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ON INVOLUTIVE LIE ALGEBRAS HAVING A CARTAN DECOMPOSITION

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We introduce the concept of Cartan decomposition relative to a Cartan subalgebra H in the sense of Y. Billig and A. Pianzola for involutive complex Lie algebras L of arbitrary dimension. If L has such a decomposition and is infinite dimensional and simple, we show it is $*$-isomorphic to a direct limit of classical finite dimensional simple involutive Lie algebras of the same type A, B, C or D.

1. PRELIMINARIES

Let L be a complex Lie algebra. An *involution* on L is a conjugate-linear map, $* : L \to L$ $(x \mapsto x^*)$, such that $(x^*)^* = x$ and $[x,y]^* = [y^*, x^*]$ for any $x, y \in L$. A Lie algebra furnished with an involution is an *involutive Lie algebra.* A *selfadjoint* subset of an involutive algebra is a subset globally invariant by the involution. If L_i $(i = 1, 2)$ are involutive Lie algebras and $f : L_1 \longrightarrow L_2$ is a morphism of Lie algebras, we say that f is a *$*$-morphism* whenever $f(x^*) = f(x)^*$ for all $x \in L_1$. We define the *Annihilator* of an involutive Lie algebra L as the selfadjoint ideal given by $\mathrm{Ann}(L) = \{x \in L : [x,y] = 0 \text{ for all } y \in L\}$. We shall say that L is *simple* if the product is nonzero and its only ideals are $\{0\}$ and L.

Billig and Pianzola introduced in [**2**] the concept of Cartan subalgebra for Lie algebras L of arbitrary dimension as follows:

DEFINITION 1.1: A subalgebra H of L is called a *Cartan subalgebra* if

(1) The elements of H act locally ad-nilpotently on H.

(2) H is its own normaliser in L, that is, $N_L(H) = H$.

If L is finite dimensional, then H is nilpotent by Engel's theorem and the classical definition of Cartan subalgebra is recovered.

In the framework of involutive Lie algebras we are interested in selfadjoint Cartan subalgebras of L. From here, unless otherwise stated, *throughout the paper H shall*

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denote a selfadjoint Cartan subalgebra of an involutive complex Lie algebra of arbitrary dimension L.

A *root* of L relative to H is a linear form commuting with the involution

$$\alpha : (H, *) \to (\mathbb{C}, ^{-}),$$

that is, $\alpha(h^*) = \overline{\alpha(h)}$ for any $h \in H$, (where $^-$ denotes the conjugation operator on $\mathbb{C}$), such that there exists $v_\alpha \in L$, $v_\alpha \neq 0$ satisfying $[h, v_\alpha] = \alpha(h)v_\alpha$ for any $h \in H$. The *root space* associated to α is the subspace $L_\alpha = \{v_\alpha \in L : [h, v_\alpha] = \alpha(h)v_\alpha \text{ for any } h \in H\}$. It is easy to prove that the root space associated to the zero root is contained in the Cartan subalgebra and, by the Jacobi identity, that if $\alpha + \beta$ is a root then $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$, and if $\alpha + \beta$ is not a root then $[L_\alpha, L_\beta] = 0$. Let us also note that $(L_\alpha)^* = L_{-\alpha}$. Indeed, for any $h \in H$ and $v_\alpha \in L_\alpha$, $[h, v_\alpha]^* = (\alpha(h)v_\alpha)^* = \overline{\alpha(h)}v_\alpha^*$, and from here $[h^*, v_\alpha^*] = -\overline{\alpha(h)}v_\alpha^* = -\alpha(h^*)v_\alpha^*$, the facts $H^* = H$ and $*^2 = *$ let us conclude easily the assertion. Given a set S of nonzero roots of L, we shall denote by $\mathrm{Sp}_{\mathbb{Z}} S$ the set of mappings

$$\mathrm{Sp}_{\mathbb{Z}} S = \left\{ \sum_{i=1}^{n} p_i \alpha_i : p_i \in \mathbb{Z} \text{ and } \alpha_i \in S \right\}.$$

DEFINITION 1.2: We shall call that L has a *Cartan decomposition* relative to H if

(1) $L = H \oplus \left(\bigoplus_{\alpha \in \Lambda} L_\alpha \right)$, where Λ is the set of all nonzero roots of L relative to H.

(2) Each L_α, $\alpha \in \Lambda$, is finite dimensional.

(3) For any finite set $S \subset \Lambda$ we have $\mathrm{Sp}_{\mathbb{Z}} S \cap \Lambda$ is also finite.

(4) There exists $v_\alpha \in L_\alpha$ such that $\alpha([v_\alpha, v_\alpha^*]) \in \mathbb{R}^+ - \{0\}$ for any $\alpha \in \Lambda$.

By using the ideas in [**11, 10, 16**] one could characterise infinite dimensional simple involutive Lie algebras over a field $\mathbb{K}$ of characteristic zero, however, we use entirely different methods to describe the complex case. In fact, the introduction of new techniques, such as the connections of roots to construct a direct system of adequate finite dimensional simple involutive Lie algebras, in the study of infinite dimensional Lie algebras is perhaps the most interesting novelty in this paper.

DEFINITION 1.3: Let $(I, \leqslant)$ be a directed set and $\{L_i\}_{i \in I}$ a family of involutive Lie algebras such that for $i \leqslant j$ there exists a $*$-monomorphism $e_{ji} : L_i \longrightarrow L_j$ such that $e_{ji}e_{ik} = e_{jk}$ and $e_{ii} = Id$ for all $i, j, k \in I$ with $k \leqslant i \leqslant j$. Then we shall say that $\mathcal{S} := (\{L_i\}_{i \in I}, \{e_{ji}\}_{i \leqslant j})$ is a *direct system* of involutive Lie algebras.

DEFINITION 1.4: Given $\mathcal{S}$ we define a *direct limit*, $\varinjlim \mathcal{S}$, as a couple $(L, \{e_i\}_{i \in I})$ where L is an involutive Lie algebra, $e_i : L_i \longrightarrow L$ is a $*$-monomorphism that satisfies $e_i = e_j e_{ji}$ and $(L, \{e_i\}_{i \in I})$ is universal for this property in the sense that if $(B, \{t_i\}_{i \in I})$ is another such couple, then there exists a unique $*$-monomorphism $\theta : L \longrightarrow B$ such that $t_i = \theta e_i$, $i \in I$.

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As in [**3**], we can prove that any direct system of involutive Lie algebras $\mathcal{S}$ has a direct limit. It is also clear that $\varinjlim \mathcal{S}$ is unique up to $*$-isomorphisms.

2. The description theorem

Unless otherwise stated, throughout this section L shall denote an infinite dimensional involutive Lie algebra with zero annihilator having a Cartan decomposition respect to H, and Λ the set of all nonzero roots.

Lemma 2.1. *The following assertions hold:*

(1) $\alpha(h_\alpha) \neq 0$ *for any* $0 \neq h_\alpha \in [L_\alpha, L_\alpha^*]$, $\alpha \in \Lambda$.

(2) *If* $[L_\alpha, L_\beta] = [L_{-\alpha}, L_\beta] = 0$ *then* $\beta(h_\alpha) = 0$ *for any* $h_\alpha \in [L_\alpha, L_\alpha^*]$, $\alpha, \beta \in \Lambda$.

Proof: 1. Similar to [**5**, Corollary 1], that is, if $h_\alpha = [v_\alpha, w_\alpha^*]$ with $v_\alpha, w_\alpha \in L_\alpha - \{0\}$ we first observe that for any $\beta \in \Lambda$ the following equation holds

$$\beta(h_\alpha) = r\alpha(h_\alpha) \tag{1}$$

with $r \in \mathbb{Q}$, this fact being consequence of $V := \mathcal{L}(L_{\beta+j\alpha} : j \in \mathbb{Z})$, the linear space generated by $\{L_{\beta+j\alpha} : j \in \mathbb{Z}\}$, is a finite dimensional vector space invariant for $\operatorname{ad}(v_\alpha)$, $\operatorname{ad}(w_\alpha^*)$ and $\operatorname{ad}(h_\alpha) = \operatorname{ad}(v_\alpha)\operatorname{ad}(w_\alpha^*) - \operatorname{ad}(w_\alpha^*)\operatorname{ad}(v_\alpha)$ on which the trace of $\operatorname{ad}(h_\alpha)$ is 0 and so $m\beta(h_\alpha) + k\alpha(h_\alpha) = 0$ with $m \neq 0$ and $m, k \in \mathbb{Z}$. Second, if $\alpha(h_\alpha) = 0$ then by equation (1), $\beta(h_\alpha) = 0$ for all nonzero root β and so $[h_\alpha, L_\beta] = 0$. As $h_\alpha \in [L_\alpha, L_{-\alpha}] \subset L_0$, we also have $[h_\alpha, H] = 0$ and then $[h_\alpha, L] = 0$. Hence, $h_\alpha \in \operatorname{Ann}(L)$ and so $h_\alpha = 0$.

2. It is an easy consequence of the Jacobi identity and the fact $L_\alpha^* = L_{-\alpha}$. ∎

Lemma 2.2. *For any* $\alpha \in \Lambda$ *we have* $\dim L_\alpha = 1$ *and* $\mathbb{Z}\alpha \cap \Lambda = \pm\alpha$.

Proof: We argue as in [**15**, Proposition I.6], that is, Lemma 2.1 gives us, for any nonzero elements $v_\alpha \in L_\alpha$, $w_\alpha^* \in L_\alpha^*$ such that $[v_\alpha, w_\alpha^*] \neq 0$, that $\alpha([v_\alpha, w_\alpha^*]) \neq 0$ and so the subalgebra $\operatorname{span}_{\mathbb{C}}\{v_\alpha, w_\alpha^*, [v_\alpha, w_\alpha^*]\}$ is isomorphic to $\mathrm{sl}(2,\mathbb{C})$, we may without loss of generality assume that $\alpha([v_\alpha, w_\alpha^*]) = 2$. Condition 3 in Definition 1.2 implies the operators $\operatorname{ad}(v_\alpha)$ and $\operatorname{ad}(w_\alpha^*)$ are locally nilpotent on L, by using now the same arguments as for $\mathrm{sl}(2,\mathbb{C})$ (see [**9**, Proposition 2.4.7]) we obtain L is a locally finite $\operatorname{span}_{\mathbb{C}}\{v_\alpha, w_\alpha^*, [v_\alpha, w_\alpha^*]\}$-module with respect to the adjoint representation. Let us consider the $\operatorname{span}_{\mathbb{C}}\{v_\alpha, w_\alpha^*, [v_\alpha, w_\alpha^*]\}$-submodule of L, $V := \mathbb{C}w_\alpha^* + H + \sum_{n=1}^{\infty} L_{n\alpha}$. As a submodule of a locally finite module, V is also locally finite. Hence the representation theory of $\mathrm{sl}(2,\mathbb{C})$ implies that the set of h_α-eigenvalues on V is symmetric with

$$\dim V^\mu(h_\alpha) = \dim V^{-\mu}(h_\alpha)$$

for each $\mu \in \mathbb{C}$. Now $V^{-2}(h_\alpha) = \mathbb{C}w_\alpha^*$ implies that $\dim V^2(h_\alpha) = \dim L_\alpha = 1$ and furthermore that

$$\dim V^{2n}(h_\alpha) = \dim L_{n\alpha} = 0$$

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for $n > 1$. Since we can replace α by $-\alpha$ in the argument, we have both conclusions of the lemma. ∎

Lemma 2.2 and condition 4 in Definition 1.2 show that given $\alpha \in \Lambda$ there exists a unique nonzero element of $L_0 \subset H$ of the form

$$h_\alpha = [e_\alpha, e_\alpha^*] \tag{2}$$

with $e_\alpha \in L_\alpha - \{0\}$, and such that $\alpha(h_\alpha) = 2$. Let us observe that e_α is unique up to a scalar factor of modulus 1. From now on h_α shall denote this element.

DEFINITION 2.3: A subset Λ_0 of Λ is called a *root system* (relative to H) if it satisfies the conditions: $\alpha \in \Lambda_0$ implies $-\alpha \in \Lambda_0$; and $\alpha, \beta \in \Lambda_0$, $\alpha + \beta \in \Lambda$ implies $\alpha + \beta \in \Lambda_0$. If we define H_{Λ_0} as $\operatorname{span}_{\mathbb{C}}\{h_\alpha : \alpha \in \Lambda_0\}$ and $V_{\Lambda_0} = \bigoplus_{\alpha \in \Lambda_0} L_\alpha$, it is straightforward to verify that $L_{\Lambda_0} = H_{\Lambda_0} \oplus V_{\Lambda_0}$ is an involutive Lie subalgebra of L, with Cartan subalgebra $H_{\Lambda_0} = H \cap L_{\Lambda_0}$, whose roots relative to H_{Λ_0} are precisely the roots in Λ_0. We shall say that L_{Λ_0} is the involutive Lie subalgebra *associated* to the root system Λ_0. Let us observe that if Λ_0 is finite then L_{Λ_0} is finite dimensional.

Our next goal is to prove the following result.

THEOREM 2.4. *Let L be an infinite dimensional simple involutive Lie algebra having a Cartan decomposition respect to H. Then there exists a direct system of finite dimensional simple involutive Lie subalgebras $\mathcal{S} := \big(\{L_i\}_{i \in I}, \{e_{ji}\}_{i \leqslant j}\big)$, with Cartan subalgebras $H_i = H \cap L_i$ and satisfying*

(1) *If $i \leqslant j$ then L_i is an involutive Lie subalgebra of L_j, e_{ji} is the inclusion mapping and each root space of L_i relative to H_i, different to H_i, is a root space of L_j.*
(2) $\varinjlim \mathcal{S} = L$.

The arguments we are going to use in the proof of Theorem 2.4 are close to the ones developed in [**6**, Section IV]. For the convenience of the reader we summarise some of the results in [**6**, Section IV] with a sketch of the proofs, and some auxiliary lemmas before proving Theorem 2.4.

LEMMA 2.5. *Let L_{Λ_0} be the involutive Lie subalgebra associated to a finite root system Λ_0. Write $\langle \cdot, \cdot \rangle$ the Killing form on L_{Λ_0}. Then the following assertions hold:*

(1) $\langle h_\alpha, h_\alpha \rangle \neq 0$ *for any* $\alpha \in \Lambda_0$.
(2) $\langle h, v_\alpha \rangle = 0$ *for any* $h \in H_{\Lambda_0}$ and $v_\alpha \in L_\alpha$, $\alpha \in \Lambda_0$.
(3) $\langle v_\alpha, v_\beta \rangle = 0$ *for any* $v_\alpha \in L_\alpha$, $v_\beta \in L_\beta$, $\alpha, \beta \in \Lambda_0$ *and* $\beta \neq -\alpha$.
(4) $\langle v_\alpha, v_{-\alpha} \rangle \neq 0$ *for any* $0 \neq v_{i\alpha} \in L_{i\alpha}$, $i \in \{\pm 1\}$ *and* $\alpha \in \Lambda_0$.

PROOF: 1. We have $\langle h_\alpha, h_\alpha \rangle = \operatorname{trz}\big(\operatorname{ad}(h_\alpha) \circ \operatorname{ad}(h_\alpha)\big) = \alpha(h_\alpha)^2 + \sum_{\gamma \in \Lambda_0 - \{\alpha\}} \gamma(h_\alpha)^2$. As in the proof of Lemma 2.1-1 we obtain $\gamma(h_\alpha) = r_\gamma \alpha(h_\alpha)$ with $r_\gamma \in \mathbb{Q}$, and finally we conclude from $\alpha(h_\alpha) = 2$ that $\langle h_\alpha, h_\alpha \rangle = 4 + 4 \sum_{\gamma \in \Lambda_0 - \{\alpha\}} r_\gamma^2 \neq 0$.

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2. Since $\langle\cdot,\cdot\rangle$ is invariant in the sense of [8, p. 69], we have

$$\langle h, v_\alpha\rangle = \frac{1}{2}\langle h, [h_\alpha, v_\alpha]\rangle = \frac{1}{2}\langle [h, h_\alpha], v_\alpha\rangle = 0.$$

3. It is clear that $\langle v_\alpha, v_\beta\rangle = \operatorname{trz}\big(\operatorname{ad}(v_\alpha)\circ\operatorname{ad}(v_\beta)\big) = 0$.
4. Since $L_\alpha^* = L_{-\alpha}$,

$$\langle h_\alpha, h_\alpha\rangle = \big\langle [e_\alpha, e_\alpha^*], h_\alpha\big\rangle = \big\langle e_\alpha, [h_\alpha, e_\alpha^*]\big\rangle = -2\langle e_\alpha, e_\alpha^*\rangle.$$

By applying 1. we have $\langle e_\alpha, e_\alpha^*\rangle \neq 0$. Hence, as $\dim L_{\pm\alpha} = 1$ we conclude $\langle v_\alpha, v_{-\alpha}\rangle \neq 0$. ∎

LEMMA 2.6. *Under the hypothesis of Lemma 2.5, if $\langle x, L_{\Lambda_0}\rangle = 0$ for some $x \in L_{\Lambda_0}$ then $x \in H_{\Lambda_0}$.*

PROOF: Write $x = h + \sum_{\alpha\in\Lambda_0} w_\alpha \in L_{\Lambda_0}$, with $h \in H_{\Lambda_0}$ and $w_\alpha \in L_\alpha$. Since $\langle x, v_{-\alpha}\rangle = 0$ for any $v_{-\alpha} \in L_{-\alpha}$, $\alpha \in \Lambda_0$, Lemma 2.5–2,3 shows $\langle w_\alpha, v_{-\alpha}\rangle = 0$, and therefore $w_\alpha = 0$ by Lemma 2.5-4. ∎

PROPOSITION 2.7. *The involutive Lie subalgebra L_{Λ_0} associated to a finite root system Λ_0 in L is semisimple.*

PROOF: Let us firstly observe that if we denote by $[L_{\Lambda_0}, L_{\Lambda_0}] := \operatorname{span}_{\mathbb{C}}\{[x,y] : x, y \in L_{\Lambda_0}\}$, then $[L_{\Lambda_0}, L_{\Lambda_0}] = L_{\Lambda_0}$ and

$$\operatorname{Rad}(L_{\Lambda_0}) \subset H_{\Lambda_0}.$$

Indeed, if $x \in L_{\Lambda_0}$ then

$$x = \sum_{\alpha\in\Lambda_0}\lambda_\alpha h_\alpha + \sum_{\alpha\in\Lambda_0} v_\alpha = \sum_{\alpha\in\Lambda_0}\lambda_\alpha [e_\alpha, e_\alpha^*] + \frac{1}{2}\sum_{\alpha\in\Lambda_0}[h_\alpha, v_\alpha] \in [L_{\Lambda_0}, L_{\Lambda_0}]$$

and so $[L_{\Lambda_0}, L_{\Lambda_0}] = L_{\Lambda_0}$. Since the radical of a finite dimensional Lie algebra L' is characterised as the ideal $\operatorname{Rad}(L') = \Big\{x \in L' : \big\langle x, [L', L']\big\rangle = 0\Big\}$, where $\langle\cdot,\cdot\rangle$ denotes the Killing form (see [8, p. 73]), the fact $[L_{\Lambda_0}, L_{\Lambda_0}] = L_{\Lambda_0}$ and Lemma 2.6 show $\operatorname{Rad}(L_{\Lambda_0}) \subset H_{\Lambda_0}$.

Secondly, we assert that

$$\operatorname{Rad}(L_{\Lambda_0}) = \operatorname{Ann}(L_{\Lambda_0}).$$

Indeed, $\operatorname{Ann}(L_{\Lambda_0})$ is a solvable ideal and therefore is included in $\operatorname{Rad}(L_{\Lambda_0})$. If $h \in \operatorname{Rad}(L_{\Lambda_0}) \subset H_{\Lambda_0}$ we have, by the character of ideal of $\operatorname{Rad}(L_{\Lambda_0})$, $[h, v_\alpha] \in \operatorname{Rad}(L_{\Lambda_0}) \subset H_{\Lambda_0}$ for any $0 \neq v_\alpha \in L_\alpha$ and $\alpha \in \Lambda_0$, therefore $\alpha(h) = 0$. Hence, we have for any $x \in L_{\Lambda_0}$,

$$[h, x] = \Big[h, \sum_{\alpha\in\Lambda_0}\lambda_\alpha h_\alpha + \sum_{\alpha\in\Lambda_0} v_\alpha\Big] = \Big[h, \sum_{\alpha\in\Lambda_0}\lambda_\alpha h_\alpha\Big] + \sum_{\alpha\in\Lambda_0}[h, v_\alpha] = 0$$

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and so $h \in \mathrm{Ann}(L_{\Lambda_0})$.

Finally, as by Levi's theorem, ([**8**, p. 91]),

$$L_{\Lambda_0} = \mathrm{Rad}(L_{\Lambda_0}) \oplus T_{\Lambda_0},$$

with T_{Λ_0} a semisimple subalgebra of L_{Λ_0}, we have

$$L_{\Lambda_0} = [L_{\Lambda_0}, L_{\Lambda_0}] = \left[\mathrm{Ann}(L_{\Lambda_0}) \oplus T_{\Lambda_0}, \mathrm{Ann}(L_{\Lambda_0}) \oplus T_{\Lambda_0}\right] \subset [T_{\Lambda_0}, T_{\Lambda_0}] \subset T_{\Lambda_0}$$

and so $L_{\Lambda_0} = T_{\Lambda_0}$, the proof is complete. ∎

We shall say that a finite set of nonzero roots $\{\alpha_i\}$ of L is *linearly independent* if the set $\{h_{\alpha_i}\}$ is linearly independent. We also recall that an L^*-algebra is defined, (see [**13, 14, 7**]), as a complex involutive Hilbert-Lie algebra for which the inner product $(\cdot \mid \cdot)$ satisfies the *H^*-identities*

$$\big([x,y] \mid z\big) = \big(y \mid [x^*, z]\big) = \big(x \mid [z, y^*]\big).$$

J.R. Schue introduced for any non zero root α of a semisimple L^*-algebra L' with inner product $(\cdot \mid \cdot)$ and with a Cartan decomposition $L' = \overline{H' + \sum L'_\alpha}$, the elements $0 \neq h'_\alpha \in [L'_\alpha, L'_{-\alpha}]$ satisfying $\alpha(h') = (h' \mid h'_\alpha)$ for any $h' \in H'$ (see [**1**, pp. 513–514] or [**13**, pp. 71–72]). It is well known, (see [**4**, Proof of Proposition 3.1] or the ideas in [**12**]), that any complex finite dimensional semisimple Lie algebra with a Cartan decomposition $L = H + \sum L_\alpha$ and with the expression $L = \bigoplus_{j=1}^{m} L_j$, where L_j are simple Lie algebras, admits an, essentially unique, involution $*'$ and inner product $(\cdot \mid \cdot)$ that make L an L^*-algebra admitting the same Cartan decomposition $L = H + \sum L_\alpha$, and such that $(L_i \mid L_j) = 0$ for $i \neq j$. Since we can see a finite dimensional semisimple involutive Lie algebra having a Cartan decomposition $L = H + \sum L_\alpha$ as a Lie algebra which also admits the Cartan decomposition (in the classical sense) $L = H + \sum L_\alpha$, the above considerations imply in this framework $h'_\alpha = k h_\alpha$ with $0 \neq k \in \mathbb{C} - \{0\}$, and joint with [**1**, Lemma 1] and [**1**, Corollary 2] give us the following two results:

Lemma 2.8. *Let L be a finite dimensional semisimple involutive Lie algebra having a Cartan decomposition relative to H. Write L as $L = \bigoplus_{j=1}^{m} L_j$ where L_j are simple Lie algebras. If α is a nonzero root relative to H, then L_α belongs precisely to one L_j. If we denote by*

$$\Lambda_j = \{\alpha : L_\alpha \subseteq L_j\},$$

then $\mathrm{span}_{\mathbb{C}}\{[v_\alpha, v_{-\alpha}] : \alpha \in \Lambda_j\}$ is a Cartan subalgebra H_j of L_j and the restrictions to H_j of the $\alpha \in \Lambda_j$ are precisely the roots of L_j.

Corollary 2.9. *Let L be as in Lemma 2.8. Let us suppose $\{\alpha_{1,}, \dots, \alpha_n\}$ is a linearly independent set of nonzero roots of L. If there exists a root γ of L such that $\gamma = \sum_{i=1}^{n} c_i \alpha_i$ $c_i \neq 0$, then all α_i and γ are roots of the same simple component L_j.*

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Lemma 2.10. *Let L be as in Lemma 2.8. If α and β are two nonzero roots such that $\alpha \neq \pm\beta$ then α and β are linearly independent.*

Proof: Suppose α and β are not linearly independent, then $h_\alpha = ch_\beta$ with $0 \neq c \in \mathbb{C}$. Let consider L as an L^*-algebra with inner product $(\cdot \mid \cdot)$. By the above observation, there exist non zero elements $h'_\alpha, h'_\beta \in H$ such that $\alpha(h) = (h \mid h'_\alpha)$ and $\beta(h) = (h \mid h'_\beta)$ for any $h \in H$ and $h'_\alpha = k_\alpha h_\alpha$, $h'_\beta = k_\beta h_\beta$ with $k_\alpha, k_\beta \in \mathbb{C} - \{0\}$. Hence,

$$\alpha(h) = (h \mid k_\alpha h_\alpha) = (h \mid k_\alpha c h_\beta) = (h \mid k_\alpha c k_\beta^{-1} h'_\beta) = \overline{k_\alpha c k_\beta^{-1}}\beta(h)$$

for any $h \in H$. From the theory of finite dimensional split semisimple Lie algebras, this is only possible if $\alpha = \pm\beta$. ∎

Definition 2.11: Let α and β be two nonzero roots of an involutive Lie algebra with zero annihilator, we shall say that α and β are *connected* if there exist $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n\}$$

is a family of nonzero roots, α_1 is a fixed element of $\{\alpha, -\alpha\}$ and $\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n = \beta$. We shall also say that $\{\alpha_1, \dots, \alpha_n\}$ is a *connection* from α to β.

It is clear that

$$\alpha_p \neq \pm\sum_{i=1}^{p-1}\alpha_i, \quad p = 2, .., n. \tag{3}$$

We denote by

$$\Lambda_\alpha := \{\beta \in \Lambda : \alpha \text{ and } \beta \text{ are connected}\}$$

Let us observe that $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \in \Lambda_\alpha$.

Lemma 2.12. *Under the hypothesis of Lemma 2.8, and if in addition α and β are two connected nonzero roots, then L_α and L_β belong to the same simple Lie algebra L_j.*

Proof: We have $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$\{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n\}$$

are nonzero roots, α_1 is a fixed element of $\{\alpha, -\alpha\}$ and $\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n = \beta$. If we consider α_1, α_2, and $\alpha_1 + \alpha_2$, by (3) $\alpha_2 \neq \pm\alpha_1$, then Lemma 2.10 gives us that α_1 and α_2 are linearly independent and finally Corollary 2.9 let us conclude $L_{\alpha_1}, L_{\alpha_2}$and $L_{\alpha_1+\alpha_2}$ belong to the same simple Lie algebra L_j. The same argument with $\alpha_1 + \alpha_2$, α_3 and $\alpha_1 + \alpha_2 + \alpha_3$ gives us $L_{\alpha_3}, L_{\alpha_1+\alpha_2+\alpha_3} \subset L_j$. Following this process we finally obtain $L_\alpha, L_\beta \subset L_j$. ∎

Proposition 2.13. *Let L be an infinite dimensional simple involutive Lie algebra having a Cartan decomposition respect to H, and let α be a nonzero root. Then the following assertions hold:*

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(1) Λ_α *is a root system.*
(2) *There exists* $\beta \in \Lambda_\alpha$ *such that* $\beta \neq \pm\alpha$.
(3) *If* γ *is a nonzero root such that* $\gamma \notin \Lambda_\alpha$, *then* $[L_\beta, L_\gamma] = 0$ *and* $\gamma(h_\beta) = 0$ *for any* $\beta \in \Lambda_\alpha$.

PROOF: 1. If $\beta \in \Lambda_\alpha$ then there exists a connection $\{\alpha_1, \dots, \alpha_n\}$ from α to β. It is easy to check that $\{-\alpha_1, \dots, -\alpha_n\}$ is a connection from α to $-\beta$ and therefore $-\beta \in \Lambda_\alpha$. If $\beta, \gamma \in \Lambda_\alpha$ and $\beta + \gamma \in \Lambda$, then there exists a connection $\{\alpha_1, \dots, \alpha_n\}$ from α to β. Hence, $\{\alpha_1, \dots, \alpha_n, \gamma\}$ is a connection from α to $\beta + \gamma$ and so $\beta + \gamma \in \Lambda_\alpha$.

2. Firstly, let us observe that there exists $\gamma \in \Lambda$, $\gamma \neq \pm\alpha$ such that either $[L_\alpha, L_\gamma] \neq 0$ or $[L_{-\alpha}, L_\gamma] \neq 0$. Indeed, if we suppose $[L_\alpha, L_\gamma] = [L_{-\alpha}, L_\gamma] = 0$ for any $\gamma \in \Lambda$, $\gamma \neq \pm\alpha$, as $L_{-\alpha} = L_\alpha^*$ then by Lemma 2.1-2 we have $\gamma(h_\alpha) = 0$ for any $\gamma \in \Lambda, \gamma \neq \pm\alpha$. Let us consider

$$I := \mathbb{C}h_\alpha \oplus L_\alpha \oplus L_{-\alpha}.$$

By the above, it is easy to prove that $[I, L] \subset I$, therefore I is a nonzero finite dimensional ideal of an infinite dimensional simple involutive Lie algebra L, a contradiction. Hence, there exists a nonzero root $\gamma \neq \pm\alpha$ such that either $[L_\alpha, L_\gamma] \neq 0$ or $[L_{-\alpha}, L_\gamma] \neq 0$. In the first case, $\{\alpha, \gamma\}$ is a connection from α to $\beta := \alpha + \gamma$, therefore $\beta \in \Lambda_\alpha$ and $\beta \neq \pm\alpha$. In the second case we argue similarly.

3. Let us suppose there exists $\beta \in \Lambda_\alpha$ such that $[L_\beta, L_\gamma] \neq 0$. If $\{\alpha_1, \dots, \alpha_n\}$ is a connection from α to β, we have $\{\alpha_1, \dots, \alpha_n, \gamma\}$ is a connection from α to $\beta + \gamma$. Since Λ_α is a root system then $\gamma \in \Lambda_\alpha$, a contradiction. Therefore $[L_\beta, L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. As $-\beta \in \Lambda_\alpha$ for any $\beta \in \Lambda_\alpha$, we also have $[L_{-\beta}, L_\gamma] = 0$. Finally, by Lemma 2.1-2 we conclude $\gamma(h_\beta) = 0$. ∎

PROPOSITION 2.14. *Let* L *be an infinite dimensional simple involutive Lie algebra having a Cartan decomposition respect to* H. *Then there exists a connection from* α *to* β *for any* $\alpha, \beta \in \Lambda$.

PROOF: Let consider the root system Λ_α and the involutive Lie subalgebra associated

$$L_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}.$$

We assert that L_{Λ_α} is a nonzero ideal of L. Indeed, by Proposition 2.13-3 we have $[L_\beta, L_\gamma] = 0$ and $[h_\beta, L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. Hence,

$$[L_{\Lambda_\alpha}, L] = \Big[\sum_{\beta\in\Lambda_\alpha} \mathbb{C}h_\beta + \sum_{\beta\in\Lambda_\alpha} L_\beta, H + \Big(\sum_{\gamma\in\Lambda_\alpha} L_\gamma\Big) + \Big(\sum_{\gamma\notin\Lambda_\alpha} L_\gamma\Big)\Big] \subset L_{\Lambda_\alpha}.$$

The simplicity of L implies $L_{\Lambda_\alpha} = L$ and therefore $\Lambda_\alpha = \Lambda$. ∎

COROLLARY 2.15. *Let* L *be as in Proposition 2.14. Then, for a fixed* $\alpha_0 \in \Lambda$, *we have*

$$L = \operatorname*{span}_{\mathbb{C}}\{h_\beta : \beta \in \Lambda_{\alpha_0}\} + \sum_{\beta\in\Lambda_{\alpha_0}} L_\beta.$$

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DEFINITION 2.16: From now on, we shall consider the classical finite dimensional simple Lie algebras (of types A, B, C, or D) endowed with the *standard involution* given by $(a_{ij})^* = (\overline{a_{ji}})$. These algebras become involutive Lie algebras with the standard involution and will be called *classical finite dimensional simple involutive Lie algebras.*

Given a classical finite dimensional simple involutive Lie algebra L of a fixed type A, B, C or D, we shall give the name *canonical* Cartan subalgebra of L to the one described in [**8**, Chapter IV, 6] for each type.

PROOF OF THEOREM 2.4: 1. Let S be a non empty finite subset of Λ, from condition 3 in Definition 1.2, $\mathrm{Sp}_{\mathbb{Z}} S \cap \Lambda$ is a finite root system and then we can consider the finite dimensional involutive Lie subalgebra associated $L_{(\mathrm{Sp}_{\mathbb{Z}} S\cap\Lambda)}$, that we shall denote by $L_S := L_{(\mathrm{Sp}_{\mathbb{Z}} S\cap\Lambda)}$. By Proposition 2.7, L_S is semisimple. It is well known from the theory of finite dimensional semisimple Lie algebras that L_S can be written

$$L_S = \bigoplus_{i=1}^{n_S} L_{S_i},$$

with L_{S_i}, $i = 1, \dots, n_S$, finite dimensional simple Lie algebras. By Lemma 2.12, we conclude that for any nonzero root α of L_S respect to $H \cap L_S$, $L_{\pm\alpha}$ belong precisely to one L_{S_i} and so any L_{S_i} is an involutive Lie algebra. Hence, we can consider the family of finite dimensional simple involutive Lie subalgebras of L,

$$\{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}},$$

where $\mathcal{F}$ denotes the family of all non empty finite subset of Λ. We wish to prove that

$$\mathcal{S} := \left(\{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}}, \{i_{S_i,T_j}\}\right),$$

where $\{i_{S_i,T_j}\}$ are the inclusion mappings is the required direct system. We assert that given

$$L_{S_i}, L_{T_j} \in \{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}},$$

there exists

$$L_{Q_{i_0}} \in \{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}}$$

such that $L_{S_i}, L_{T_j} \subset L_{Q_{i_0}}$. Indeed, let us fix $\alpha_0 \in S_i$. By Proposition 2.14, for any $\beta \in S_i \cup T_j$ there exists a connection from α_0 to β, which we denote by $C_{\alpha_0,\beta}$. We have that $Q := \bigcup_{\beta\in S_i\cup T_j} C_{\alpha_0,\beta}$ is a finite set of Λ and therefore we can consider the finite dimensional semisimple involutive Lie subalgebra associated L_Q. Write $L_Q = \bigoplus_{i=1}^{n_Q} L_{Q_i}$, L_{Q_i} being simple subalgebras of L_Q. By Lemma 2.8, there exists $L_{Q_{i_0}}$ such that $L_{\alpha_0} \subset L_{Q_{i_0}}$. Finally, by Lemma 2.12, $L_{S_i}, L_{T_j} \subset L_{Q_{i_0}}$. Therefore, $\mathcal{S}$ is a direct system with the inclusion which clearly satisfies assertion 1. of the theorem.

2. Let us denote $\varinjlim \mathcal{S} = (L', \{e_j\}_j)$. As $(L, \{i_j\}_j)$, where i_j denotes the inclusion mapping, satisfies the conditions of the direct limit for $\mathcal{S}$, the universal property

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of the direct limits shows the existence of a unique $*$-monomorphism $\Phi : L' \to L$ such that $\Phi \circ e_j = i_j$. Since $L' = \bigcup_j e_j(L_j)$, (see for instance [3]), we have $\Phi(L') = \Phi\Big(\bigcup_j e_j(L_j)\Big) = \bigcup_j L_j$, and therefore Φ is a $*$-isomorphism from L' onto $\bigcup_j L_j$. Finally, we assert that $L = \bigcup_j L_j$. Indeed, if $x \in L$, by Proposition 2.14 and Corollary 2.15, $x = \sum_{i=1}^{n} \lambda_{\alpha_i} h_{\alpha_i} + \sum_{j=1}^{m} v_{\gamma_j}$ with $\alpha_i, \gamma_j \in \Lambda$, $v_{\gamma_j} \in L_{\gamma_j}$ and $\lambda_{\alpha_i} \in \mathbb{C}$. Consider $T = \{\alpha_i : i = 1, \dots, n\} \cup \{\gamma_j : j = 1, \dots, m\} \subset \Lambda$ and, following the above notation, $T' = \bigcup_{\beta \in T} C_{\delta_0, \beta}$, δ_0 being a fixed element of T. We have T' is a finite set of Λ that gives us the semisimple finite dimensional involutive Lie algebra associated $L_{T'}$. Write $L_{T'} = \bigoplus_{i=1}^{r} L_{T_i'}$, where $L_{T_i'}$, $i = 1, \dots, r$ are simple finite dimensional involutive Lie algebras. As $\mathcal{S}$ is a direct system for the inclusion then there exists a finite dimensional simple involutive Lie subalgebra L_{P_0} such that $\bigcup_{i=1}^{r} L_{T_i'} \subseteq L_{P_0}$ and therefore $x \in L_{P_0}$. The proof of 2. is complete. ∎

Theorem 2.17. *Let L be an infinite dimensional simple involutive Lie algebra having a Cartan decomposition respect to H. Then L is $*$-isomorphic to a direct limit of classical finite dimensional simple involutive Lie algebras of the same type A, B, C or D.*

Proof: Let us consider the direct system of finite dimensional simple involutive Lie algebras $\mathcal{S}$ given in Theorem 2.4. We can suppose all of the L_i are isomorphic to classical simple Lie algebras of a same type A, B, C or D. Indeed, the infinite dimensional character of L let us remove the exceptional Lie algebras of $\mathcal{S}$, and secondly that (i) each L_i is contained in one isomorphic to one of type A or else (ii) there exists L_{i_0} such that $L_i \supset L_{i_0}$ implies that L_i is isomorphic to one of type B, C or D. In each of the two cases is possible to define a subsystem satisfying assertions 1. and 2. of Theorem 2.4.

If all of the L_i are isomorphic to classical simple Lie algebras of type A and we denote by $\phi_i : L_i \to A_i$ such isomorphisms, we assert that if consider A_i as an involutive Lie algebra with its standard involution, then there exists a $*$-isomorphism ξ_i from L_i onto A_i. Indeed, ϕ_i induces on A_i a unique Cartan decomposition $A_i = H' \oplus \Big(\bigoplus_{\alpha' \in \Lambda_i'} (A_i)_{\alpha'}\Big)$ and involution $*'$ that make ϕ_i a $*$-isomorphism. On the other hand, if we consider A_i with its canonical Cartan decomposition given in [8, p. 136–137], $A_i = H'' \oplus \Big(\bigoplus_{\alpha'' \in \Lambda_i''} (A_i)_{\alpha''}\Big)$, it is well known from the theory of finite dimensional Lie algebras, see [8, Chapter IX, Theorem 3], that there exists an automorphism $\mu_i : A_i \to A_i$ satisfying $\mu_i(H') = H''$. As a consequence, we can express the roots α'' as $\alpha''(h'') = \alpha'\big(\mu_i^{-1}(h'')\big)$ for a certain root α'. This gives us a a bijection $\alpha' \to \alpha''$ satisfying that the Cartan matrices associated to a fixed simple system of roots $(\alpha_1', \dots, \alpha_n')$, $\big(2\langle \alpha_i', \alpha_j' \rangle / \langle \alpha_i', \alpha_i' \rangle\big)$ and the one associated to $(\alpha_1'', \dots, \alpha_n'')$ are identical. Let $e_{\alpha_i'}$, $(e_{\alpha_i'})^{*'}$, $h_{\alpha_i'}$ as in (2), the canonical generators for

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A_i associated to $(\alpha'_1, \ldots, \alpha'_n)$, and $E_{pq} \in L_{\alpha''_i}$, $E_{qp} \in L_{-\alpha''_i}$, $E_{pp} - E_{qq} \in H''$, (where E_{rs} denotes the elemental matrix), the canonical generators for A_i associated to $(\alpha''_1, \ldots, \alpha''_n)$ (see [**8**, p. 136–137]). By applying the Isomorphism Theorem, [**8**, Theorem 2 on p. 127], there exists a unique automorphism η_i of A_i mapping $e_{\alpha'_i}$ on E_{pq}, $(e_{\alpha'_i})^{*'}$ on E_{qp} and $h_{\alpha'_i}$ on $E_{pp} - E_{qq}$. Moreover, as $\{e_{\alpha'_i}, (e_{\alpha'_i})^{*'}, h_{\alpha'_i}\}$ generates A_i, ([**8**, Property XVIII on p. 123]), we can assert η_i is a $*$-automorphism from $(A_i, *')$ onto (A_i, τ), τ being the standard involution $(a_{i,j})^\tau := (\overline{a_{j,i}})$. Finally, we have $\xi_i := \eta_i \circ \phi_i$ is $*$-isomorphism from L_i onto the classical simple involutive Lie algebra A_i as we wished to prove.

If all of the L_i are isomorphic to classical Lie algebras X_i of a same type B, C or D, we argue as in the previous case to find a $*$-isomorphism ξ_i from L_i onto the classical simple involutive Lie algebra X_i.

From now on X denotes a classical simple involutive Lie algebra of a fixed type $X = A, B, C$ or D. For any couple $i, j \in I$ with $i \leqslant j$, let e_{ji} be the inclusion mapping and f_{ji} the unique $*$-monomorphism making commutative the following diagram

$$\begin{array}{ccc} & \xi_j & \\ L_j & \rightarrow & X_j \\ e_{ji}\ \uparrow & & \uparrow\ f_{ji} \\ L_i & \rightarrow & X_i \\ & \xi_i & \end{array} \tag{4}$$

It is clear that

$$\mathcal{S}^\sharp = \left(\{X_i\}_{i\in I}, \{f_{ji}\}_{i,j\in I, i\leqslant j}\right)$$

is a direct system of classical finite dimensional simple involutive Lie algebras of a same type X. Finally, since for any $i, j \in I$ with $i \leqslant j$, we have the $*$-isomorphisms $\xi_i : L_i \to X_i$, $\xi_j : L_j \to X_j$ and the commutativity of the diagrams (4) we conclude $\varinjlim \mathcal{S}$ is $*$-isomorphic to $\varinjlim \mathcal{S}^\sharp$ and the proof is complete. ∎

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