Can. J. Math., Vol. XXXIII, No. 2, 1979, pp. 485–499

THE POINCARÉ DUAL OF A GEODESIC ALGEBRAIC CURVE IN A QUOTIENT OF THE 2-BALL

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Introduction. We shall consider an irreducible, non-singular, totally geodesic holomorphic curve N in a compact quotient $M = \Gamma\backslash D$ of the unit ball $D = \{(z, w) : |z|^2 + |w|^2 < 1\}$ in $\mathbf{C}^2$ with the Kahler structure provided by the Bergman metric. The main result of this paper is an explicit construction of the harmonic form of type (1, 1) which is dual to N. Our construction is as follows. Let $p : D \rightarrow \Gamma\backslash D$ be the universal covering map. Choose a component D_1 in the inverse image of N under p. The choice of D_1 corresponds to choosing an embedding of the fundamental group of N into Γ. We denote the image by Γ_1. Let $\pi : D \rightarrow D_1$ be the fiber bundle obtained by exponentiating the normal bundle of D_1 in D. Let μ be the volume form of D_1. We define a family of closed (1, 1) forms ψ, depending on a complex parameter s, by the formula:

$$(0.1) \quad \psi(z, s) = \frac{s}{4\pi} \|\pi^*\mu\|^s \left\{ *\pi^*\mu - \frac{1}{s} \pi^*\mu \right\}.$$

We define a meromorphic family of closed forms on M by the following series, convergent for $\operatorname{Re} s > 1$:

$$(0.2) \quad \Omega(z, s) = \sum_{\Gamma_1\backslash\Gamma} \gamma^*\psi(z, s).$$

We then have the following theorem.

THEOREM. (i) $\Omega(z, s)$ *has a meromorphic extension to all of* $\mathbf{C}$ *and satisfies the differential functional equation:*

$$(0.3) \quad \Delta\Omega(z, s) + (s^2 - 1)\Omega(z, s) = (s^2 - 1)\Omega(z, s + 1).$$

(ii) $s = 1$ *is not a pole for* Ω *and* $\Omega(z, 1)$ *is the harmonic form of type* (1, 1) *dual to* N; *accordingly,* $\Omega(z, 1)$ *is never identically zero.*

The above represents the special case of the general theory of the hyperbolic Eisenstein series for pairs $\{G, G_\sigma\}$, where G_σ is the centralizer of an involution σ of G, corresponding to the pair $\{SU(2, 1), U(1, 1)\}$. This case is significant because of the curvature correction term $-1/s\pi^*\mu$ in (0.1). Such a term did not appear in [**4**] and [**5**] where the pairs

Received November 23, 1979. The first author was supported by N.S.F. Grant No. MCS78-02817, the second author was supported by NSERC Grant No. A3509 and the Sloan Foundation.

https://doi.org/10.4153/CJM-1981-042-x Published online by Cambridge University Press

$\{SO(n, 1), SO(k, 1)\}$ were considered. The reason for the appearance of this correction term is that the normal bundle of D_1 in D is not flat.

The above theorem generalizes easily to compact cycles in non-compact finite volume quotients in the ball; however, unlike the case of a closed geodesic in a Riemann surface, the dual form is never a cusp form. In fact, the dual of an irreducible variety N in a general non-compact finite volume quotient of a domain is never a cusp-form because there is an invariant form τ (some power of the Kahler form) which has a nonzero period over N. This period is exactly the inner product of $*\tau$ and the dual form of N. But $*\tau$ is a residue of an Eisenstein series. The generalization of our theory to non-compact cycles will require additional work.

A discussion of a representation-theoretic interpretation of our work is the content of the remark following Lemma 2.5. In particular the form $\psi(z, 1)$ gives rise to an embedding of the non-holomorphic discrete series representation $\pi_{1,1}$ of $SU(2, 1)$ in the square-integrable harmonic functions on $U(1, 1)\backslash SU(2, 1)$. It can be shown that $\pi_{1,1}$ occurs exactly once in this space. Finally, analogous results could be obtained using totally real curves. This corresponds to the pair $\{SU(2, 1), SO(2, 1)\}$. In this case, the dual form is the imaginary part of a holomorphic 2-form. This form is never zero provided N is orientable.

We would like to thank Blaine Lawson and Jim Simons for helpful conversations.

1. Topological preliminaries. We now explain what is meant by the term "dual" in the first paragraph of this paper. The algebraic curve N gives rise to an element, also denoted N, in $H_2(M; \mathbf{R})$. But we have the Poincaré duality isomorphism $H_2(M; \mathbf{R}) \rightarrow H^2(M; \mathbf{R})$. By the de Rham theorem we can regard $H^2(M; \mathbf{R})$ as the space of closed 2-forms modulo the exact 2-forms. We say any closed form ω whose class in $H^2(M; \mathbf{R})$ is the image of N under Poincaré duality is dual to N. The cohomology class of such a form is characterized by either of the following two properties:

(i) for any oriented 2-cycle c' we have

$$\int_{c'} \omega = c' \cdot N$$

(here $c' \cdot N$ denotes the intersection number of the cycles c' and N.)

(ii) for any closed 2-form η we have

$$\int_M \eta \wedge \omega = \int_N \eta.$$

Remark. Since N is algebraic it is enough to verify that (ii) is true for

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all η of type $(1, 1)$. By the Hodge theorem there exists a unique harmonic form dual to N which we will refer to as the dual of N.

Since $p: D \to M$ is the universal cover of M, the fundamental group of M acts holomorphically (hence isometrically) on D and we obtain a representation $\rho: \Gamma \to PSU(2, 1)$ with image (isomorphic to Γ) a torsion-free, co-compact discrete subgroup of $PSU(2, 1)$, the group of automorphisms of D. Since N is totally geodesic in M, the fundamental group Γ_1 of N injects into Γ. Choosing an inverse image D_1 of N under p, is equivalent to choosing an embedding $\Gamma_1 \to \Gamma$ and consequently an embedding $\Gamma_1 \to PSU(2, 1)$. We assume henceforth that such a choice has been made. We note that D_1 must be a transform of the set $\{(z, 0) : |z| < 1\}$ by an element of $PSU(2, 1)$, since all totally geodesic holomorphic copies of the 1-ball in D are obtained in this fashion. As a consequence, we see that the automorphism group of D_1 is isomorphic to $PSU(1, 1)$, so that the following diagram of inclusion is commutative.

$$\begin{array}{ccc} G_1 & \longrightarrow & SU(2, 1) \\ \uparrow & & \uparrow \\ \Gamma_1 & \longrightarrow & \Gamma \end{array}$$

We begin our problem of constructing the dual of N by reducing the problem to that of constructing a special form on the cylinder $E = \Gamma_1 \backslash D$.

LEMMA 1.1. *Suppose ϕ is a closed integrable 2-form on E satisfying the following conditions:*

(i) *if η is any bounded closed 2-form of type $(1, 1)$ then*

$$\int_E \eta \wedge \phi = \int_N \eta,$$

(ii) *if $\{\gamma_\alpha : \alpha \in I\}$ is a set of coset representatives for Γ_1 in Γ then the series $\sum_{\alpha \in I} \gamma_\alpha{}^* \phi$ converges (we will henceforth denote such sums by $\sum_{\Gamma_1 \backslash \Gamma} \gamma^* \phi$). Then $\omega = \sum_{\Gamma_1 \backslash \Gamma} \gamma^* \phi$ projects to a form on M which is dual to N.*

Proof. In order to have reasonable notation we will identify an invariant form under an equivalence relation with its projection to the quotient space. Let $\mathscr{D}$ be a fundamental domain for Γ in D and $\mathscr{D}_1$ be a fundamental domain for Γ_1 in D. We observe

$$\mathscr{D}_1 = \bigcup_{\alpha \in I} \gamma_\alpha(\mathscr{D}).$$

We now prove that ω is dual to N by proving Property (ii) of paragraph

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1 of this section. Let η be a closed form on M. Then:

$$\int_M \eta \wedge \omega = \int_{\mathcal{D}} \eta \wedge \omega = \sum_{\alpha \in I} \int_{\mathcal{D}} \eta \wedge \gamma_\alpha{}^* \phi$$
$$= \sum_{\alpha \in I} \int_{\mathcal{D}} \gamma_\alpha{}^*(\eta \wedge \phi) = \sum_{\alpha \in I} \int_{\gamma_\alpha \mathcal{D}} \eta \wedge \phi$$
$$= \int_{\mathcal{D}_1} \eta \wedge \phi = \int_E \eta \wedge \phi = \int_N \eta.$$

With this the lemma is proved.

We are left with the problem of constructing a form ϕ satisfying the conditions (i) and (ii). We first note that we may identify the manifold E to the normal bundle of N in M by exponentiating the normal fibers using the Riemannian exponential map in M. We also obtain a fibering $\pi : D \rightarrow D_1$ which is a homogeneous G_1 bundle, consequently is invariant under Γ_1 and induces the bundle E over N. We next note that condition (i) of Lemma 1.1 expresses the fact that ϕ is a dual of the zero section of E. By abstract algebraic topology we would expect any form representing the Thom class of E to satisfy (i). Motivated by the previous discussion, we will call a form satisfying the hypotheses of Lemma 1.1 a *Thom form*. We will construct the Thom form in the case of need in Section 2.

We now introduce a very useful operator.

Let ω be a form on $\Gamma_1 \backslash D$. Then we define $\Pi\omega$ by the formula:

$$\Pi\omega = \frac{1}{\operatorname{Vol} \Gamma_1 \backslash G_1} \int_{\Gamma_1 \backslash G_1} g^* \pi^* \omega.$$

Π is a projection operator onto the image of the G_1-invariant forms in $\Gamma_1 \backslash D$.

LEMMA 1.2. *Suppose ϕ is a G_1-invariant form. Then to verify the condition* (i) *of Lemma* 1.1 *it is sufficient to assume η is G_1-invariant.*

Proof. Π is an orthogonal projection; hence, a self-adjoint idempotent.

$$\int_E \eta \wedge \phi = (*\eta, \phi) = (*\eta, \Pi\phi) = (\Pi*\eta, \phi) = (*\Pi\eta, \phi)$$

and the lemma is proved.

2. The Thom form and coordinate computations. We now introduce normal coordinates. Choose a base-point p on D_1 and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal frame for the tangent space to D at p so that e_1 and e_2 are tangent to D_1 and e_3, e_4 are perpendicular to D_1. We extend e_1, e_2, e_3, e_4 to an orthonormal frame E_1, E_2, E_3, E_4 for $T(D)|D_1$ by using radial parallel translation from p in D. However, since D_1 is totally geodesic in D the vector fields E_1 and E_2 are always tangent to D_1 and the vector fields E_3 and E_4 are normal to D_1. We now assign the coordinates (x_1, x_2, x_3, x_4) to the point

$$\exp_q (x_3E_3(q) + x_4E_4(q))$$

where

$$q = \exp_p(x_1e_1 + x_2e_2).$$

We then change the rectangular coordinates x_3, x_4 for the fiber to polar coordinates by setting

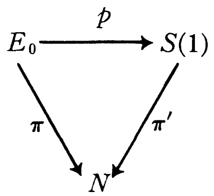
$$r = \sqrt{x_3^2 + x_4^2} \quad \text{and} \quad \theta = \text{arc tan}\, x_4/x_3.$$

We now construct a basis for the invariant (1, 1) forms on D. Let μ be the Riemannian volume form on D_1. We define a form ν on E by the formula that for $v, w \in T_y(E)$ we have:

$$\nu(v, w) = \text{vol}_F(p_F v, p_F w)$$

where p_F is the projection on the vertical vectors of π given by the Riemannian connection, F is the fiber through y and vol_F is the volume form on the Riemannian manifold F. Clearly ν and $\pi^*\mu$ are the required basis.

We now compute ν and $\pi^*\mu$ in normal coordinates. We let $S(r)$ for $r > 0$ denote the sub-bundle of E consisting of circles of radius r. We have a projection mapping $p:E_0 \to S(1)$ induced by the mapping $x \to x/\|x\|$ on the fibers where E_0 is the sub-bundle of vectors which are not zero; that is, the complement of $\Gamma_1\backslash D_1$ in $\Gamma_1\backslash D$. Then we have a commutative diagram:



where π' is the bundle mapping induced by π.

Now on $S(1)$ we have the connection form ω for the Riemannian connection on π. By [**3**], volume 2, page 277, the normal bundle has curvature $-\frac{1}{2}$ (note that the curvature transformation of the normal bundle is the restriction of the ambient curvature tensor). We choose the sign of the connection form so that $d\omega = -\frac{1}{2}\pi'^*\mu$. For this normalization the restriction of ω to the fiber is $-d\theta$, where $d\theta$ is the volume element of the circle (recall that the transgression of the fundamental class of the fiber is the negative of the Euler class). We define a form $\tilde{\omega}$ on E_0 by the formula $\tilde{\omega} = p^*\omega$. Then we have:

(2.1) $$d\tilde{\omega} = -\tfrac{1}{2}\pi^*\mu.$$

LEMMA 2.1. *Let K be the horizontal distribution of E. Then, we have:*
(i) *If $y \in \pi^{-1}(p)$, then $K|_y$ is spanned by $\partial/\partial x_1|_y$, $\partial/\partial x_2|_y$.*
(ii) *$K|_y$ is the orthogonal complement of the tangents along the fibers for all $y \in E$.*

Proof. It is sufficient to prove (ii) for points in $\pi^{-1}(p)$ by G_1 invariance. Hence we suppose $y \in \pi^{-1}(p)$. Suppose

$$y = \exp_p(a_2e_3 + a_4e_4).$$

We define a section s of E by the formula

$$s(q) = a_3E_3(q) + a_4E_4(q).$$

Since E_3 and E_4 are covariant constant at p, the section s is covariant constant at p and accordingly $K|_y$ is spanned by $ds|_p(e_1)$ and $ds|_p(e_2)$. But

$$ds|_p(e_1) = \left.\frac{\partial}{\partial x_1}\right|_y \quad \text{and} \quad ds|_p(e_2) = \left.\frac{\partial}{\partial x_2}\right| .$$

Thus we have proved (i).

The statement (ii) will follow if we can prove that for $1 \leqq i \leqq 2$, $3 \leqq j \leqq 4$ we have:

$$\left\langle \left.\frac{\partial}{\partial x_i}\right|_y, \left.\frac{\partial}{\partial x_j}\right|_y \right\rangle = 0.$$

There is an involutive isometry σ_2 of D, fixing $\pi^{-1}(p)$ and mapping D_1 into itself. We now check that σ_2 commutes with π. Since

$$\pi^{-1}(q) = \{\exp v : v \in T_q(D_1)^\perp\},$$

it is enough to prove

$$\sigma_2 \exp_q v = \exp_{\sigma_2 q} d\sigma_2(v).$$

But this follows because σ_2 is an isometry. Thus $d\pi \circ d\sigma_2 = d\sigma_2 \circ d\pi$. Since σ_2 is an isometry, it maps K to itself. Suppose now $y \in \pi^{-1}(p)$ and $v \in K|_y$. Then $d\pi(v) \in T_p(D_1)$ and hence

$$d\sigma_2 \circ d\pi(v) = -d\pi(v) = d\pi \circ d\sigma_2(v).$$

Hence $d\sigma_2(v)$ is the unique horizontal vector projecting to $-d\pi(v)$. But $-v$ is another such vector. Hence $d\sigma_2(v) = -v$ and we find $d\sigma_2|K_y$ is the negative of the identity map. But $d\sigma_2$ restricted to the vertical is the identity and the lemma follows because distinct eigenspaces of an isometry are orthogonal.

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LEMMA 2.2. *Suppose $y \in E$ satisfies $\pi(y) = p$. Then we have*

(i) $\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle \Big|_y = 0$

(ii) $\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right\rangle \Big|_y = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2} \right\rangle \Big|_y = \left(\cosh \frac{r}{2}\right)^2.$

Proof. We recall a standard result (see [**1**], page 90). Let ξ be the normal vector to D_1 at p with $\exp_p \xi = y$. Let α be the geodesic given by

$$\alpha(s) = \exp_p \xi/\|\xi\|.$$

Let $\|\xi\| = r$. For $j = 1, 2$, define Jacobi fields $V_j(s)$ along α by the initial conditions:

$$V_j(0) = e_j,\ V_j'(0) = 0.$$

Then:

$$\frac{\partial}{\partial x_j}\Big|_y = V_j(r).$$

Assuming this result, the lemma follows by solving the Jacobi equation along α. This equation is:

$$\nabla_T^2 V_j - R_{T,V_j} T = 0$$

where T is the unit tangent field to α.

If we let X_j, for $j = 1, 2$, be the parallel translate of e_j along α then we know by Lemma 2.1 that there exist functions f_{j1}, f_{j2} so that

$$V_j(r) = f_{j1}(r)X_1(r) + f_{j2}(r)X_2(r).$$

Now recalling that the curvature tensor R for D is parallel and using the formula of [**3**], page 277, for R we find:

$$R_{T(r),X_j(r)}T(r) = \tfrac{1}{4}X_j(r)$$

and consequently

$$f_{11}(r) = f_{22}(r) = \cosh(r/2) \quad \text{and} \quad f_{12}(r) = f_{21}(r) = 0.$$

The lemma follows.

COROLLARY. *If $\pi(y) = p$ then we have:*

(i) $\langle dx_1, dx_2\rangle|_y = 0$

(ii) $\langle dx_1, dx_1\rangle|_y = \langle dx_2, dx_2\rangle|_y = 1\Big/\left(\cosh \frac{r}{2}\right)^2$

(iii) $\langle \pi^*\mu, \pi^*\mu\rangle|_y = \langle dx_1 \wedge dx_2, dx_1 \wedge dx_2\rangle = 1\Big/\left(\cosh \frac{r}{2}\right)^4.$

Remark. By G_1 invariance, the formula

$$\|\pi^*\mu\| = 1 \Big/ \left(\cosh \frac{r}{2}\right)^2$$

is valid for all $y \in E$.

LEMMA 2.3.

(i) $\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle \Big|_y = \sinh r$ *for all* $y \in E_0$.

(ii) $\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial r} \right\rangle \Big|_y = 0$ *for all* $y \in E_0$.

Proof. For each fiber $\pi^{-1}(q)$, the functions $\{r, \theta\}$ are the usual geodesic polar coordinates with center q. But each fiber is a hyperbolic plane with curvature -1. The lemma is now immediate.

LEMMA 2.4. *If* $\pi(y) = p$ *and* $y \neq p$, *then* $\tilde{\omega}|_y = -d\theta|_y$.

Proof. We have seen

$$\tilde{\omega}\left(\frac{\partial}{\partial x_1}\Big|_y\right) = 0 \quad \text{and} \quad \tilde{\omega}\left(\frac{\partial}{\partial x_2}\Big|_y\right) = 0$$

in Lemma 2.2. But for all y in E_0 we have

$$\theta\left(\frac{\partial}{\partial x_1}\Big|_y\right) = 0 \quad \text{and} \quad \theta\left(\frac{\partial}{\partial x_2}\Big|_y\right) = \frac{\pi}{2};$$

hence

$$d\theta\left(\frac{\partial}{\partial x_1}\Big|_y\right) = 0 \quad \text{and} \quad d\theta\left(\frac{\partial}{\partial x_2}\Big|_y\right) = 0.$$

Since $d\theta$ and $\tilde{\omega}$ are defined as pull-backs of forms on $S(1)$ they must both annihilate $\partial/\partial r$. Thus if $y \in \pi^{-1}(p)$ there is a smooth function λ on $\pi^{-1}(p)$ so that

$$\tilde{\omega}|_y = \lambda(y) d\theta|_y.$$

Noting that $\lambda \circ p = \lambda$ and $\lambda \equiv -1$ on the unit circle in $\pi^{-1}(p)$, the lemma is proved.

COROLLARY (i) $\langle \tilde{\omega}, dr \rangle|_y = 0$ *for all* y *in* E_0

(ii) $\nu = -\sinh r\, dr \wedge \tilde{\omega}$

(iii) $d\nu = -\frac{1}{2} \sinh r\, dr \wedge \pi^*\mu$.

Proof. The first two formulas follow from the observation that they are G_1 invariant and for $y \in \pi^{-1}(p)$ reduce to:

(i)′ $\langle dr, d\theta \rangle = 0$

(ii)′ $\nu = \sinh r\, dr \wedge d\theta$

where they follow from Lemma 2.3.

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The last formula follows from the formula (2.1).

Remark. Though we won't make use of it, we can compute $\tilde{\omega}$ explicitly. Indeed, since $\{dx_1, dx_2, dr, d\theta\}$ span $T_y^*(E_0)$ for all y, there are smooth functions λ_1, λ_2 on E_0 so that

$$\tilde{\omega} = -d\theta + \lambda_1 dx_1 + \lambda_2 dx_2.$$

But noting that $\tilde{\omega} + d\theta$ annihilates vertical vectors and is invariant under dilations of the fiber, we see that the form $\lambda_1 dx_1 + \lambda_2 dx_2$ is the lift of a form from the base; that is, there exist smooth functions μ_1, μ_2 on D_1 so that $\lambda_1 = \pi^*\mu_1, \lambda_2 = \pi^*\mu_2$. But we have:

$$E_3^*\tilde{\omega}\left(\frac{\partial}{\partial x_1}\right) = \mu_1$$

$$E_3^*\tilde{\omega}\left(\frac{\partial}{\partial x_2}\right) = \mu_2.$$

Hence, if we define, for $i = 1, 2$:

$$\Gamma_{13}^{\ 4} = \langle \nabla_{\partial/\partial x_1} E_3, E_4\rangle \quad \text{and} \quad \Gamma_{23}^{\ 4} = \langle \nabla_{\partial/\partial x_2} E_3, E_4\rangle$$

we have $\mu_1 = \Gamma_{13}^4$ and $\mu_2 = \Gamma_{23}^4$ and hence

$$\tilde{\omega} = -d\theta + \pi^*\mu_1 dx_1 + \pi^*\mu_2 dx_2.$$

Changing to polar coordinates $\rho = \sqrt{x_1^2 + x_2^2}$, $\psi = \arctan x_2/x_1$ we have, noting E_3 is parallel along rays emanating from ρ:

$$\tilde{\omega} = -d\theta + B(\rho)d\psi$$

and noting $E_3^*d\tilde{\omega} = -\frac{1}{2}\sinh \rho d\rho \wedge d\psi$ we find:

$$\tilde{\omega} = -d\theta - \tfrac{1}{2}\cosh \rho d\psi - \tfrac{1}{2}d\psi. \tag{2.2}$$

Note that since $\|d\theta\| = \|\tilde{\omega}\| = 1/\sinh r$, the forms $d\theta$ and $\tilde{\omega}$ do not extend smoothly to E.

We are finally able to produce the Thom form.

PROPOSITION 2.1. (i) *For any s with* $\operatorname{Re} s > 1$, *the form*

$$\psi(z, s) = \frac{s}{4\pi}\|\pi^*\mu\|^s\left\{*\pi^*\mu - \frac{1}{s}\pi^*\mu\right\}$$

is a Thom form.

(ii) *ψ satisfies the differential equation:*

$$\Delta\psi(z, s) = -(s^2 - 1)\psi(z, s) + (s^2 - 1)\psi(z, s + 1).$$

Proof. We first note that the previous coordinate computations imply

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that ψ has the coordinate representation:

$$\psi(z, s) = -\frac{s}{4\pi}\left\{\frac{\sinh r}{\left(\cosh \frac{r}{2}\right)^{2s+2}} dr \wedge \tilde{\omega} + \frac{1}{s}\frac{1}{\left(\cosh \frac{r}{2}\right)^{2s}} \pi^*\mu\right\}. \tag{2.3}$$

We note that $\|\psi\|$, the pointwise norm of ψ, is given by:

$$\|\psi\| = \alpha(s)\frac{1}{\left(\cosh \frac{r}{2}\right)^{2s+2}}$$

where $\alpha(s)$ depends only on s and is finite if $s \neq 0$.

From the above we see that ψ has the following properties:

(i) ψ is G_1 invariant for all s
(ii) ψ is closed for all s
(iii) $\|\psi\|$ is integrable on E provided Re $s > 1$
(iv) $\|\psi\| = o\left(1 \Big/ \left(\cosh \frac{r}{2}\right)^4\right)$ provided Re $s > 1$
(v) $\int_E \psi = 1$ where F is any fiber of π.

We now show these properties imply ψ is a Thom form if Re $s > 1$.

Let η be a closed and bounded form on E. We must show:

$$\int_E \eta \wedge \psi = \int_N \eta.$$

By Lemma 1.2 we may suppose η is G_1 invariant. Since ν and $\pi^*\mu$ span the G_1 invariant (1, 1) forms we may write

$$\eta = B_1(r)dr \wedge \tilde{\omega} + B_2(r)\pi^*\mu.$$

We then see:

$$\begin{aligned} d\eta &= -B_1(r)dr \wedge d\tilde{\omega} + B_1'(r)dr \wedge \pi^*\mu \\ &= [\tfrac{1}{2}B_1(r) + B_2'(r)]dr \wedge \pi^*\mu. \end{aligned}$$

Thus η is closed if and only if

$$B_1(r) = -2B_2'(r)$$

and we may rewrite η as

$$\eta = -2B'(r)dr \wedge \tilde{\omega} + B(r)\pi^*\mu.$$

We note that since $\|\eta\|$ is bounded we have

$$|B(r)| \leqq C \cosh r \quad \text{for some constant } C.$$

Since ψ is closed the above argument shows that there exists a function $A(r, s)$ so that:

$$\psi = -2A'(r, s)dr \wedge \tilde{\omega} + A(r, s)\pi^*\mu.$$

Clearly

$$A(r, s) = -\frac{1}{4\pi}\frac{1}{\left(\cosh\frac{r}{2}\right)^{2s}} = \beta(s)\|\pi^*\mu\|^s.$$

We then have

$$\begin{aligned}\eta \wedge \psi &= -2[A'(r, s)B(r) + A(r, s)B'(r)]\pi^*\mu \wedge dr \wedge \tilde{\omega} \\ &= 2[A'(r, s)B(r) + A(r, s)B'(r)]\pi^*\mu \wedge dr \wedge d\theta.\end{aligned}$$

Since ψ is integrable we have

$$\int_E \eta \wedge \psi = \lim_{\substack{r\to\infty \\ \epsilon\to 0}} \int_{E(r,\epsilon)} \eta \wedge \psi$$

where $E(r, \epsilon)$ is the bundle of annuli of points in E located in the tube with boundary $S(r)$, the circle bundle of radius r, and the circle bundle $S(\epsilon)$ of radius ϵ. Now in $E(r, \epsilon)$ we have (abbreviating $A(r, s)$ by $A(r)$):

$$\eta \wedge \psi = 2d[A(r)B(r)\pi^*\mu \wedge d\theta] = 2d\left[\frac{A(r)B(r)}{(\sinh r)\left(\cosh\frac{r}{2}\right)^2} i_{\partial/\partial r}\operatorname{vol}_E\right].$$

Thus:

$$\int_E \eta \wedge \psi = 2\left[\lim_{r\to\infty}\int_{S(r)} \frac{A(r)B(r)}{(\sinh r)\left(\cosh\frac{r}{2}\right)^2}\operatorname{vol}_{S(r)} - \lim_{\epsilon\to 0}\int_{S(\epsilon)} \frac{A(\epsilon)B(\epsilon)}{(\sinh \epsilon)\left(\cosh\frac{\epsilon}{2}\right)^2}\operatorname{vol}_{S(\epsilon)}\right].$$

But:

$$\int_{S(r)} \frac{A(r)B(r)}{(\sinh r)\left(\cosh\frac{r}{2}\right)^2}\operatorname{vol}_{S(r)} = \frac{A(r)B(r)}{(\sinh r)\left(\cosh\frac{r}{2}\right)^2}\operatorname{vol} S(r)$$
$$= 2\pi A(r)B(r)\operatorname{vol} N.$$

Now, for $\operatorname{Re}(s) > 1$,

$$B(r) = O(\cosh r) \quad \text{and} \quad A(r) = o(1/\cosh r);$$

hence, the first limit is zero.

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Similarly, we observe:

$$2\int_{S(\epsilon)} \frac{A(\epsilon)B(\epsilon)}{(\sinh \epsilon)\left(\cosh \frac{\epsilon}{2}\right)^2} \operatorname{vol}_{S(\epsilon)} = 4\pi A(\epsilon)B(\epsilon) \operatorname{vol} N$$

and noting $4\pi A(0) = 1$ we find the limit to be

$$B(0) \operatorname{vol} N = \int_N \eta.$$

The second statement of the proposition follows from a straightforward calculation. With this the proposition is proved.

Remark 2.1. In [**5**], we chose ψ to be

$$\frac{1}{\kappa(s)} \|\pi^*\mu\|^s (*\pi^*\mu).$$

In that paper, we considered totally geodesic cycles in quotients of n-dimensional hyperbolic space. Any such cycle has a flat normal bundle and consequently $\|\pi^*\mu\|^s(*\pi^*\mu)$ was a closed form. In the case considered in this paper it was necessary to subtract the correction term $1/s\|\pi^*\mu\|^s\pi^*\mu$ in order to make ψ closed.

We now define the hyperbolic Eisenstein series Ω for the pair $\{SU(2, 1), U(1, 1)\}$, or more precisely, for the pair $\{\Gamma, \Gamma_1\}$, by the formula

$$(2.4)\quad \Omega(z, s) = \sum_{\Gamma_1\backslash\Gamma} \gamma^*\psi(z, s).$$

Since the Riemannian volume element, vol_E, for E is given by

$$\operatorname{vol}_E = \left(\cosh \frac{r}{2}\right)^2 (\sinh r)\, \pi^*\mu \wedge dr \wedge d\theta$$

we see that $\|\psi\|$ is integrable provided $\operatorname{Re} s > 1$ and we obtain:

LEMMA 2.5. *The series in* (2.4) *is absolutely convergent provided* $\operatorname{Re} s > 1$.

Remark 2.1. Consider the form

$$\phi = \frac{1}{4\pi} \|\pi^*\mu\| \{*\pi^*\mu - \pi^*\mu\}.$$

Then ϕ is easily seen to be square integrable and harmonic in $\Gamma_1\backslash G/K$. The lift of ϕ to G via the projection $p: G \to \Gamma_1\backslash G/K$ can be identified (by using left invariant forms on G) with a collection of functions on G which are right K equivariant, left invariant under G_1 and are annihilated by the Casimir (since ϕ is harmonic). This collection of functions can be

https://doi.org/10.4153/CJM-1981-042-x Published online by Cambridge University Press

projected to $G_1\backslash G$ to obtain a collection of harmonic, square-integrable K-finite functions on $G_1\backslash G$, also to be denoted ϕ; we note square integrability follows from the compactness of the fibers in the diagram:

$$\begin{array}{ccccc} K & & & & \Gamma_1\backslash G_1 \\ & \searrow & & \swarrow & \\ & & \Gamma_1\backslash G & & \\ & \swarrow & & \searrow & \\ G_1\backslash G & & & & \Gamma_1\backslash G/K \end{array}$$

A general representation theoretic argument then leads to the conclusion that ϕ generates a copy of the discrete series representation $\pi_{1,1}$ in the square-integrable harmonic functions on $G_1\backslash G$. By analogy with the usual theory of Eisenstein series one could rewrite Ω as $\Omega(z, s; \phi)$ and

$$(2.5) \quad \Omega(z, s; \phi) = \sum_{\Gamma_1\backslash\Gamma} \gamma^*\psi(z, s).$$

This general Ω would then be an intertwining operator from a space of functions on $G_1\backslash G$ to a space of tensors on $\Gamma\backslash G/K$. We do not yet know the correct domain for Ω.

3. The meromorphic continuation of the hyperbolic Eisenstein for the $\{SU(2, 1), U(1, 1)\}$ pair.

THEOREM 3.1. (i) *The hyperbolic Eisenstein series may be meromorphically continued to the entire complex plane and satisfies the differential functional equation:*

$$\Delta\Omega(z, s) + (s^2 - 1)\Omega(z, s) = (s^2 - 1)\Omega(z, s + 1).$$

(ii) $s = 1$ *is a regular value and* $\Omega(z, 1)$ *is the harmonic* $(1, 1)$ *form dual to c; hence, we may define the dual as* $\lim_{s\to 1} \Omega(z, s)$, *the limit of convergent series.*

Proof. We first prove (i). Let $\nu_1, \nu_2, \ldots, \nu_n \ldots$ be an orthonormal basis of eigenforms of type $(1, 1)$ for Δ on M. We assume $\Delta\nu_n = \lambda_n\nu_n$. Then for Re $s > 1$ we may write

$$(3.1) \quad \Omega(z, s) = \sum_{n=0}^{\infty} a_n(s)\nu_n(z)$$

where

$$a_n(s) = (\Omega(z, s), \nu_n) = \int_{\mathscr{D}} \Omega(z, s) \wedge *\nu_n.$$

Noting that $\Omega(z, s)$ is C^{∞} on M, which is compact, we can conclude that the series in 3.1 is uniformly convergent. By integrating by parts we obtain

$$\begin{aligned} C_k(s) &= (\Delta^k\Omega, \nu_n) = (\Omega, \Delta^k\nu) \\ &= \lambda_n{}^k(\Omega, \nu_n) \\ |a_n(s)| &\leqq |C_k(s)|/\lambda_n{}^k. \end{aligned}$$

By [**2**] we have $\lambda_n \sim cn^{1/2}$, hence

$$|a_n(s)| \leqq |C_k(s)|/n^{k/2}.$$

Since $C_k(s)$ is clearly decreasing as Re s goes to ∞ we have

$$|a_n(s)| \leqq \beta_k/n^{k/2} \text{ for Re } s \geqq \alpha > 0.$$

Now we use Proposition 2.1.

$$\Delta\Omega(z, s) + (s^2 - 1)\Omega(z, s) = (s^2 - 1)\Omega(z, s + 1).$$

Differentiating 3.1 term by term we obtain

$$(3.2) \quad \lambda_n a_n(s) + (s^2 - 1)a_n(s) = (s^2 - 1)a_n(s + 1)$$

$$a_n(s) = \frac{s^2 - 1}{\lambda_n + s^2 - 1} a_n(s + 1)$$

$$a_n(s - 1) = \frac{s(s - 2)}{\lambda_n + s(s - 2)} a_n(s)$$

$$a_n(s - 1) = \frac{s(s - 2)(s - 1)(s + 1)}{(\lambda_n + s(s - 2))(\lambda_n + (s - 1)(s + 1))} a_n(s + 1).$$

In this way we can continue the functions $a_n(s)$ to the entire plane with possible poles where the polynomials on the denominator of the recursion equation vanish. In particular there is no pole at $s = 1$. Indeed it is sufficient to use (3.2), since $a_n(s + 1)$ is defined for Re $s > 0$ by (3.1). Thus:

$$a_n(0) = 0 \text{ unless } n \text{ corresponds to the zero eigenvalue.}$$

If the zero eigenvalue has multiplicity r where $r = \dim H^{1,1}(X, C)$ we have, for $j = 1, 2, \ldots, r$:

$$a_j(s) = a_j(s + 1) \text{ for } s \text{ in a neighbourhood of } 1, s \neq 1.$$

Since a_j is bounded near $s = 2$ it must be bounded near $s = 1$ and $s = 1$ is a regular point.

Using the formula $\lambda_n \sim cn^{1/2}$ we see that if we keep away from the poles of $a_n(s)$ then the continued functions satisfy an even better estimate (in the new region where they are defined by the previous argument) than the old estimate; this is because λ_n occurs in the denominator of (3.2). Thus we have arrived at a continuation of $\Omega(z, s)$ to the entire complex plane which is regular at $s = 1$. Substituting $s = 1$ into the equation

$$\Delta\Omega(z, s) + (s^2 - 1)\Omega(z, s) = (s^2 - 1)\Omega(z, s + 1)$$

and recalling that Ω is regular at $s = 1$ and $s = 2$ we obtain

$$\Delta\Omega(z, 1) = 0.$$

We have constructed the harmonic form dual to N; indeed $\Omega(z, 1)$ is harmonic and the cohomology class of $\Omega(z, s)$ is constant, since it is constant for large s.

Since N is a Kahler submanifold of a Kahler manifold, it is never a boundary; hence, $\Omega(z, 1)$ is never identically zero. With this the theorem is proved.

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