ \phi_{12},\ \phi_{22},\ \ldots,\ \phi_{n-1,n-1},\ \phi_{n1},\ \phi_{n2},\ \ldots,$$
$$\phi_{n,n-1},\ \phi_{1n},\ \phi_{2n},\ \ldots,\ \phi_{nn},\ \ldots .$$

If the subscript ij precedes the subscript mn in this ordering we write $(i,j) < (m,n)$. Given $f(P)$ in $C(X\times Y)$, $P = (x,y)$, we define

$$(4.2)\qquad a_{11} = f(r_1, r_1'),\quad s_{mn}(P) = \sum_{(i,j)\leq(m,n)} a_{ij}\phi_{ij}(P),$$
$$a_{mn} = f(r_m, r_n') - \sum_{(i,j)<(m,n)} a_{ij}\phi_{ij}(r_m, r_n').$$

We note that (ii) of §3 implies

$$(4.3)\qquad \phi_{mn}(r_i, r_j') = 1 \quad \text{if } i = m \text{ and } j = n,$$
$$= 0 \quad \text{if } (i,j) < (m,n).$$

We shall show that

$$(4.4)\qquad \left\| f - \sum a_{ij}\,\phi_{ij} \right\| = 0.$$

We note that

$$s_{mn}(P) = a_{mn}\,\phi_{mn}(P) + \sum_{(i,j)<(m,n)} a_{ij}\,\phi_{ij}(P),$$
$$s_{mn}(r_m, r_n') = a_{mn} + f(r_m, r_n') - a_{mn} = f(r_m, r_n'),$$

using (4.2) and (4.3). With (4.3) this implies that

$$(4.5)\qquad s_{mn}(r_i, r_j') = f(r_i, r_j') \quad \text{if } (i,j) \leq (m,n).$$

We first show that

$$(4.6)\qquad \| s_{nn} - f \| \to 0, \quad \| s_{n,n-1} - f \| \to 0$$

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as $n \to \infty$. Now $s_{nn}(P) \in C(X \times Y)$, and coincides with $f(P)$ at (r_i, r_j'), $i, j \leqslant n$. These points as vertices partition $X \times Y$ into $(n-1)^2$ rectangles and the density of $\{r_i\}$ in X, $\{r_i'\}$ in Y implies that the dimensions of these rectangles approach zero uniformly as $n \to \infty$. In the closures of these rectangles the X- and Y-sections of s_{mn} are linear. If P is in the closure of the rectangles with vertices P_i, i = 1, 2, 3, 4,

$$\min_k \left[s_{nn}(P_k) = f(P_k) \right] \leqslant s_{nn}(P) \leqslant \max_k \left[s_{nn}(P_k) = f(P_k) \right]$$

$$\| s_{nn} - f \| \leqslant \max_k | f(P_k) - f(P) | ,$$

and the first part of (4.6) follows from the uniform continuity of f(P). The second part follows by the same argument.

We next show that if

(4.7) $$\| s_{n-1, n-1} - f \| < \varepsilon ,$$

then

(4.8) $$\| s_{ni} - s_{n-1, n-1} \| < \varepsilon , \; i = 1, 2, \ldots, n-1.$$

Define

$$d_i(P) = s_{ni}(P) - s_{n-1,n-1}(P) = \sum_{j=1}^{i} a_{nj} \, \phi_n(x) \, \phi_j'(y).$$

Thus $|d_i(x, y)|$ assumes its maximum values for $x = r_n$ and, since $d_i(r_n, y)$ is continuous and piecewise linear between the points $y = r_j'$, $j = 1, 2, \ldots, i$, it is sufficient to prove that

(4.9) $$| d_i(r_n, r_j') | < \varepsilon , \; j = 1, 2, \ldots, i.$$

Since $d_i(r_n, r_j') = d_{i-1}(r_n, r_j')$ if $j < i$ and $|d_i(r_n, r_i')| = |f(r_n, r_i') - s_{n-1,n-1}(r_n, r_i')| < \varepsilon$ by (3.7), simple induction proves (4.9).

To complete the proof of (4.4) we show that

(4.10) $$\| s_{in} - s_{n, n-1} \| < \varepsilon , \; i = 1, 2, \ldots, n,$$

if n is sufficiently large. Define

$$f_1(x, y) = \phi_1'(y) f(x, r_1'),$$

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$$f_n(x,y) = \phi_n'(y) \left[f(x,r_n') - \sum_{i=1}^{n-1} f_i(x,r_n')\right], \ n > 1.$$

Since $\phi_n'(r_i') = 0$ if $n > i$,

$$\sum_{i=1}^{n} f_i(x,r_j') = f(x,r_j'); \ j = 1,2,\ldots,n; \ n = 1,2,\ldots,$$

and since for each x, $\sum_{i=1}^{n} f_i(x,r_j')$ is linear between adjacent points of the set $y = r_i'$, $i = 1,2,\ldots,n$ the fact that $\{r_i'\}$ is dense in Y and f(x, y) is uniformly continuous implies that

$$\|f - \sum_1^\infty f_i\| = 0,$$

and $\|f_n\| < \varepsilon$ if n is sufficiently large. We note that, for each n, $f_n(x,r_n') \in C(X)$ and

$$f_n(x,r_n') = \sum_1^\infty a_{in} \phi_i(x).$$

Now, if n is sufficiently large that $\|f_n\| < \varepsilon$,

$$|s_{in}(P) - s_{n,n-1}(P)| = |\sum_{j=1}^{i} a_{jn}\phi_j(x)\phi_n'(y)| \leq |\sum_{j=1}^{i} a_{jn}\phi_j(x)| .$$

The last expression on the right is the absolute value of a Schauder partial sum for $f_n(x,r_n')$ in C(X) and is therefore bounded by

$$\sup\{|f_n(x,r_n)| : x \in X\} \leq \|f_n\| < \varepsilon .$$

The uniqueness of the coefficients is then established as before using (4.3) and the continuous linear functionals e_{ij} defined by $e_{ij}(f) = f(r_i, r_j')$.

5. C(X) with X compact. In this section we describe briefly a simple countable basis for C(X) when X is a compact (that is a closed and bounded) subset of the line. Let α = g.l.b. X, β = l.u.b. X. Then the complement of X relative to $[\alpha, \beta]$ is open and therefore can be expressed as $\bigcup_1^n (\alpha_i, \beta_i)$, $n \leq \infty$, with the open intervals (α_i, β_i) disjoint. Define

$$x_0(P) = (P - \alpha)/(\beta - \alpha), \ \alpha \leq P \leq \beta ;$$

$$x_1(P) = -(P - \beta)/(\beta - \alpha) .$$

Proceeding by induction, the points α_i, β_i, $i = 1,2,\ldots,n-1$

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partition $[\alpha, \beta]$ into intervals. If (α_n, β_n) falls in the interval (β', α'), define

$$x_{2n}(P) = (P - \beta')/(\alpha_n - \beta'); \quad \beta \leq P \leq \alpha_n;$$

$$= 0 \text{ elsewhere}; \qquad n = 1, 2, \ldots;$$

$$x_{2n+1}(P) = -(P - \alpha')(\alpha_n - \beta')^{\prime}; \quad \beta \leq P \leq \alpha_n;$$

$$= 0 \text{ elsewhere}; \qquad n = 1, 2, \ldots.$$

With the coefficients chosen as in (2.2) with $\{\alpha, \beta, \alpha_i, \beta_i\}$ replacing $\{r_i\}$, $s_{2n}(P)$ will coincide with $f(P) \in C(X)$ at $\alpha, \beta, \alpha_1, \beta_1, \ldots, \alpha_{n-1}, \beta_{n-1}, \alpha_n$ and $s_{2n+1}(P) = f(P)$ at these points plus β_n. If X is nowhere dense in $[\alpha, \beta]$ the functions x_i, $i = 1, 2, \ldots$ will form a countable basis for C(X). If X is not nowhere dense in $[\alpha, \beta]$, then for some value or values of n one or more of the closed intervals complementary to $\bigcup_1^n (\alpha_i, \beta_i)$ on $[\alpha, \beta]$ will be contained in X. Completing the one dimensional Schauder basis for each such closed interval, there is an obvious ordering whereby the original elements plus all the additional elements for all the closed intervals, is a countable basis for C(X) with (i), (ii) and the equivalent of (iii) all holding. If $X = X_1 \times X_2 \ldots \times X_n$, where each X_i is a compact subset of the line, arguments similar to those given in §4 can be used to show that the set of all Cartesian products of basis elements of the above type for each $C(X_i)$ is, with a suitable ordering, a countable basis for C(X).

6. Appendix. The separability of C(X). If X is an arbitrary topological space, C(X) will denote the space of bounded continuous functions on X. If X is locally compact $C_o(X)$ denotes the space of continuous functions vanishing at infinity (i.e. such that for each $\varepsilon > 0$, $\{x : f(x) \geq \varepsilon\}$ is compact). Topological concepts will be defined as in [3]. Most of the following results are well known.

If C(X) has a countable basis it must be separable as we have seen above. Suppose that $\{f_n(x)\}$, $n = 1, 2, \ldots$ is dense in C(X). Define a pseudo-metric ρ on X by

$$\rho(x_1, x_2) = \sum_1^\infty 2^{-n} \, |f_n(x_1) - f_n(x_2)| \, / 2 \, ||f_n|| \leq 1.$$

Then ρ is a metric if and only if C(X) separates points of X. Let T denote the original topology, M the metric or pseudo-

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metric topology. It is easy to show directly that T always contains M. There exist regular Hausdorff spaces for which C(X) consists of the constant functions [3, p. 117]. Since M is then the indiscrete topology it is clear that M and T need not be the same. However, since $M \subset T$, $C(X, M) \subset C(X, T)$. Since every $f \in C(X)$ is continuous for the M-topology $C(X,M) = C(X, T)$. It is clear that a Schauder basis for one of these spaces is a Schauder basis for the other. If X is a pseudo-metric space the subspace X_o obtained by using one representative element from the closure of each one point set is, with the relative topology, a metric space and the existence of a countable basis in $C(X_o)$ implies the existence of a countable basis in C(X). Thus as far as the basis problem is concerned there is no loss of generality in assuming that X is a metric space. We note that when the continuous functions on X determine the topology of X, that is when X is completely regular, the separability of C(X) implies that T = M and T is metrizable.

THEOREM A. If X is a metric space and C(X) is separable then X is compact.

We note that if X is metric compactness is equivalent to:- For each sequence in X there is a subsequence converging to a point of X [3, p. 138].

Suppose that X is metric but not compact. A metric space is normal [3, theorem 10, p. 120] and therefore completely regular [3, p. 117]. If X is not compact there exists an infinite sequence $\{x_i\}$ of different points of X with no convergent subsequence. By induction construct a sequence $\{S_i\}$ of disjoint closed spheres with centers x_i and decreasing radii. Then $\bigcup_1^\infty S_i$ is closed. For each i the complete regularity of X and the existence of $e(x) \equiv 1$ in C(X) implies the existence of a function $f_i \in C(X)$ with $\|f_i\| = 1$ and with $f_i(x_i) = 1$, $f_i(x) = 0$, $x \in \sim S_i$. Let $\alpha = 0.a_1a_2 \ldots$ denote the dyadic expansion of an arbitrary number between 0 and 1 and let $f_\alpha(x) = \sum_1^\infty a_i f_i(x)$. Then $f_\alpha \in C(X)$. Since $\|f_{\alpha_1} - f_{\alpha_2}\| = 1$ if $\alpha_1 \neq \alpha_2$, C(X) cannot be separable giving a contradiction.

COROLLARY. If X is a subset of n-dimensional Euclidean space with the relative topology and C(X) is separable then X is closed and bounded.

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THEOREM B. If X is a compact metric space $C(X)$ is separable [3, S(d), p. 245].

A proof closely related to that of the Stone-Weierstrass theorem can be given. It is based on the following lemma.

LEMMA. Let A be a set of real-valued continuous functions on a compact space X which is closed under the lattice operations $f \cup g = \max [f(t), g(t)]$, $f \cap g = \min [f(t), g(t)]$. Then the uniform closure of A contains every function in $C(X)$ which can be approximated at every pair of points by a function of A [4, p. 8].

To prove the theorem it is thus sufficient to construct a <u>countable</u> set A closed under the lattice operations and such that every function in $C(X)$ can be approximated at every pair of points by a function of A. A compact metric space is normal and separable. Let $\{x_i\}$, $i = 1, 2, \ldots$ be dense in X. For $i \neq j$ let S_i, S_j denote the closed spheres with centers x_i and x_j and radii $\rho(x_i, x_j)/3$. By Urysohn's lemma [3, p. 115] there exists $f_{ij}(x) \in C(X)$, vanishing in S_i and unity in S_j. Let A_o denote the countable set of functions $r_1 + r_2 f_{ij}(x)$ where i, j runs through the set of pairs of different positive integers and r_1, r_2 run through the rationals. Let A denote the closure of A_o under the lattice operations. Then A is countable.

Let $f(x)$ be an arbitrary element of $C(X)$, p, q arbitrary points of X. Given $\varepsilon > 0$, r sufficiently small, $|f(p) - f(x)| < \varepsilon/2$ if $\rho(p, x) < r$; $|f(q) - f(x)| < \varepsilon/2$ if $\rho(q, x) < r$. If $r < \rho(p, q)/6$ and $x_i \in \{x : \rho(x, p) < r\}$, $x_j \in \{x : \rho(x, q) < r\}$, then $p \in S_i$ and $q \in S_j$ and

$$f_{ij}(p) = f_{ij}(x_i), \quad f_{ij}(q) = f_{ij}(x_j). \tag{6.1}$$

Let $h(x) = r_1 + r_2 f_{ij}(x)$ where r_1 and r_2 are chosen so that $|f(x_k) - h(x_k)| < \varepsilon/2$, $k = 1, 2$. Then $h \in A_o \subset A$ and $|f(p) - h(p)| < \varepsilon$, $|f(q) - h(q)| < \varepsilon$.

In the above argument the metric has been used to obtain (6.1). It is of interest to note that there exist separable compact Hausdorff spaces X for which $C(X)$ is not separable [3, p. 164]. Such an X is completely regular and would be metrizable if $C(X)$ were separable.

If X is a metric locally compact space that is not compact, $C(X)$ is not separable. However, if X is separable, as is the case when X is the union of a countable collection of compact sets, and if $X^* = X \cup \{\infty\}$ is the one point compactification of X [3, p. 150], the argument given above can be modified to show that the subset of $C(X^*)$ consisting of the functions vanishing at infinity is separable and this implies that $C_0(X)$ is separable.

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Summer Research Institute
and
Queen's University