Nagoya Math. J., **248** (2022), 849–864
DOI 10.1017/nmj.2022.11

MAGNETIC (QUASI-)MODULAR FORMS

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Abstract. A (folklore?) conjecture states that no *holomorphic* modular form $F(\tau)=\sum_{n=1}^{\infty} a_n q^n \in q\mathbb{Z}[[q]]$ exists, where $q=e^{2\pi i\tau}$, such that its anti-derivative $\sum_{n=1}^{\infty} a_n q^n/n$ has integral coefficients in the q-expansion. A recent observation of Broadhurst and Zudilin, rigorously accomplished by Li and Neururer, led to examples of *meromorphic* modular forms possessing the integrality property. In this note, we investigate the arithmetic phenomenon from a systematic perspective and discuss related transcendental extensions of the differentially closed ring of quasi-modular forms.

§1. Introduction

One of the arithmetic features of modular and quasi-modular forms is integrality of the coefficients in their Fourier expansions. This is trivially seen on the generators

$$E_2(\tau)=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},\quad E_4(\tau)=1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},\quad E_6(\tau)=1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} \tag{1}$$

of the ring of quasi-modular forms, as well as on the *discriminant* cusp form

$$\Delta(\tau)=q\prod_{m=1}^{\infty}(1-q^m)^{24}=\frac{E_4^3-E_6^2}{1728},$$

where $q=q(\tau)=e^{2\pi i\tau}$ for τ from the upper half-plane $\operatorname{Im}\tau>0$. All q-expansions above converge for q inside the unit disk, and in fact have polynomial growth of the coefficients. A more surprising fact, brought to the mathematical community by Ramanujan [21] more than 100 years ago, is that the three Eisenstein series in (1) satisfy the algebraic system of differential equations

$$\delta E_2=\frac{1}{12}(E_2^2-E_4),\quad \delta E_4=\frac{1}{3}(E_2E_4-E_6),\quad \delta E_6=\frac{1}{2}(E_2E_6-E_4^2), \tag{2}$$

where

$$\delta=\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}=q\frac{\mathrm{d}}{\mathrm{d}q}.$$

Ramanujan's notation for the Eisenstein series (1) was $P(q),Q(q),R(q)$, respectively, as he mainly viewed them as functions of the q-nome. Since the functions E_2,E_4,E_6 are algebraically independent over $\mathbb{C}$, and even over $\mathbb{C}(q)$ and over $\mathbb{C}(\tau,q)$ [19,22], this fine structure gives rise to remarkable applications in transcendental number theory to the values of quasi-modular forms. One particular notable example in this direction is a famous theorem of Nesterenko [20], which states that, given a complex number q with $0<|q|<1$,

Received October 8, 2020. Revised February 3, 2022. Accepted April 30, 2022.
2020 Mathematics subject classification: 11F33, 11F11, 11F32, 11F37, 13N99.
Keywords: Ramanujan's mathematics, modular forms, quasi-modular forms, Shimura–Borcherds lift.

at least three of the four quantities $q, P(q), Q(q), R(q)$ are algebraically independent over $\mathbb{Q}$.

Establishing transformation properties of a double integral, which characterizes the output voltage of a Hall plate affected by the shape of the plates and sizes of the contacts and which is—for this reason—dubbed *magnetic*, in the work [7], Broadhurst and Zudilin came across a *meromorphic* modular form (on a congruence subgroup), whose anti-derivative had integral coefficients in its q-expansion and was not a modular object itself. This arithmetic observation was subsequently proved by Li and Neururer in [17] who also noticed that the formal anti-derivative

$$\tilde{F}_{4a} = \delta^{-1}\left(\frac{\Delta}{E_4^2}\right) = \int_0^q \frac{\Delta}{E_4^2}\,\frac{\mathrm{d}q}{q}$$

of the meromorphic modular form $F_{4a}(\tau) = \Delta/E_4^2$ has integer coefficients in its q-expansion. (They proved a slightly weaker version about the integrality of the anti-derivative of $64\Delta/E_4^2$.) The function $F_{4a}(\tau)$ has weight 4 and possesses the double pole at $\tau = \rho = e^{2\pi i/3}$ in the fundamental domain, and a simple analysis reveals that it is not the image under δ of an element from the (differentially closed) field $\mathbb{C}(q, E_2, E_4, E_6)$. This implies that the anti-derivative $\tilde{F}_{4a} = \delta^{-1}F_{4a}$ is transcendental over the field, and hence the addition of $\tilde{F}_{4a}$ to the latter increases the transcendence degree by 1. Following the background in [7], Li and Neururer coined the name *magnetic modular form* to a meromorphic modular form like F_{4a}. A principal goal of this note is to investigate the *magnetic modular* phenomenon further and to give more examples of those.

THEOREM 1. *The meromorphic modular forms $F_{4a}(\tau) = \Delta/E_4^2$ and $F_{4b}(\tau) = E_4\Delta/E_6^2$ of weight 4 are magnetic. In other words, their anti-derivatives $\delta^{-1}F_{4a}$ and $\delta^{-1}F_{4b}$ have integral q-expansions.*

THEOREM 2. *The meromorphic modular form $F_6(\tau) = E_6\Delta/E_4^3$ of weight 6 is doubly magnetic: its first and second anti-derivatives $\delta^{-1}F_6$ and $\delta^{-2}F_6$ have integral q-expansions.*

There are other instances in the literature of related integrality phenomena; however, the existing methods of proofs seem to be quite different from what we use below. Investigating the solution space of the linear differential equation

$$D_k f(\tau) = 0, \quad \text{where } D_k = \delta^2 - \frac{k+1}{6}E_2(\tau)\delta + \frac{k(k+1)}{12}\delta E_2(\tau),$$

in [13], Honda and Kaneko found that, when $k = 4$, it is spanned by E_4 and

$$\tilde{E}_4 = E_4 \cdot \delta^{-1}\left(\frac{\Delta^{5/6}}{E_4^2}\right) \in q^{5/6}\mathbb{Q}[[q]].$$

They numerically observed and proved some related results about the p-integrality of $\tilde{E}_4$ for primes $p \equiv 1 \bmod 3$. This theme was later analyzed and generalized in [2,11,12]. Bringing some parallel to that investigations, it is easy to check that the functions E_4 and $E_4\,\delta^{-1}(\Delta/E_4^2)$ (both with integer coefficients in their q-expansions!) span the solution space of the differential equation $Df = 0$, where

$$D = \delta^2 - E_2\delta + \frac{1}{36}\left(7E_2^2 - 5E_4 - 2\frac{E_2E_6}{E_4}\right) = D_5 + \frac{1}{6}\left(E_2\frac{\delta E_4}{E_4} - 5\delta E_2\right).$$

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At the same time, the only quasi-modular solutions of $D_5 y = 0$ are spanned by δE_4 (see [15, Th. 2]).

A somewhat different account of strong divisibility of the coefficients of modular forms shows up in the context of arithmetic properties of traces of singular moduli initiated in Zagier's work [23]. As this topic remains quite popular, we only list a selection of contributions [1,3,8,9,10,14]. The methods involved make use of the Shimura correspondence, which is also the main ingredient of our proof of Theorems 1 and 2.

§2. Magnetic quasi-modular forms

In this part, we formalize the notion of magnetic forms and give results, which may be thought of as generalizations of Theorems 1 and 2, but use the theorems as principal steps.

Consider the family

$$f_{a,b,c} = E_2^a E_4^b E_6^c, \quad \text{where } a,b,c \in \mathbb{Z},\ a \ge 0,$$

of meromorphic quasi-modular forms. Their q-expansions all belong to $\mathbb{Z}[[q]]$. For $k \in \mathbb{Z}$ even, denote by W_k the $\mathbb{Q}$-vector space in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[[q]]$ (the q-series $f \in \mathbb{Q}[[q]]$ with $Nf \in \mathbb{Z}[[q]]$ for some $N \in \mathbb{Z}_{>0}$) spanned by the q-expansions of the forms $f_{a,b,c}$ of weight k, that is, with $2a + 4b + 6c = k$. Because

$$\delta f_{a,b,c} = \frac{k-a}{12} f_{a+1,b,c} - \frac{a}{12} f_{a-1,b+1,c} - \frac{b}{3} f_{a,b-1,c+1} - \frac{c}{2} f_{a,b+2,c-1}, \tag{3}$$

the differential operator δ defines a well-defined map $W_k \to W_{k+2}$. Clearly, the image δW_k in W_{k+2} is a $\mathbb{Q}$-subspace in $\mathbb{Q} \otimes_{\mathbb{Z}} q\mathbb{Z}[[q]]$; we call W_{k+2}^0 the cuspidal subspace of W_{k+2}, that is, the set of all elements in W_{k+2} with vanishing constant term in their q-expansion.

We say that an element $v \in W_k^0$ is *magnetic* if its formal anti-derivative

$$\delta^{-1} v = \int_0^q v \, \frac{\mathrm{d}q}{q} \in \mathbb{Q} \otimes_{\mathbb{Z}} q\mathbb{Z}[[q]].$$

We also call it *strongly magnetic* if $\delta^{-1} v \in q\mathbb{Z}[[q]]$. With the magnetic property, we can associate the equivalence relation $\sim$ on W_k writing $v \sim w$ if and only if the difference $v - w$ is in W_k^0 and is magnetic.

Let V_k (respectively, V_k^0) be the $\mathbb{Q}$-vector subspace of W_k (respectively, of W_k^0) generated by the forms $f_{a,b,c}$ with $a \in \{0, 1, \dots, k-2\}$. According to relation (3), this range of a makes the subspace V_k *stable* under the δ-differentiation. Notice that $\delta V_2 \subseteq V_4^0$.

THEOREM 3. *Any element of V_4^0 is magnetic.*

REMARK 1. It seems that the elements of W_4^0 with $a > 2$ (i.e., outside the range assumed in V_4^0) with the magnetic property are those that come as linear combinations of δ-derivatives of elements from W_2. In other words, we expect that the choice of V_4^0 in the theorem as a magnetic space of weight 4 to be sharp.

Derivation of Theorem 3 from Theorem 1. It follows from Theorem 1 that the forms

$$f_{0,1,0} - f_{0,-2,2} = 1728 F_{4a} \quad \text{and} \quad f_{1,2,-1} - f_{0,1,0} = 6\delta f_{0,2,-1} - 5184 F_{4b}$$

are magnetic; in other words, we have the equivalences $f_{0,-2,2} \sim f_{0,1,0}$ and $f_{1,2,-1} \sim f_{0,1,0}$.

Any element in V_4 can be written as $E_2^a P(E_4, E_6)/(E_4^m E_6^n)$, for some a, m, n non-negative integers, $a \le 2$, and $P(x,y) \in \mathbb{Q}[x,y]$. Such an expression clearly splits into a linear

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combination of the form $f_{a,b,c}\in V_4$ with $0\le a\le 2$ and either $b\ge 0$ or $c\ge 0$. If both $b\ge 0$ and $c\ge 0$, then we get only two elements in V_4, namely, $f_{0,1,0}$ and $f_{2,0,0}=f_{0,1,0}+12\delta f_{1,0,0}$, both equivalent to $f_{0,1,0}$. Therefore, we only need to prove the theorem in two situations: $b\ge 0$ and $c<0$, or $b<0$ and $c\ge 0$.

If $b\ge 0$ and $c<0$, then there is only one form $f_{a,b,c}\in V_4$ with $c=-1$. Indeed, solving $4=2a+4b+6c=2a+4b-6$, we get $a=1$ and $b=2$. By the hypothesis, this form $f_{1,2,-1}\sim f_{0,1,0}$. For $c\le -2$, we use equation (3) (with $k=2$) in the form

$$\frac{c+1}{2}f_{a,b,c}=-\delta f_{a,b-2,c+1}-\frac{a}{12}f_{a-1,b-1,c+1}-\frac{b-2}{3}f_{a,b-3,c+2}-\frac{a-2}{12}f_{a+1,b-2,c+1},$$

and induction on $-c$ to conclude that $f_{a,b,c}$ is equivalent to a linear combination of $f_{1,2,-1}$ and $f_{0,1,0}$, and hence to $f_{0,1,0}$ alone. (Notice that prefactors $a/12$ and $(a-2)/12$ leave the terms on the right-hand side in V_4.)

If $b<0$ (and $c\ge 0$), we use equation (3) in the form

$$\frac{b+1}{3}f_{a,b,c}=-\delta f_{a,b+1,c-1}-\frac{a-2}{12}f_{a+1,b+1,c-1}-\frac{a}{12}f_{a-1,b+2,c-1}-\frac{c-1}{2}f_{a,b+3,c-2}. \qquad (4)$$

When $b=-1$ and $b=-2$, the only forms $f_{a,b,c}\in V_4$ possible with $c\ge 0$ are $f_{1,-1,1}$ and $f_{0,-2,2}$, respectively. Substituting $a=0$, $b=-2$, and $c=2$ in (4) leads to

$$-\frac{1}{3}f_{0,-2,2}=-\delta f_{0,-1,1}+\frac{1}{6}f_{1,-1,1}-\frac{1}{2}f_{0,1,0}$$

implying $f_{1,-1,1}\sim f_{0,-2,2}\sim f_{0,1,0}$ from the hypothesis. For $b\le -3$, we use (4) to conclude by induction on $-b$ that any such $f_{a,b,c}$ is equivalent to a linear combination of $f_{0,-2,2}$, $f_{1,-1,1}$, and $f_{0,1,0}$, and hence to $f_{0,1,0}$. This completes the proof of the theorem. ∎

Remark 2. It follows from the proof that we can replace the generator $f_{0,-2,2}-f_{0,1,0}$ with $f_{1,-1,1}-f_{0,1,0}$. Furthermore, alternative choices for $f_{0,-2,2}-f_{0,1,0}$ and $f_{1,2,-1}-f_{0,1,0}$ are $\tilde F_j=E_2\cdot(\delta E_j)/E_j$ or $\hat F_j=(\delta^2E_j)/E_j$, for $j=4,6$.

For weight 6, the situation is slightly different. Only the following is true.

Theorem 4. *Let U_6 be the subspace of V_6 spanned over $\mathbb{Q}$ by $f_{a,b,c}$ with the additional constraint $c\ge 0$, and $U_6^0=U_6\cap V_6^0$ its cuspidal subspace. Then any element of U_6^0 is magnetic.*

Remark 3. In fact, it seems that the space U_6^0 possesses the *strongly* magnetic property: the anti-derivative of any difference of two $f_{a,b,c}$ from U_6 has an integral q-expansion.

Proof. For $c=0$, we only have two elements $f_{3,0,0}$ and $f_{1,1,0}$ in U_6, and $f_{3,0,0}\sim f_{1,1,0}$ since $f_{3,0,0}-f_{1,1,0}=6\delta f_{2,0,0}$. Moreover, they are both strongly equivalent to $f_{0,0,1}$, because $f_{1,1,0}-f_{0,0,1}=3\delta E_4$.

For $c=1$, we find out that $f_{0,0,1}$, $f_{2,-1,1}$, and $f_{4,-2,1}$ are in U_6. Then $f_{4,-2,1}$ is strongly equivalent to any of $f_{3,0,0}$, $f_{1,1,0}$, and $f_{0,0,1}$ in accordance with $f_{4,-2,1}-f_{3,0,0}=3\delta f_{4,-1,0}$ and the above. With the help of Theorem 2 and derivation

$$f_{2,-1,1}-f_{0,0,1}=4\delta f_{1,-1,1}-4\delta f_{0,-2,2}+2(f_{1,1,0}-f_{0,0,1})-4608F_6,$$

we see that the same is true for $f_{2,-1,1}$.

We have just shown that any element in the subspace U_6^0 generated by $f_{a,b,c}$ with $c\in\{0,1\}$ does have the (strongly) magnetic property. For the rest of our theorem, we proceed by

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induction over c using the following consequence of equation (3) when $k=4$:

$$\frac{b}{3}f_{a,b-1,c+1} = -\delta f_{a,b,c} + \frac{4-a}{12}f_{a+1,b,c} - \frac{a}{12}f_{a-1,b+1,c} - \frac{c}{2}f_{a,b+2,c-1}. \qquad \square$$

§3. A magnetic extension of the field of quasi-modular forms

The functions τ, q, E_2, E_4, E_6 are algebraically independent over $\mathbb{C}$ (see [19,22]). We can identify the differential field $\mathbb{C}\langle\tau, q, E_2, E_4, E_6\rangle$ generated by them over $\mathbb{C}$ with the differential field $\mathcal{K} = \mathbb{C}\langle\tau, q, X, Y, Z\rangle$ equipped with the derivation

$$D = \frac{1}{2\pi i}\frac{\partial}{\partial\tau} + q\frac{\partial}{\partial q} + \frac{1}{12}(X^2 - Y)\frac{\partial}{\partial X} + \frac{1}{3}(XY - Z)\frac{\partial}{\partial Y} + \frac{1}{2}(XZ - Y^2)\frac{\partial}{\partial Z}.$$

Our goal is to demonstrate that the elements

$$v_1 = \frac{XZ}{Y} - Y \quad \text{and} \quad v_2 = \frac{XY^2}{Z} - Z$$

corresponding to $f_{1,-1,1} - f_{0,1,0}$ and $f_{1,2,-1} - f_{0,1,0}$ do not have D-anti-derivatives in $\mathcal{K}$ (not even in $\mathcal{K}\langle D^{-1}v_2\rangle$ and $\mathcal{K}\langle D^{-1}v_1\rangle$, respectively). This follows trivially from noticing that $\operatorname{ord}_Y v_1 = -1$ and $\operatorname{ord}_Z v_2 = -1$, so that if either $D^{-1}v_1$ or $D^{-1}v_2$ existed, then $\operatorname{ord}_Y D^{-1}v_1 < 0$ and $\operatorname{ord}_Z D^{-1}v_2 < 0$, hence $\operatorname{ord}_Y v_1 = \operatorname{ord}_Y D(D^{-1}v_1) \le -2$ and similarly $\operatorname{ord}_Z v_2 \le -2$, contradiction.

By [16, Lem. 3.9] applied twice, the anti-derivatives

$$\tilde{E}_{4a} = \delta^{-1}(f_{1,-1,1} - f_{0,1,0}) \quad \text{and} \quad \tilde{E}_{4b} = \delta^{-1}(f_{1,2,-1} - f_{0,1,0})$$

are algebraically independent over the field $\mathbb{C}\langle\tau, q, E_2, E_4, E_6\rangle$, and the extended differential field

$$\mathbb{C}\langle\tau, q, E_2, E_4, E_6, \tilde{E}_{4a}, \tilde{E}_{4b}\rangle$$

has transcendence degree 7 over $\mathbb{C}$ and is a Picard–Vessiot extension of the differential field $\mathbb{C}\langle\tau, q, E_2, E_4, E_6\rangle$. Again, by identifying the former through the isomorphism

$$\varphi\colon E_2 \mapsto X,\ E_4 \mapsto Y,\ E_6 \mapsto Z,\ \tilde{E}_{4a} \mapsto S,\ \tilde{E}_{4b} \mapsto T$$

with the differential field $\hat{\mathcal{K}} = \mathbb{C}\langle\tau, q, X, Y, Z, S, T\rangle$ equipped with the derivation

$$\begin{aligned}\hat{D} = {}& \frac{1}{2\pi i}\frac{\partial}{\partial\tau} + q\frac{\partial}{\partial q} + \frac{1}{12}(X^2 - Y)\frac{\partial}{\partial X} + \frac{1}{3}(XY - Z)\frac{\partial}{\partial Y} + \frac{1}{2}(XZ - Y^2)\frac{\partial}{\partial Z} \\ & + \left(\frac{XZ}{Y} - Y\right)\frac{\partial}{\partial S} + \left(\frac{XY^2}{Z} - Z\right)\frac{\partial}{\partial T},\end{aligned}$$

we want to demonstrate that the element

$$v_3 = \frac{X^2 Z}{Y} - Z$$

corresponding to $f_{2,-1,1} - f_{0,0,1}$ does not have a $\hat{D}$-anti-derivative in $\hat{\mathcal{K}}$.

Assume on the contrary that there is an element $u_3 \in \hat{\mathcal{K}}$ such that $\hat{D}u_3 = v_3$. Notice that the functions τ, $q = e^{2\pi i\tau}$, $E_2(\tau)$, $E_4(\tau)$, and $E_6(\tau)$ are all analytic at $\tau = \rho = e^{2\pi i/3}$, the

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latter three having the values

$$E_2(\rho)=\frac{2\sqrt{3}}{\pi},\quad E_4(\rho)=0,\quad E_6(\rho)=\left(\frac{3\Gamma(\frac{1}{3})^6}{8\pi^4}\right)^3.$$

With the help of Ramanujan's system (2), we find out that

$$E_4(\tau)=-\frac{2\pi i}{3}E_6(\rho)(\tau-\rho)+O\big((\tau-\rho)^2\big)\quad \text{as } \tau\to\rho,$$

so that

$$\left.\begin{aligned} f_{1,-1,1}-f_{0,1,0}&=\frac{3iE_2(\rho)}{2\pi}\,\frac{1}{\tau-\rho}+O(1),\\ f_{2,-1,1}-f_{0,0,1}&=\frac{3iE_2(\rho)^2}{2\pi}\,\frac{1}{\tau-\rho}+O(1)\end{aligned}\right\}\quad \text{as } \tau\to\rho$$

and $f_{1,2,-1}-f_{0,1,0}$ is analytic at $\tau=\rho$. In turn, this implies that

$$\left.\begin{aligned} \tilde{E}_{4a}&=-3E_2(\rho)\ln(\tau-\rho)+g_1(\tau),\\ \delta^{-1}(f_{2,-1,1}-f_{0,0,1})&=-3E_2(\rho)^2\ln(\tau-\rho)+g_3(\tau)\end{aligned}\right\}\quad \text{as } \tau\to\rho$$

for some functions $g_1(\tau)$ and $g_3(\tau)$ analytic at $\tau=\rho$, while $\tilde{E}_{4b}(\tau)$ is analytic there. To summarize, the function

$$\delta^{-1}(f_{2,-1,1}-f_{0,0,1})-\frac{2\sqrt{3}}{\pi}\tilde{E}_{4a}(\tau)=\delta^{-1}(f_{2,-1,1}-f_{0,0,1})-E_2(\rho)\tilde{E}_{4a}(\tau)$$

is analytic at $\tau=\rho$, and hence only representable as a rational function of $\tau,q,E_2,E_4,E_6,\tilde{E}_{4b}$. Using the isomorphism φ, we conclude that

$$u=u_3-\frac{2\sqrt{3}}{\pi}S\in\hat{\mathcal{K}}$$

is a polynomial in τ,q,X,Y,Z,T. The latter is seen to be impossible after the operator $\hat{D}$ is applied to u and to $u_3-\frac{2\sqrt{3}}{\pi}S$ leading to a rational expression of S in terms of the other generators of $\hat{\mathcal{K}}$. The contradiction we arrive at implies that the anti-derivative

$$\tilde{E}_6=\delta^{-1}(f_{2,-1,1}-f_{0,0,1})$$

is transcendental over the field $\mathbb{C}\langle\tau,q,E_2,E_4,E_6,\tilde{E}_{4a},\tilde{E}_{4b}\rangle$. On replacing the generators of the latter with the anti-derivatives of magnetic modular forms from Theorems 1 and 2, we obtain the following result.

THEOREM 5. *The differentially closed field*

$$\mathbb{C}\langle\tau,q,E_2,E_4,E_6,\tilde{F}_{4a},\tilde{F}_{4b},\tilde{F}_6\rangle,$$

generated by τ, $q=e^{2\pi i\tau}$, *the Eisenstein series* (1), *and the anti-derivatives*

$$\tilde{F}_{4a}=\delta^{-1}\left(\frac{\Delta}{E_4^2}\right),\quad \tilde{F}_{4b}=\delta^{-1}\left(\frac{E_4\Delta}{E_6^2}\right),\quad \tilde{F}_6=\delta^{-1}\left(\frac{E_6\Delta}{E_4^3}\right)$$

with integral coefficients in their q-*expansions, has transcendence degree* 8 *over* $\mathbb{C}$.

REMARK 4. Another way to see that no u_3 exists in $\hat{\mathcal{K}}$ such that $\hat{D}u_3 = v_3$ is by casting u_3 in the form p/q with p, q in the ring $\mathcal{R}[S]$, where $\mathcal{R} = \mathbb{C}\langle \tau, q, X, Y, Z, T\rangle$, and $\gcd(p,q) = 1$. After clearing the denominators in $\hat{D}(p/q) = v_3$ and comparing the degree in S on both sides, one concludes that $\hat{D}q = uq$ for some $u \in \mathcal{R}$ (i.e., independent of S). This leads to conclusion $q \in \mathcal{R}$, so that u_3 is a polynomial in S. Finally, the equation $\hat{D}u_3 = X^2Z/Y - Z$ is seen to be impossible by comparing the order in Y on both sides.

EXERCISE 1. We leave to the reader the exercise to prove that the anti-derivative of $\tilde{F}_6$ (in turn, the second anti-derivative of F_6) is transcendental over the field in Theorem 5.

§4. Half-integral weight weakly holomorphic modular forms

Following the ideas in [17], we cast magnetic modular forms of weight $2k$ as the images of weakly holomorphic eigenforms of weight $k+1/2$ under the Shimura–Borcherds (SB) lift. In our settings, an input for the lift is a form $f(\tau) = \sum_{n\gg-\infty} a(n)q^n$ from the Kohnen plus space $M^{!,+}_{k+1/2}$ (meaning that $a(n)$ vanishes when $(-1)^k n \not\equiv 0, 1 \bmod 4$); the output is the meromorphic modular form $\Psi(f)(\tau) = \sum_{n>0} A(n)q^n$ with

$$A(n) = \sum_{d|n} \left(\frac{d}{D}\right) d^{k-1} a(|D|n^2/d^2), \tag{5}$$

where $D = D_k = 1$ for k even (so that the Kronecker–Jacobi symbol $\left(\frac{d}{D}\right)$ is always 1) and $D = D_k = -3$ for k odd. In other words,

$$\Psi = \Psi_k \colon f = \sum_{n\gg-\infty} a(n)q^n \mapsto F = \sum_{n>0} q^n \sum_{d|n} \left(\frac{d}{D_k}\right) d^{k-1} a(|D_k|n^2/d^2), \tag{6}$$

and the latter expression is just $F = \sum_{n>0} q^n \sum_{d|n} d^{k-1} a(n^2/d^2)$ when k is even. We also distinguish the Kohnen plus cuspidal space $S^{!,+}_{k+1/2}$ in $M^{!,+}_{k+1/2}$ by imposing the additional constraint $a(0) = 0$.

Our examples of forms from $M^{!,+}_{k+1/2}$ with $k = 2$ involved in the proof of Theorem 1 are the following three:

$$\begin{aligned} g_0(\tau) &= \theta(\tau)\,(\theta(\tau)^4 - 20E_{2,4}(\tau)) \\ &= 1 - 10q - 70q^4 - 48q^5 - 120q^8 - 250q^9 - \cdots - 550q^{16} - \cdots \\ &\quad - 1210q^{25} - \cdots - 1750q^{36} - \cdots - 3370q^{49} - \cdots, \\ g_1(\tau) &= \frac{\theta(\tau)E_4(4\tau)^2E_6(4\tau)}{\Delta(4\tau)} \\ &= q^{-4} + 2q^{-3} + 2 - 196884q^4 - \cdots - 85975040q^9 - \cdots \\ &\quad - 86169224844q^{16} - \cdots - 51186246451200q^{25} - \cdots \\ &\quad - 35015148280961780q^{36} - \cdots - 21434928162930081792q^{49} - \cdots, \end{aligned}$$

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$$
\begin{aligned}
g_2(\tau) &= \frac{g_0(\tau)E_4(4\tau)^3}{\Delta(4\tau)} \\
&= q^{-4} - 10q^{-3} + 674 - 7488q + 144684q^4 - \cdots - 224,574,272q^9 - \cdots \\
&\quad - 42882054732q^{16} - \cdots - 63793268216640q^{25} - \cdots \\
&\quad - 31501841125150388q^{36} - \cdots - 22385069000981561664q^{49} - \cdots,
\end{aligned}
$$

where $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$ and

$$
E_{2,4}(\tau) = \frac{-E_2(\tau) + 3E_2(2\tau) - 2E_2(4\tau)}{24} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} q^n \sum_{d\mid n} d. \tag{7}
$$

The modular form $g_0(\tau)$ is known by the name of normalized Cohen–Eisenstein series of weight $5/2$.

LEMMA 1. (a) *The weight* $5/2$ *weakly holomorphic modular form*

$$
f_{4a}(\tau) = \frac{7}{8}g_0(\tau) + \frac{1}{768}g_1(\tau) - \frac{1}{768}g_2(\tau) = \frac{1}{64}q^{-3} + q - 506q^4 + \cdots
$$

lies in the Kohnen plus cuspidal space $S^{!,+}_{5/2}$, *and its SB lift* $\Psi(f_{4a})$ *is* $F_{4a} = \Delta/E_4^2$.
(b) *The weight* $5/2$ *weakly holomorphic modular form*

$$
f_{4b}(\tau) = \frac{19}{18}g_0(\tau) - \frac{5}{648}g_1(\tau) - \frac{1}{648}g_2(\tau) = -\frac{1}{108}q^{-4} + q + 1222q^4 + \cdots
$$

lies in the Kohnen plus cuspidal space $S^{!,+}_{5/2}$, *and its SB lift* $\Psi(f_{4b})$ *is* $E_4\Delta/E_6^2$.
Moreover, $f_{4a} \in \frac{1}{64}q^{-3}\mathbb{Z}[[q]]$ *and* $f_{4b} \in \frac{1}{108}q^{-4}\mathbb{Z}[[q]]$.

The identification $\Psi(f_{4a}) = F_{4a}$ is already in Borcherds' [5, Exam. 14.4].

Proof. Indeed, we only need to check that f_{4a}, f_{4b} have vanishing constant term and that the first three coefficients in the q-expansions of $\Psi(f_{4a})$, $\Psi(f_{4b})$ agree with those of the predicted meromorphic modular forms; we choose to check the first seven coefficients.

For the integrality statement, we use the alternative expressions

$$
64f_{4a}(\tau) = \frac{f^*_{14+1/2}(\tau)}{\Delta(4\tau)}
$$

and

$$
-108f_{4b}(\tau) = \frac{f_{14+1/2}(\tau)\,(j(4\tau) - 674) + 10f^*_{14+1/2}(\tau)}{\Delta(4\tau)},
$$

where the forms $f_{b+1/2}(\tau), f^*_{b+1/2}(\tau)$ are the holomorphic modular forms of weight $b + 1/2$ with integral q-expansions from the table in [8, Appendix] and $j(\tau) = E_4(\tau)^3/\Delta(\tau)$ is the elliptic modular invariant. ∎

As we see further, for certain forms $\sum_{n\gg-\infty} a(n)q^n \in S^{!,+}_{5/2}$ with integral q-expansions (in particular, for the forms $64f_{4a}$ and $108f_{4b}$), one can make use of Hecke operators to conclude with the divisibility $n \mid a(n^2)$ for $n > 0$. This readily implies that $64F_{4a}$ and $108F_{4b}$

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in Theorem 1 are strongly magnetic modular forms, since the relation in (5) translates the divisibility into

$$\frac{A(n)}{n} = \sum_{d|n} \frac{a(n^2/d^2)}{n/d} = \sum_{d|n} \frac{a(d^2)}{d} \in \mathbb{Z}.$$

A detailed analysis below reveals that the factors 64 and 108 can be also removed.

§5. The square part and Hecke operators

We refer the reader to [6,8] for the definition of Hecke operators $\mathcal{T}_p$ and T_{p^2} on integral weight $2k$ and half-integral weight $k+1/2$ modular forms (including weakly holomorphic or meromorphic), respectively. As in the case of the SB lift $\Psi = \Psi_k$ in (6), these definitions make perfect sense for *any* Laurent series $f = \sum_{n \gg -\infty} a(n)q^n$, not necessarily of modular origin, but with the weight $2k$ or $k+1/2$ additionally supplied. We refer to the finite sum $\sum_{n<0} a(n)q^n$ as to the principal part of *f*. We take

$$f \,|\, U_p = \sum_{n \gg -\infty} a(np)q^n, \quad f \,|\, V_p = \sum_{n \gg -\infty} a(n)q^{np}, \quad f \,|\, \chi = \sum_{n \gg -\infty} \chi(n)a(n)q^n$$

for a character $\chi\colon \mathbb{Z} \to \mathbb{C}$, and define

$$f \,|\, \mathcal{T}_p = f \,|\, (\mathcal{T}_p, 2k) = f \,|\, U_p + p^{2k-1}V_p$$

and

$$f \,|\, T_{p^2} = f \,|\, (T_{p^2}, k+1/2) = f \,|\, U_p^2 + p^{k-1}\chi_p + p^{2k-1}V_p^2,$$

where $\chi_p(n) = \chi_{p,k}(n) = \big(\frac{(-1)^k n}{p}\big)$ is the Kronecker–Jacobi symbol.

A simple calculation shows that $\Psi_k(f) \,|\, (\mathcal{T}_p, 2k) = \Psi_k\big(f \,|\, (T_{p^2}, k+1/2)\big)$, which we can reproduce in a simplified form

$$\Psi(f) \,|\, \mathcal{T}_p = \Psi(f \,|\, T_{p^2})$$

when k is fixed.

LEMMA 2. *Given a positive integer k, assume that there are no cusp forms of weight* $2k$. *For a prime p, let* $f \in M^{!,+}_{k+1/2}$ *have p-integral coefficients and satisfy* $p^2 > -\operatorname{ord}_q(f)$. *Then*

$$f \,|\, T^n_{p^2} \equiv 0 \bmod p^{(k-1)n}.$$

Proof. Following the argument in [6, Proof of Lem. 3.1], we can write

$$T^n_{p^2} = \sum_{\substack{a,b,c,r \ge 0 \\ a+b+c=n \\ r \le \min\{a,c\}}} \alpha_{a,b,c,r} \cdot p^{(2k-1)c+(k-1)b} \cdot U^{a-r}_{p^2} \chi^b_p V^{c-r}_{p^2}, \tag{8}$$

where $\alpha_{a,b,c,r}$ are some integers. This writing can be easily deduced from $V_{p^2}\chi_p = \chi_p U_{p^2} = 0$ and the fact that $V_{p^2}U_{p^2}$ is the identity. We only need to analyze the principal part of $f \,|\, T^n_{p^2}$ which, by the hypothesis $\dim S_{2k} = 0$, determines it uniquely.

If $r < a$, then $f \,|\, U^{a-r}_{p^2} \chi^b_p V^{c-r}_{p^2}$ has no principal part, because the latter is killed by a single action of U_{p^2} (since $a_{-p^2 m} = 0$ for any $m \ge 0$). Therefore, we may assume that $a = r \le c$. This implies that $(2k-1)c + (k-1)b \ge (k-1)(2c+b) \ge (k-1)n$, and hence the principal of $f\,|\,T_{p^2}^n$ part is divisible by $p^{(k-1)n}$. This, in turn, implies that $f\,|\,T_{p^2}^n = p^{(k-1)n}\cdot g$ for some $g\in\mathcal{M}_{k+1/2}^{!,+}$ with p-integral coefficients, since there is a basis $\{g_m = q^m + O(q) : m\in\mathbb{Z},\, (-1)^k m\equiv 0\}$ of $M_{k+1/2}^{!,+}$ whose elements have all coefficients integral (see [8, Prop. 2]). ∎

In parallel with (6), define

$$\Phi=\Phi_k\colon g=\sum_{n\gg-\infty} b(n)q^n \mapsto \sum_{n>0} q^{|D_k|n^2}\sum_{d|n}\left(\frac{d}{D_k}\right)d^{k-1}\mu(d)b(n/d),$$

where $\mu(\cdot)$ is the Möbius function and, as before, $D_k = 2\cdot(-1)^k-1\in\{1,-3\}$. We further define the *square part* of a Laurent series $f=\sum_{n\gg-\infty}a(n)q^n$ as

$$f^{\square}=\sum_{n>0}a(|D_k|n^2)q^{|D_k|n^2}.$$

The definitions immediately lead to the following conclusions.

LEMMA 3. *We have $\Phi(\Psi(f))=f^{\square}$. In particular, if $\Psi(f)\in q\mathbb{Z}[[q]]$, then $f^{\square}\in q\mathbb{Z}[[q]]$.*

Notice that $f_{4a}^{\square}, f_{4b}^{\square}\in q\mathbb{Z}[[q]]$ by this lemma, because both $F_{4a}=\Psi(f_{4a})$ and $F_{4b}=\Psi(f_{4b})$ are in $q\mathbb{Z}[[q]]$.

In addition to this, we list some other easily verifiable properties about the interaction of Hecke operators and square parts.

LEMMA 4. *Given a Laurent series $f=\sum_{n\gg-\infty}a(n)q^n$ and positive integer k, the following statements are true.*

(a) $\Psi(f)\,|\,\mathcal{T}_p^n=\Psi(f\,|\,T_{p^2}^n)$ *for* $n=1,2,\dots$.
(b) $\Psi(f)=\Psi(f^{\square})$.
(c) $(f\,|\,T_{p^2})^{\square}=f^{\square}\,|\,T_{p^2}$ *termwise, that is,* $(f\,|\,U_{p^2})^{\square}=f^{\square}\,|\,U_{p^2}$, $(f\,|\,V_{p^2})^{\square}=f^{\square}\,|\,V_{p^2}$, *and* $(f\,|\,\chi_p)^{\square}=f^{\square}\,|\,\chi_p$.
(d) *If the coefficients of f are integral and $k\ge 2$, then* $f\,|\,T_{p^2}\equiv f\,|\,U_p^2 \bmod p$.

Proof of Theorem 1. Consider $f\in\{f_{4a},f_{4b}\}$. For a prime $p\ge 5$, the form f is p-integral and we have $\operatorname{ord}_q(f)\ge -4$; therefore, Lemma 2 with $k=2$ applies to result in

$$f\,|\,T_{p^2}^n\equiv 0 \bmod p^n.$$

Applying SB map (6), we deduce that, for $F=\Psi(f)\in\{F_{4a},F_{4b}\}$, we have $F\,|\,\mathcal{T}_p^n\equiv 0 \bmod p^n$ for all $n\ge 1$, and hence $F\,|\,U_p^n\equiv 0\bmod p^n$; in other words, $F=\sum_{m>0}A(m)q^m$ has the strong p-magnetic property:

$$p^n\mid m \implies p^n\mid A(m) \tag{9}$$

for any prime $p\ge 5$. This argument also works for $f=f_{4a}$ in the case of $p=3$, because f_{4a} is 3-integral.

Consider now $p=3$ and $f=f_{4b}$, in which case we only know that $27f$ is 3-integral. Take the (unique!) element $g_r\in M_{5/2}^{!,+}$ with q-expansion $g_r=q^{-4\cdot 9^r}+O(q)$; by [8, Prop. 2], it has integral coefficients. We first show that $g_0^{\square}\,|\,T_9^n\equiv 0 \bmod 3^{n+3}$. For $n=0$, this is true, because $g_0=-108\cdot f_{4b}$ and $f_{4b}^{\square}$ is in $q\mathbb{Z}[[q]]$. For $n=1$, we observe that $\Psi(-\frac{1}{108}g_0\,|\,T_9)=F_{4b}\,|\,\mathcal{T}_3$ and

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$F_{4b} \equiv \Delta \bmod 3$ (since both $E_4, E_6 \equiv 1 \bmod 3$). This implies that $F_{4b} \,|\, \mathcal{T}_3 \equiv \Delta \,|\, \mathcal{T}_3 \equiv 0 \bmod 3$, and hence

$$-\frac{1}{108} g_0^{\square} \,|\, T_9 = \Phi(F_{4b} \,|\, \mathcal{T}_3) \equiv 0 \bmod 3,$$

meaning that $g_0^{\square} \,|\, T_9^n \equiv 0 \bmod 3^{n+3}$ is true when $n = 1$. Since $g_0 \,|\, T_9 = 27g_1 - 3g_0$, we also deduce from this that $g_1^{\square} \equiv 0 \bmod 3$.

For n general, we want to write $g_0 \,|\, T_9^n$ as a $\mathbb{Z}$-linear combination of g_r with $r = 0, 1, \ldots, n$. Looking at the principal part of $g_0 \,|\, T_9^n$, one finds out that only terms of the form $q^{-4\cdot 3^{2m}}$ appear, so that subtracting the related linear combination of f_r leads to a holomorphic cusp form, which then must vanish. To examine this linear combination in more details, we proceed as in the proof of Lemma 2:

$$g_0 \,|\, T_9^n = \sum_{a,b,c,r} \alpha_{a,b,c,r} \cdot 3^{3c+b} \cdot g_0 \,|\, U_9^{a-r} \chi_3^b V_9^{c-3}$$

(see (8)). As already noticed in that proof, only the terms with $r = a \le c$ contribute to the principal part, and thus to the linear combination; the terms with $r = a$ contribute by the subsum

$$\sum_{a,b,c} \alpha_{a,b,c,a} \cdot (-1)^b \cdot 3^{3c+b} \cdot g_{c-a}.$$

Now, notice that if $2c \ge a + 3$, then the coefficient is divisible by 3^{n+3}. In the remaining situations, we have $2a \le 2c < a + 3$, in particular $a \in \{0, 1, 2\}$, and we use the following analysis:

(a) If $a = 2$, then the inequalities imply that $c = 2$, and hence $b = n - 4$; the corresponding term is then a multiple of $3^{3\cdot 2+n-4} g_0$.
(b) If $a = 1$, then $c = 1$, and hence $b = n - 2$; the corresponding term happens to be a multiple of $3^{3\cdot 1+n-2} g_0$.
(c) If $a = 0$, then $c \in \{0, 1\}$. The term corresponding to $c = 0$ is a multiple of $3^n g_0$, whereas the term corresponding to $c = 1$ is a multiple of $3^{n+2} \cdot g_1$.

Gathering all the terms, we end up with an expression

$$g_0 \,|\, T_9^n = 3^{n+3} g + 3^{n+2} \alpha \cdot g_1 + 3^n \beta \cdot g_0,$$

where g is integral and both α and β are integers. Taking the square parts on both sides and using the results for $n = 0, 1$, we deduce that $g_0^{\square} \,|\, T_9^n \equiv 0 \bmod 3^{n+3}$ for any $n = 0, 1, \ldots$. Finally, we apply the SB map to this congruence to deduce that $F_{4b} \,|\, \mathcal{T}_3^n \equiv 0 \bmod 3^n$ for all $n \ge 0$. In other words, this implies the congruences (9) for $p = 3$.

Turning now our attention to the prime $p = 2$, notice that the Hecke operator T_4 does not respect the Kohnen plus space. However, if we define the projection

$$K^+ = K_k^+ \colon \sum_{n\in\mathbb{Z}} a(n) q^n \mapsto \sum_{\substack{n\in\mathbb{Z} \\ (-1)^k n \equiv 0,1 \bmod 4}} a(n) q^n,$$

then the operator $T_4' = K^+ \circ T_4$ maps the space $M_{k+1/2}^{!,+}$ onto itself and inherits all the properties used above for T_{p^2} when $p > 2$. We use this operator T_4' in place of T_4 to complete the proof of our Theorem 1. Notice that in both cases $f = f_{4a}$ and $f = f_{4b}$ has powers of 2

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in the denominator of its main term. For an ease of the argument, we treat the two cases separately, although the same strategy is used for both, along the line with the proof above of relation (9) for $p=3$.

When $f=f_{4b}$, we need to prove that $F_{4b}\,|\,\mathcal{T}_2^n \equiv 0 \bmod 2^n$, which is in turn implied by the congruence $f_{4b}^{\square}\,|\,T_4'^{\,n} \equiv 0 \bmod 2^n$. Introduce $g_r = q^{-4\cdot 4^r}+O(q) \in M_{5/2}^{!,+}$ with integral q-expansions for $r=0,1,\dots$ and notice that $f_{4b} = -\frac{1}{108}\cdot g_0$. The induction on $r\ge 0$ shows that the recursion $g_r\,|\,T_4' = 8g_{r+1}+g_{r-1}$ takes place, with the convention that $g_{-1}=0$. This in turn leads to

$$g_0\,|\,T_4'^{\,n} = 2^{n+2}g + 2^{n+1}\alpha\cdot g_1 + 2^n\beta\cdot g_0$$

for some integral $g\in M_{5/2}^{!,+}$ and $\alpha,\beta\in\mathbb{Z}$. Taking the square parts on both sides and using that $F_{4b}\equiv\Delta \bmod 8$, and hence $\Phi(F_{4b}\,|\,\mathcal{T}_2)\equiv\Phi(\Delta\,|\,\mathcal{T}_2)\equiv 0 \bmod 8$, we conclude with $g_0^{\square}\,|\,T_4'^{\,n}\equiv 0 \bmod 2^{n+2}$, and hence with (9) for $p=2$ and $F=F_{4b}$.

For $f=f_{4a}$, we introduce the family $g_r = q^{-3\cdot 4^r}+O(q)\in M_{5/2}^{!,+}$, where $r=0,1,\dots$, which is invariant under the action of the operator T_4', and proceed similarly to get exactly the same recursion $g_r\,|\,T_4' = 8g_{r+1}+g_{r-1}$ for $r\ge 0$ with $g_{-1}=0$. On using $g_0=\frac{1}{64}f_{4a}$,

$$g_0\,|\,T_4'^{\,n} = 2^{n+6}g + 2^{n+5}\alpha\cdot g_2 + 2^{n+4}\beta\cdot g_1 + 2^{n+3}\gamma\cdot g_0,$$

for $n\ge 3$, and $F_{4a}\equiv\Delta \bmod 8$, we conclude with $g_0^{\square}\,|\,T_4'^{\,n}\equiv 0 \bmod 2^{n+6}$ implying $F_{4a}\,|\,\mathcal{T}_2^n \equiv 0 \bmod 2^n$ as required. □

Proof of Theorem 2. We now work with $k=3$. Consider

$$f(\tau) = -\frac{1}{384}\,\frac{f_{15+1/2}^*(\tau)}{\Delta(4\tau)} \in \mathcal{M}_{k+1/2}^{!,+},$$

where $f_{b+1/2}^*$ is the weight $b+1/2$ modular form from the table in [8, Appendix]. One can easily check (through the first few coefficients) that $\Psi(f)=F_6$ and from the expression above we also know that f has p-integral coefficients for any $p\ge 5$. It follows from Lemma 2 (applied this time with $k=3$) that $f\,|\,T_{p^2}^n \equiv 0 \bmod p^{2n}$. Therefore, $F_6\,|\,\mathcal{T}_p^n\equiv 0 \bmod p^{2n}$ for all $n\ge 0$ implying that $F_6\,|\,U_p^n \equiv 0 \bmod p^{2n}$ and that, for $F_6=\sum_{m>0}A(m)q^m$, we have

$$p^n \mid m \implies p^{2n} \mid A(m) \tag{10}$$

for any prime $p\ge 5$.

Since $384 = 3\cdot 2^7$, for $p=3$, we see that $3f$ is 3-integral. Repeating the argument from Lemma 2 and using the fact that f is a multiple of the unique element in $\mathcal{M}_{7/2}^{!,+}$ with the integral q-expansion $q^{-1}+O(q)$, we deduce that $f\,|\,T_9^n = 3^{2n}\cdot(g+\alpha f)$ with α an integer and g a 3-integral modular form. Indeed, the principal part of $f\,|\,T_9^n$ is a $\mathbb{Z}$-linear combination of the principal parts of

$$3^{(2\cdot 3-1)c+(3-1)b}\cdot f\,|\,\chi_3^b V_9^{c-a} = 3^{2n}\cdot(3^{c-a}f)\,|\,\chi_3^b V_9^{c-a}.$$

If $c-a\ge 1$, the principal part of $(3^{c-a}f)\,|\,\chi_3^b V_9^{c-a}$ is 3-integral; when $c=a$, the principal part of $f\,|\,\chi_3^b$ is an integral multiple of the principal part of f. Thus, $f\,|\,T_9^n = 3^{2n}\cdot(g+\alpha\cdot f)$ implies (applying the SB lift to both sides) that $F_6\,|\,\mathcal{T}_3^n\equiv 0 \bmod 3^{2n}$, and hence we deduce that (10) is true also for $p=3$.

To prove the relation (10) for $p=2$, we proceed as in the proof of Theorem 1. We introduce the T_4'-invariant family of weight $7/2$ weakly holomorphic modular forms $g_r = q^{-4^r}+O(q)$

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with integral q-expansions with the help of [8, Prop. 2]. Again, we write the expression of $g_0\,|\,T_4'^{\,n}$ as $\mathbb{Z}$-linear combination of g_r with $r=0,1,\dots,n$ and analyze the powers of 2 appearing in the coefficients; similarly, we can prove that $g_0^{\square}\,|\,T_4'^{\,n}\equiv 0 \bmod 2^{2n+7}$ for any $n\geq 0$. For $n=0$, this comes from the integrality of $f^{\square}$, whereas for $n=1$, we get it, again, by noticing that $F_6\equiv E_6\Delta \bmod 2^4$ while $E_6\Delta$ being an eigenform of weight 18 with slope 4 at the prime 2. The induction argument follows *mutatis mutandis* as in the proof of Theorem 1. ∎

§6. Miscellania on half-integral weight modular forms

In this part, not well related to the proofs of Theorems 1 and 2, we indicate a different strategy of constructing half-integral weight weakly holomorphic modular forms using a traditional rising operator.

Standard examples of weight 1/2 modular forms (see [4, Sec. 14, Exam. 2]) include the theta function $\theta(\tau)=\sum_{n\in\mathbb{Z}}q^{n^2}$ and

$$\begin{aligned}h_0(\tau)&=\frac{E_{2,4}(\tau)\theta(\tau)\,(\theta(\tau)^4-2E_{2,4}(\tau))\,(\theta(\tau)^4-16E_{2,4}(\tau))\,E_6(4\tau)}{\Delta(4\tau)}+56\theta(\tau)\\&=q^{-3}-248q+26752q^4+\cdots,\end{aligned}$$

where $E_{2,4}(\tau)$ is given in (7). The images of 12θ and $4\theta+h_0$ under the multiplicative Borcherds lift

$$\Psi^{\mathrm{mult}}\colon \sum_{n\gg-\infty}c(n)q^n\mapsto q^{-h}\prod_{n>0}(1-q^n)^{c(n^2)}$$

are the modular forms $\Delta(\tau)$ and $E_4(\tau)$, respectively (see [4, Th. 14.1] for the definition of h). Although it is not useful for our results in this note, we remark that the two weakly holomorphic modular forms can serve as constructors of some weight 5/2 modular forms from §4.

LEMMA 5. *The raising operator*

$$\mathcal{D}=\mathcal{D}_k\colon f\mapsto \delta f-\frac{2k+1}{6}E_2(4\tau)\cdot f$$

maps $M^{!,+}_{k+1/2}$ *onto* $M^{!,+}_{k+5/2}$.

Proof. Observe that $E_2(\tau)-4E_2(4\tau)$ is a modular form of weight 2 for $\Gamma_0(4)$, so that the difference between the usual raising operator and $\mathcal{D}$ is the multiplication by a weight 2 modular form, thus indeed $\mathcal{D}\colon M^!_{k+1/2}\to M^!_{k+5/2}$. On the other hand, both δ and multiplication by any modular form $f(4\tau)$ preserve the Kohnen plus space condition, and the lemma follows. ∎

For the functions g_0, f_{4a}, and f_{4b} in §4, we find out that

$$g_0=-6\mathcal{D}\theta,\quad 64f_{4a}=-\frac{6}{19}\mathcal{D}h_0$$

and

$$108f_{4b}=-\frac{6}{25}\mathcal{D}\left(2h_0-1006\theta-\frac{\theta E_6(4\tau)^2}{\Delta(4\tau)}\right).$$

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Table 1. Strong magnetic modular forms of weight 4 *(where* $f_m = q^{-m} + O(q)$ *denotes the unique weakly holomorphic cusp form in* $M^{!,+}_{5/2}$*).*

SB lift	Description in terms of E_4 and $j = E_4^3/\Delta$
$\Psi(3^{-3}f_7)$	$E_4\,\dfrac{19j - 8\cdot 15^3}{(j+15^3)^2}$
$\Psi(-2^{-3}f_8)$	$E_4\,\dfrac{101j - 3\cdot 20^3}{(j-20^3)^2}$
$\Psi(2^{-6}f_{11})$	$E_4\,\dfrac{43j - 6\cdot 32^3}{(j+32^3)^2}$
$\Psi(48^{-2}f_3 \mid T_4)$	$E_4\,\dfrac{14j + 18\cdot 15^3}{(j-2\cdot 30^3)^2}$
$\Psi(108^{-1}f_4 \mid (1-\frac{1}{2}T_4))$	$E_4\,\dfrac{611j + 404\cdot 33^3}{(j-66^3)^2}$
$\Psi(3^{-3}f_7 \mid (2-\frac{1}{2}T_4))$	$E_4\,\dfrac{82451j + 5272\cdot 255^3}{(j-255^3)^2}$
$\Psi(12^{-3}f_{19})$	$E_4\,\dfrac{25j - 2\cdot 96^3}{(j+96^3)^2}$
$\Psi(12^{-3}f_{43})$	$E_4\,\dfrac{11329j - 578\cdot 960^3}{(j+960^3)^2}$
$\Psi(12^{-3}f_{67})$	$E_4\,\dfrac{1221961j - 49442\cdot 5280^3}{(j+5280^3)^2}$
$\Psi(12^{-3}f_{163})$	$E_4\,\dfrac{908855380249j - 23238932978\cdot 640320^3}{(j+640320^3)^2}$
$\Psi(15^{-1}f_{15})$	$E_4\,\dfrac{785j^3 - 15219684j^2 + 28709816985j + 837864\cdot 495^3}{(j^2+191025j-495^3)^2}$
$\Psi(-80^{-1}f_{20})$	$E_4\,\dfrac{733j^3 + 72767680j^2 - 984198615040j + 123\cdot 20^3\cdot 880^3}{(j^2-158\cdot 20^3 j - 880^3)^2}$
$\Psi(-f_{23})$	$E_4\,\dfrac{P_{23}(j)}{(j^3+27934\cdot 5^3 j^2 - 329683\cdot 5^6 j + 187^3\cdot 5^9)^2}$

where

$$\begin{aligned} P_{23}(j) &= 141826j^5 - 286458244\cdot 5^3 j^4 + 5214621227\cdot 5^6 j^3 + 3414887843776\cdot 5^9 j^2 \\ &\quad - 47816219216827\cdot 5^{12} j + 4378632\cdot 187^3\cdot 5^{15}. \end{aligned}$$

§7. Concluding remarks

Although we expect that our discussion above exhausts all elements with the magnetic property in W_4^0, many such exist for W_{2k}^0 with $k > 2$, for example, the q-series $E_2^m\cdot(\delta E_j)/E_j$ for $j = 4, 6$ and $m = 1,2,3,4,6$ (but not for $m = 5$). Constructing magnetic *modular* forms—meromorphic ones with poles at quadratic irrationalities from the upper half-plane—is a routine on the basis of SB lift (6); Table 1 lists a few instances of this production explicitly in terms of the j-invariant $j(\tau) = E_4^3/\Delta$. Generating the forms with multiple magnetic property in higher weights is a tougher task; one such example $E_4^2(j - 3\cdot 2^{10})/j^2$ can be

https://doi.org/10.1017/nmj.2022.11 Published online by Cambridge University Press

found in the more recent work [18] of Löbrich and Schwagenscheidt; another example of a triply magnetic form of weight 8 is

$$E_4^2\,\frac{13j^3-443556j^2+1610452125j-98280\cdot 15^6}{(j+15^3)^4}.$$

We have observed that in all such instances the related numerators, viewed as polynomials in j, have real zeros only. Furthermore, there are weaker divisibility conditions (resembling the Honda–Kaneko congruences [13]) for individual summands of magnetic forms; for example, the anti-derivatives of

$$\frac{E_4 j}{(j-2\cdot 30^3)^2} \quad \text{and} \quad \frac{E_4}{(j-2\cdot 30^3)^2}$$

are already p-integral for primes $p\equiv 5 \bmod 6$. We have not tried to investigate this arithmetic subphenomenon.

There is a good reason to believe that all such magnetic forms originate from suitable Shimura–Borcherds lifts. However, maybe, there is more in this story—then time will show.

Acknowledgment. We thank the anonymous referee for their critical reading and valuable comments: those really helped to reduce errors and improve the exposition.

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