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DOI:10.4153/S0008439519000511

Stability of almost periodic Nicholson's blowflies model involving patch structure and mortality terms *

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Abstract: Taking into account the effects of patch structure and nonlinear density-dependent mortality terms, we explore a class of almost periodic Nicholson's blowflies model in this paper. Employing the Lyapunov function method and differential inequality technique, some novel assertions are developed to guarantee the existence and exponential stability of positive almost periodic solutions for the addressed model, which generalize and refine the corresponding results in some recent published literatures. Particularly, an example and its numerical simulations are arranged to support the proposed approach.

Keywords: Nicholson's blowflies model; patch structure; density-dependent mortality term; almost periodic solution; stability.

AMS(2010) Subject Classification: 34C25; 34K13

1 Introduction

The qualitative theory of differential equations model has been an attractive topic because of their significance and applications in areas such as physics, mathematical biology, and control theory [1, 2, 3, 4]. In population systems, due to the factors such as seasonal variation of weather, mating, harvesting and so on, the periodic fluctuations are widely occurring process and play key roles in modeling [5, 6, 7]. However, when there are nonintegral mul-

*This work was supported by the National Natural Science Foundation of China (Nos. 11971076, 11861037, 11771059, 51839002), the Scientific Research Fund of Hunan Provincial Education Department (No. 16C0036).

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tiples periods (also called as incommensurable) for different components of the temporally nonuniform environment, more and more scientists realize that assume the environment have almost periodicity instead of periodicity might be a better candidature [8, 9, 10, 11]. Nowadays, the investigations of almost periodic dynamics systems have been the new world-wide focus (see Refs. [12, 13, 14, 15, 16, 17]). In particular, the existence and global stability of almost periodic solutions for the famous scalar Nicholson's blowflies model with a nonlinear density-dependent mortality term:

$$x'(t) = -a(t) + b(t)e^{-x(t)} + \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))}, \tag{1.1}$$

and th Nicholson's blowflies systems with patch structure and nonlinear density-dependent mortality terms:

$$\begin{aligned} x_i'(t) &= -a_{ii}(t) + b_{ii}(t)e^{-x_i(t)} + \sum_{j=1, j\neq i}^{n} (a_{ij}(t) - b_{ij}(t)e^{-x_j(t)}) \\ &\quad + \sum_{j=1}^{m} \beta_{ij}(t)x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))},\ i \in Q := \{1, 2 \cdots, n\}, \end{aligned} \tag{1.2}$$

has been extensively investigated in previous studies [14, 18] and [19], respectively. Here, the information on the delay and coefficient functions presented in (1.1) and (1.2) can be available from [20, 21, 22, 24] and the references cited therein. For the feedback function xe^{-x} and its derivative $\frac{1-x}{e^x}$, the author in [25] pointed out that there exist two fixed positive numbers κ and $\tilde{\kappa}$ such that

$$\kappa \approx 0.7215355,\ \widetilde{\kappa} \approx 1.342276,\ \frac{1-\kappa}{e^\kappa} = \frac{1}{e^2},\ \sup_{x\geq\kappa} |\frac{1-x}{e^x}| = \frac{1}{e^2},\ \kappa e^{-\kappa} = \widetilde{\kappa}e^{-\widetilde{\kappa}}.$$

It should be pointed out that the global exponential stability of almost periodic solutions of (1.1) has been shown in [14, 18] under the restriction that the almost periodic solution exists in a small interval $[\kappa, \widetilde{\kappa}]$, and the global exponential stability of (1.2) has been established in [19] where the authors adopted the restraint that the almost periodic solution exists in a small domain $\underbrace{[\kappa, \widetilde{\kappa}] \times [\kappa, \widetilde{\kappa}] \times \cdots \times [\kappa, \widetilde{\kappa}]}_{n}$. Obviously, the above restriction and restraint do not accord with the biological significance of the population models. In particular, to the best of our knowledge, there is not yet research work on the global stability of almost periodic solutions of Nicholson's blowflies systems with patch structure and nonlinear density-dependent mortality terms when the almost periodic solutions do not belong to the above domain.

According to the above discussions, in this paper, without adopting $\underbrace{[\kappa, \widetilde{\kappa}] \times [\kappa, \widetilde{\kappa}] \times \cdots \times [\kappa, \widetilde{\kappa}]}_{n}$ as the existence domain of almost periodic solutions, we establish the existence and global exponential stability of positive almost periodic solutions for Nicholson's blowflies systems (1.2) involving patch structure and nonlinear density-dependent mortality terms. The proposed criterion improves and complements some existing results in the recent publications [14, 18, 19, 23, 24], and its effectiveness is demonstrated by a numerical example.

2 Preliminaries

The following notations will be used throughout the rest of this paper. Denote

$$g^{sup} = \sup_{t\in[t_0,\ +\infty)} g(t), g^{inf} = \inf_{t\in[t_0,\ +\infty)} g(t), \sigma_i = \max_{1\le j\le m} \tau_{ij}^{sup}, C_+ = \prod_{i=1}^{n} C([-\sigma_i,\ 0], [0, +\infty)).$$

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define $|x| = (|x_1|, \ldots, |x_n|)$ and $||x|| = \max_{i\in Q} |x_i|$.

Definition 2.1 (see [8, 9]). Let $u(t) : \mathbb{R} \longrightarrow \mathbb{R}^n$ be continuous in t. $u(t)$ is said to be almost periodic on $\mathbb{R}$, if for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : ||u(t+\delta)-u(t)|| < \varepsilon \text{ for all } t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, such that for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $||u(t + \delta) - u(t)|| < \varepsilon, \quad \text{for all } \ t \in \mathbb{R}$.

Hereafter, for $i \in Q, j \in I = \{1, 2, \cdots, m\}$, it will be assumed that $a_{ii},\ b_{ii},\ \gamma_{ij} : \mathbb{R} \to (0, +\infty)$, $a_{ij}(i \neq j)$, $b_{ij}(i \neq j)$, $\beta_{ij}, \tau_{ij} : \mathbb{R} \to [0, +\infty)$ are almost periodic functions, and there exist two positive constants S_- and S^+ such that

$$S_- = \min_{i\in Q}\{\liminf_{t\to+\infty} \ln(\frac{b_{ii}(t) - \sum\limits_{j=1,j\neq i}^{n} b_{ij}(t)}{a_{ii}(t) - \sum\limits_{j=1,j\neq i}^{n} a_{ij}(t)})\},$$

and

$$S^+ = \max_{i\in Q}\{\limsup_{t\to+\infty} \ln(\frac{b_{ii}(t) - \sum\limits_{j=1,j\neq i}^{n} b_{ij}(t)}{a_{ii}(t) - \sum\limits_{j=1,j\neq i}^{n} (a_{ij}(t) + \sum\limits_{j=1}^{m} \frac{1}{e}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)})})\}.$$

Furthermore, it will be considered the following admissible initial conditions:

$$x_i(t_0 + \theta) = \varphi_i(\theta),\ \theta \in [-\sigma_i, 0],\ \varphi = (\varphi_1, \ldots, \varphi_n) \in C_+ \ \text{and} \ \varphi_i(0) > 0, i \in Q. \qquad (2.1)$$

We designate $x(t; t_0, \varphi)$ for a solution of the initial value problem (1.2) and (2.1), and denote the maximal right-interval of existence of $x(t; t_0, \varphi)$ by $[t_0,\ \eta(\varphi))$.

Lemma 2.1 (see [23, Lemma 2.1]) . For any two fixed positive constants ω_1 and ω_2,

$$(e^{-s}-e^{-t})\mathrm{sgn}(s-t)\leq -e^{-\omega_2}|s-t| \ \text{ where } \ s,t\in[\omega_1,\ \omega_2],\omega_1\leq\omega_2,$$

and

$$|se^{-s}-te^{-t}|\leq \max\{\frac{1}{e^2},\frac{1-\omega_1}{e^{\omega_1}}\}|s-t| \ \text{ where } \ s,t\in[\omega_1,\ +\infty).$$

Lemma 2.2. Assume that

$$b_{ii}(t)>a_{ii}(t)-\sum_{j=1,j\neq i}^{n}(a_{ij}(t)-b_{ij}(t)),\ \text{for all } t\in[t_0,\ +\infty),\ i\in Q, \tag{2.2}$$

and

$$\sup_{t\in[t_0,\ +\infty)}\{-a_{ii}(t)\sum_{j=1,j\neq i}^{n}a_{ij}(t)+\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}\frac{1}{e}\}<0,\ i\in Q. \tag{2.3}$$

Then, $x(t)=x(t;t_0,\varphi)$ is positive and bounded on $[t_0,+\infty)$, and

$$0<S_-\leq\liminf_{t\to+\infty}x_i(t)\leq\limsup_{t\to+\infty}x_i(t)\leq S^+,\quad i\in Q. \tag{2.4}$$

Proof. First, we state that

$$x_i(t)>0 \quad \text{for all} \quad t\in[t_0,\ \eta(\varphi)),\ i\in Q. \tag{2.5}$$

Otherwise, we can pick $i_0\in Q$ and $\bar{t}_{i_0}\in(t_0,\ \eta(\varphi))$ to satisfy that

$$x_{i_0}(\bar{t}_{i_0})=0,\quad x_j(t)>0 \quad \text{for all} \quad t\in[t_0,\ \bar{t}_{i_0}),\ j\in Q.$$

Apparently, (1.2) and (2.2) yield

$$\begin{aligned}
0 \ &\geq\ x'_{i_0}(\bar{t}_{i_0})\\
&=\ -a_{i_0i_0}(\bar{t}_{i_0})+b_{i_0i_0}(\bar{t}_{i_0})e^{-x_{i_0}(\bar{t}_{i_0})}+\sum_{j=1,j\neq i_0}^{n}(a_{i_0j}(\bar{t}_{i_0})-b_{i_0j}(\bar{t}_{i_0})e^{-x_j(\bar{t}_{i_0})})\\
&\quad +\sum_{j=1}^{m}\beta_{i_0j}(\bar{t}_{i_0})x_{i_0}(\bar{t}_{i_0}-\tau_{i_0j}(\bar{t}_{i_0}))e^{-\gamma_{i_0j}(\bar{t}_{i_0})x_{i_0}(\bar{t}_{i_0}-\tau_{i_0j}(\bar{t}_{i_0}))}\\
&\geq\ -a_{i_0i_0}(\bar{t}_{i_0})+b_{i_0i_0}(\bar{t}_{i_0})+\sum_{j=1,j\neq i_0}^{n}(a_{i_0j}(\bar{t}_{i_0})-b_{i_0j}(\bar{t}_{i_0}))>0,
\end{aligned}$$

which is a contradiction and results the above statement.

Now, we demonstrate that $x(t)$ is bounded on $[t_0,\eta(\varphi))$. For $t\in[t_0-\sigma_i,\eta(\varphi))$ and $i\in Q$, we define

$$M_i(t)=\max\{\xi:\xi\leq t,x_i(\xi)=\max_{t_0-\sigma_i\leq s\leq t}x_i(s)\}.$$

Suppose that $x(t)$ is unbounded on $[t_0, \eta(\varphi))$. Then, we can choose $i^* \in Q$ and a strictly monotone increasing sequence $\{\zeta_n\}_{n=1}^{+\infty}$ such that

$$x_{i^*}(M_{i^*}(\zeta_n)) = \max_{j\in Q}\{x_j(M_j(\zeta_n))\}, \ \lim_{n\to+\infty} x_{i^*}(M_{i^*}(\zeta_n)) = +\infty, \ \lim_{n\to+\infty} \zeta_n = \eta(\varphi), \qquad (2.6)$$

and then $\lim_{n\to+\infty} M_{i^*}(\zeta_n) = \eta(\varphi)$. According to (1.2) and the fact $\sup_{u\geq 0} ue^{-u} = \frac{1}{e}$, it follows from (2.6) that

$$\begin{aligned}
0 \ \leq \ & x'_{i^*}(M_{i^*}(\zeta_n)) \\
= \ & -a_{i^*i^*}(M_{i^*}(\zeta_n)) + b_{i^*i^*}(M_{i^*}(\zeta_n))e^{-x_{i^*}(M_{i^*}(\zeta_n))} \\
& + \sum_{j=1,j\neq i^*}^{n} (a_{i^*j}(M_{i^*}(\zeta_n)) - b_{i^*j}(M(\zeta_n))e^{-x_j(M_{i^*}(\zeta_n))}) \\
& + \sum_{j=1}^{m} \frac{\beta_{i^*j}(M_{i^*}(\zeta_n))}{\gamma_{i^*j}(M_{i^*}(\zeta_n))}\gamma_{i^*j}(M_{i^*}(\zeta_n))x_{i^*}(M_{i^*}(\zeta_n) - \tau_{i^*j}(M_{i^*}(\zeta_n))) \\
& \times e^{-\gamma_{i^*j}(M_{i^*}(\zeta_n))x_{i^*}(M_{i^*}(\zeta_n)-\tau_{i^*j}(M_{i^*}(\zeta_n)))} \\
\leq \ & -a_{i^*i^*}(M_{i^*}(\zeta_n)) + b_{i^*i^*}(M_{i^*}(\zeta_n))e^{-x_{i^*}(M_{i^*}(\zeta_n))} \\
& + \sum_{j=1,j\neq i^*}^{n} (a_{i^*j}(M_{i^*}(\zeta_n)) - b_{i^*j}(M(\zeta_n))e^{-x_{i^*}(M_{i^*}(\zeta_n))}) \\
& + \sum_{j=1}^{m} \frac{\beta_{i^*j}(M_{i^*}(\zeta_n))}{\gamma_{i^*j}(M_{i^*}(\zeta_n))}\frac{1}{e}, \text{ for all } M_{i^*}(\zeta_n) > t_0.
\end{aligned}$$

Taking $n \to +\infty$ leads to

$$\begin{aligned}
0 \ \leq \ & \lim_{n\to+\infty}[-a_{i^*i^*}(M_{i^*}(\zeta_n)) + \sum_{j=1,j\neq i^*}^{n} a_{i^*j}(M_{i^*}(\zeta_n)) + \sum_{j=1}^{m} \frac{\beta_{i^*j}(M_{i^*}(\zeta_n))}{\gamma_{i^*j}(M_{i^*}(\zeta_n))}\frac{1}{e}] \\
\leq \ & \sup_{t\in[t_0, \ +\infty)}[-a_{i^*i^*}(t) + \sum_{j=1,j\neq i^*}^{n} a_{i^*j}(t) + \sum_{j=1}^{m} \frac{\beta_{i^*j}(t)}{\gamma_{i^*j}(t)}\frac{1}{e}] < 0,
\end{aligned}$$

which is absurd and suggests that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. By [Theorem 2.3.1, 26], we easily show $\eta(\varphi) = +\infty$.

Next, we validate that (2.4) is true. Designate $i^l, i^L \in Q$ such that

$$l = \liminf_{t\to+\infty} x_{i^l}(t) = \min_{i\in Q}\liminf_{t\to+\infty} x_i(t), \ L = \limsup_{t\to+\infty} x_{i^L}(t) = \max_{i\in Q}\limsup_{t\to+\infty} x_i(t).$$

By the fluctuation By lemma [27, Lemma A.1.], we can select two sequences $\{t_k^*\}_{k=1}^{+\infty}$ and $\{t_k^{**}\}_{k=1}^{+\infty}$ satisfying

$$\lim_{k\to+\infty} t_k^* = +\infty, \quad \lim_{k\to+\infty} x_{i^l}(t_k^*) = l = \liminf_{t\to+\infty} x_{i^l}(t), \quad \text{and} \quad \lim_{k\to+\infty} x'_{i^l}(t_k^*) = 0, \qquad (2.7)$$

and

$$\lim_{k\to+\infty} t_k^{**} = +\infty, \quad \lim_{k\to+\infty} x_{i^L}(t_k^{**}) = L = \limsup_{t\to+\infty} x_{i^L}(t), \quad \text{and} \quad \lim_{k\to+\infty} x'_{i^L}(t_k^{**}) = 0, \qquad (2.8)$$

respectively. From the almost periodicity of (1.2), we can select a subsequence of $\{k\}_{k\geq 1}$, still denoted by $\{k\}_{k\geq 1}$, such that $\lim\limits_{k\to+\infty} a_{i^l j}(t_k^*)$, $\lim\limits_{k\to+\infty} b_{i^l j}(t_k^*)$, $\lim\limits_{k\to+\infty} \beta_{i^l q}(t_k^*)$, $\lim\limits_{k\to+\infty} \gamma_{i^l q}(t_k^*)$, $\lim\limits_{k\to+\infty} x_j(t_k^*)$, $\lim\limits_{k\to+\infty} x_{i^l}(t_k^* - \tau_{i^l q}(t_k^*))$, $\lim\limits_{k\to+\infty} a_{i^L j}(t_k^{**})$, $\lim\limits_{k\to+\infty} b_{i^L j}(t_k^{**})$, $\lim\limits_{k\to+\infty} \beta_{i^L q}(t_k^{**})$, $\lim\limits_{k\to+\infty} \gamma_{i^L q}(t_k^{**})$, $\lim\limits_{k\to+\infty} x_j(t_k^{**})$ and $\lim\limits_{k\to+\infty} x_{i^L}(t_k^{**} - \tau_{i^L q}(t_k^{**}))$ exist for all $j \in Q, q \in I$. Furthermore, by taking limits, we have from (2.7) and (2.8) that

$$\begin{aligned} 0 &= \lim_{k\to+\infty} x'_{i^l}(t_k^*) \\ &\geq -\lim_{k\to+\infty} a_{i^l i^l}(t_k^*) + \lim_{k\to+\infty} b_{i^l i^l}(t_k^*) e^{-l} + \sum_{j=1, j\neq i^l}^{n} (\lim_{k\to+\infty} a_{i^l j}(t_k^*) - \lim_{k\to+\infty} b_{i^l j}(t_k^*) e^{-l}), \end{aligned}$$

and

$$\begin{aligned} 0 &= \lim_{k\to+\infty} x'_{i^L}(t_k^{**}) \\ &= -\lim_{k\to+\infty} a_{i^L i^L}(t_k^{**}) + \lim_{k\to+\infty} b_{i^L i^L}(t_k^{**}) e^{-L} \\ &\quad + \sum_{j=1, j\neq i^L}^{n} (\lim_{k\to+\infty} a_{i^L j}(t_k^{**}) - \lim_{k\to+\infty} b_{i^L j}(t_k^{**}) e^{-\lim\limits_{k\to+\infty} x_j(t_k^{**})}) \\ &\quad + \sum_{j=1}^{m} \lim_{k\to+\infty} \frac{\beta_{i^L j}(t_k^{**})}{\gamma_{i^L j}(t_k^{**})} \lim_{k\to+\infty} \gamma_{i^L j}(t_k^{**}) x_{i^L}(t_k^{**} - \tau_{i^L j}(t_k^{**})) e^{-\lim\limits_{k\to+\infty} \gamma_{i^L j}(t_k^{**}) \lim\limits_{k\to+\infty} x_{i^L}(t_k^{**} - \tau_{i^L j}(t_k^{**}))} \\ &\leq -\lim_{k\to+\infty} a_{i^L i^L}(t_k^{**}) + \lim_{k\to+\infty} b_{i^L i^L}(t_k^{**}) e^{-L} + \sum_{j=1, j\neq i^L}^{n} (\lim_{k\to+\infty} a_{i^L j}(t_k^{**}) - \lim_{k\to+\infty} b_{i^L j}(t_k^{**}) e^{-L}) \\ &\quad + \sum_{j=1}^{m} \lim_{k\to+\infty} \frac{\beta_{i^L j}(t_k^{**})}{\gamma_{i^L j}(t_k^{**})} \frac{1}{e}, \end{aligned}$$

which entail that

$$S_- \leq \liminf_{t\to+\infty} \ln\left(\frac{b_{i^l i^l}(t) - \sum\limits_{j=1, j\neq i^l}^{n} b_{i^l j}(t)}{a_{i^l i^l}(t) - \sum\limits_{j=1, j\neq i^l}^{n} a_{i^l j}(t)}\right) \leq \liminf_{t\to+\infty} x_{i^l}(t) \leq \liminf_{t\to+\infty} x_i(t)$$

and

$$\limsup_{t\to+\infty} x_i(t) \leq \limsup_{t\to+\infty} x_{i^L}(t) \leq \limsup_{t\to+\infty} \ln\left(\frac{b_{i^L i^L}(t) - \sum\limits_{j=1, j\neq i^L}^{n} b_{i^L j}(t)}{a_{i^L i^L}(t) - \sum\limits_{j=1, j\neq i^L}^{n} (a_{i^L j}(t) - \sum\limits_{j=1}^{m} \frac{1}{e} \frac{\beta_{i^L j}(t)}{\gamma_{i^L j}(t)})}\right) \leq S^+,$$

for all $i \in Q$. This ends the proof of Lemma 2.2.

Lemma 2.3. Assume that (2.2), (2.3) and

$$\lim_{t\to+\infty}\sup\{-b_{ii}(t)e^{-S^+}+\sum_{j=1,j\neq i}^{n} b_{ij}(t)e^{-S_-}+\sum_{j=1}^{m}\beta_{ij}(t)\max\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{inf}S_-}{e^{\gamma_{ij}^{inf}S_-}}\}\}<0, i\in Q, \quad (2.9)$$

hold. Moreover, suppose that $x(t)=x(t;t_0,\varphi)$. Then for any $\varepsilon>0$, there exists $l=l(\varepsilon)>0$, such that each interval $[\alpha,\ \alpha+l]$ includes at least one number δ for which there exists $\hat{\Lambda}>0$ satisfies

$$\|x(t+\delta)-x(t)\|\leq\varepsilon,\quad \text{for all}\quad t>\hat{\Lambda}. \quad (2.10)$$

Proof. According to (2.9), for all $i\in Q$ it is easy to see that there exists $t_0^*\geq t_0$ such that

$$\sup_{t\geq t_0^*}\Big\{-b_{ii}(t)e^{-S^+}+\sum_{j=1,j\neq i}^{n} b_{ij}(t)e^{-S_-}+\sum_{j=1}^{m}\beta_{ij}(t)\max\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{inf}S_-}{e^{\gamma_{ij}^{inf}S_-}}\Big\}<0. \quad (2.11)$$

Set

$$\begin{aligned} H_i(u,v) &= \sup_{t\geq t_0^*}\{-[b_{ii}(t)e^{-(S^++v)}-u]+\sum_{j=1,j\neq i}^{n} b_{ij}(t)e^{-(S_--v)} \\ &\quad +\sum_{j=1}^{m}\beta_{ij}(t)\max\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{inf}(S_--v)}{e^{\gamma_{ij}^{inf}(S_--v)}}\}e^{u\sigma_i}\}, u,v\in[0,\ 1],\ i\in Q. \end{aligned}$$

Furthermore, let $B=\frac{1}{2}\min_{i\in Q}|H_i(0,0)|$, then $B<0$. According to the continuity of $H_i(u,v)$, one can pick a sufficiently small constant $0<\eta<1$ such that

$$H_i(u,v)<-B \text{ for all } (u,v)\in[0,\ \eta]\times[0,\ \eta],\ i\in Q,$$

and then fixed $\lambda\in[0,\ \eta]$, we have

$$\begin{aligned} H_i(\lambda,\varepsilon) &= \sup_{t\geq t_0^*}\{-[b_{ii}(t)e^{-(S^++\varepsilon)}-\lambda]+\sum_{j=1,j\neq i}^{n} b_{ij}(t)e^{-(S_--\varepsilon)} \\ &\quad +\sum_{j=1}^{m}\beta_{ij}(t)\max\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{inf}(S_--\varepsilon)}{e^{\gamma_{ij}^{inf}(S_--\varepsilon)}}\}e^{\lambda\sigma_i}\}<0,\ \text{for all } \varepsilon\in(0,\eta],\ i\in Q, \end{aligned} \quad (2.12)$$

and

$$\max_{i\in Q}\{\sup_{\varepsilon\in[0,\eta]} H_i(\lambda,\varepsilon)\}=-B<0.$$

Without loss of generality, to prove Lemma 2.3, we only need to show that (2.10) holds for $\varepsilon \in (0, \min\{\eta, S_-\})$. For $i \in Q$, $t \in (-\infty, t_0 - \sigma_i]$, we add the definition of $x_i(t)$ with $x_i(t) \equiv x_i(t_0 - \sigma_i)$. Set

$$\begin{aligned} A_i(\delta, t) \;=\;& [b_{ii}(t+\delta) - b_{ii}(t)]e^{-x_i(t+\delta)} - \sum_{j=1, j\neq i}^{n} [b_{ij}(t+\delta) - b_{ij}(t)]e^{-x_j(t+\delta)} \\ &+ \sum_{j=1}^{m} [\beta_{ij}(t+\delta) - \beta_{ij}(t)] x_i(t+\delta - \tau_{ij}(t+\delta)) e^{-\gamma_{ij}(t+\delta)x_i(t+\delta-\tau_{ij}(t+\delta))} \\ &+ \sum_{j=1}^{m} \beta_{ij}(t)[x_i(t+\delta - \tau_{ij}(t+\delta)) e^{-\gamma_{ij}(t+\delta)x_i(t+\delta-\tau_{ij}(t+\delta))} \\ &- x_i(t - \tau_{ij}(t) + \delta) e^{-\gamma_{ij}(t+\delta)x_i(t-\tau_{ij}(t)+\delta)}] \\ &+ \sum_{j=1}^{m} \beta_{ij}(t)[x_i(t - \tau_{ij}(t) + \delta) e^{-\gamma_{ij}(t+\delta)x_i(t-\tau_{ij}(t)+\delta)} \\ &- x_i(t - \tau_{ij}(t) + \delta) e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t)+\delta)}] - [a_{ii}(t+\delta) - a_{ii}(t)] \\ &+ \sum_{j=1, j\neq i}^{n} [a_{ij}(t+\delta) - a_{ij}(t)], \qquad t \in \mathbb{R}. \end{aligned} \tag{2.13}$$

For any $\varepsilon \in (0, \min\{\eta, S_-\})$, it follows from Lemma 2.1 that there exists $T_\varphi > t_0^*$ such that

$$S_- - \varepsilon < x_i(t) < S^+ + \varepsilon, \quad \text{for all} \quad t \in [T_\varphi - \sigma_i, \ +\infty), \ i \in Q, \tag{2.14}$$

which implies that the right side of (1.1) is also bounded, and $x_i'(t)$ is a bounded function on $[t_0, +\infty)$. Thus, with the help of the fact that $x_i(t) \equiv x_i(t_0 - \sigma_i)$ for $t \in (-\infty, \ t_0 - \sigma_i]$, we gain that $x_i(t)$ is uniformly continuous on $\mathbb{R}$. From uniformly almost periodic family theory in [[8], Corollary 2.3, p. 19], for each $\varepsilon \in (0, \min\{\eta, S_-\})$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l] \subseteq \mathbb{R}$, includes a δ for which

$$|A_i(\delta, t)| \le \frac{1}{2} B\varepsilon, \quad \text{for all} \quad t \in \mathbb{R}, \ i \in Q. \tag{2.15}$$

Let $\Lambda_0 \ge \max\{T_\varphi + \max_{i\in Q} \sigma_i, \ T_\varphi + \max_{i\in Q} \sigma_i - \delta\}$. For $t \in \mathbb{R}$, denote

$$u(t) = (u_1(t), u_2(t), \cdots, u_n(t)), \ u_i(t) = x_i(t+\delta) - x_i(t),$$

and

$$U(t) = (U_1(t), U_2(t), \cdots, U_n(t)), \quad U_i(t) = e^{\lambda t} u_i(t),$$

where $i \in Q$. Let i_t be such an index that

$$|U_{i_t}(t)| = ||U(t)||. \tag{2.16}$$

Then, for all $t \geq \Lambda_0$, we have

$$
\begin{aligned}
u_i'(t) &= b_{ii}(t)[e^{-x_i(t+\delta)} - e^{-x_i(t)}] - \sum_{j=1, j\neq i}^{n} b_{ij}(t)[e^{-x_j(t+\delta)} - e^{-x_j(t)}] \\
&\quad + \sum_{j=1}^{m} \beta_{ij}(t)[x_i(t-\tau_{ij}(t)+\delta)e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t)+\delta)} \\
&\quad - x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))}] + A_i(\delta, t). \qquad (2.17)
\end{aligned}
$$

With the help of Lemma 2.1, one can show the following inequalities:

$$\gamma_{ij}^{inf}(S_- - \varepsilon) \leq \gamma_{ij}(t)x(t-\tau_{ij}(t)),\ \gamma_{ij}(t)x(t-\tau_{ij}(t)+\delta),\ i \in Q, j \in I,\ t \geq \Lambda_0,$$

$$S_- - \varepsilon \leq x_i(t),\ i \in Q, t \geq \Lambda_0,$$

$$(e^{-s} - e^{-t})\text{sgn}(s-t) \leq -e^{-(S^+ + \varepsilon)}|s-t|,\ |e^{-s} - e^{-t}| \leq e^{-(S_- - \varepsilon)}|s-t| \text{ where } s, t \in [S_- - \varepsilon, S^+ + \varepsilon],$$

and

$$|se^{-s} - te^{-t}| \leq \max\{\frac{1}{e^2}, \frac{1-\gamma_{ij}^{inf}(S_- - \varepsilon)}{e^{\gamma_{ij}^{inf}(S_- - \varepsilon)}}\}|s-t| \text{ where } s, t \in [\gamma_{ij}^{inf}(S_- - \varepsilon), +\infty), i \in Q, j \in I.$$

This, together with (2.14), (2.16) and (2.17), follows that we get

$$
\begin{aligned}
& D^-(|U_{i_s}(s)|)|_{s=t} \\
\leq\ & \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\{b_{i_t i_t}(t)[e^{-x_{i_t}(t+\delta)} - e^{-x_{i_t}(t)}]\text{sgn}(x_{i_t}(t+\delta) - x_{i_t}(t)) \\
& + \sum_{j=1, j\neq i_t}^{n} b_{i_t j}(t)|e^{-x_j(t+\delta)} - e^{-x_j(t)}| + \sum_{j=1}^{m} \beta_{i_t j}(t) \\
& \times |x_{i_t}(t-\tau_{i_t j}(t)+\delta)e^{-\gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t)+\delta)} - x_{i_t}(t-\tau_{i_t j}(t))e^{-\gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t))}| \\
& + |A_{i_t}(\delta, t)|\} \\
=\ & \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\{b_{i_t i_t}(t)[e^{-x_{i_t}(t+\delta)} - e^{-x_{i_t}(t)}]\text{sgn}(x_{i_t}(t+\delta) - x_{i_t}(t)) \\
& + \sum_{j=1, j\neq i_t}^{n} b_{i_t j}(t)|e^{-x_j(t+\delta)} - e^{-x_j(t)}| + \sum_{j=1}^{m} \frac{\beta_{i_t j}(t)}{\gamma_{i_t j}(t)} \times \\
& |\gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t)+\delta)e^{-\gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t)+\delta)} \\
& - \gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t))e^{-\gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t))}| + |A_{i_t}(\delta, t)|\} \\
\leq\ & \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\{-b_{i_t i_t}(t)e^{-(S^+ + \varepsilon)}|u_{i_t}(t)| + \sum_{j=1, j\neq i_t}^{n} b_{i_t j}(t)e^{-(S_- - \varepsilon)}|u_j(t)| \\
& + \sum_{j=1}^{m} \beta_{i_t j}(t)\max\{\frac{1}{e^2}, \frac{1-\gamma_{i_t j}^{inf}(S_- - \varepsilon)}{e^{\gamma_{i_t j}^{inf}(S_- - \varepsilon)}}\}|u_{i_t}(t-\tau_{i_t j}(t))| + |A_{i_t}(\delta, t)|\}
\end{aligned}
$$

$$
\begin{aligned}
= \; & -[b_{i_t i_t}(t)e^{-(S^+ +\varepsilon)} - \lambda]|U_{i_t}(t)| + \sum_{j=1, j\neq i_t}^{n} b_{i_t j}(t)e^{-(S_- -\varepsilon)}|U_j(t)| \\
& + \sum_{j=1}^{m} \beta_{i_t j}(t) \max\{\frac{1}{e^2}, \frac{1-\gamma_{i_t j}^{inf}(S_- -\varepsilon)}{e^{\gamma_{i_t j}^{inf}(S_- -\varepsilon)}}\} e^{\lambda \tau_{i_t j}(t)} |U_{i_t}(t-\tau_{i_t j}(t))| \\
& + e^{\lambda t}|A_{i_t}(\delta, t)| \quad \text{for all} \quad t \geq \Lambda_0. \qquad (2.18)
\end{aligned}
$$

Let

$$E(t) = \sup_{-\infty < s \leq t} \{e^{\lambda s}\|u(s)\|\}.$$

It is obvious that $e^{\lambda t}\|u(t)\| \leq E(t)$, and $E(t)$ is non-decreasing.

Now, the remaining proof will be divided into two steps.

Step one. If $E(t) > e^{\lambda t}\|u(t)\|$ for all $t \geq \Lambda_0$, we assert that

$$E(t) \equiv \|U(\Lambda_0)\|, \quad \text{for all} \quad t \geq \Lambda_0. \qquad (2.19)$$

In the contrary case, one can pick $\Lambda_1 > \Lambda_0$ such that $E(\Lambda_1) > E(\Lambda_0)$. Because

$$e^{\lambda t}\|u(t)\| \leq E(\Lambda_0) \quad \text{for all} \quad t \leq \Lambda_0,$$

there must exist $\beta^* \in (\Lambda_0, \; \Lambda_1)$ such that

$$e^{\lambda \beta^*}\|u(\beta^*)\| = E(\Lambda_1) \geq E(\beta^*),$$

which contradicts the fact that $E(\beta^*) > e^{\lambda \beta^*}\|u(\beta^*)\|$ and proves the above assertion. Then, we can select $\Lambda_2 > \Lambda_0$ satisfying

$$\|u(t)\| \leq e^{-\lambda t}E(t) = e^{-\lambda t}E(\Lambda_0) < \frac{\varepsilon}{2} \quad \text{for all} \quad t \geq \Lambda_2. \qquad (2.20)$$

Step two. If there exists $\varsigma \geq \Lambda_0$ such that $E(\varsigma) = e^{\lambda \varsigma}\|u(\varsigma)\|$, we can have from (2.18) and the definition of $E(t)$ that

$$
\begin{aligned}
0 \; \leq \; & D^-(|U_{i_s}(s)|)|_{s=\varsigma} \\
\leq \; & -[b_{i_\varsigma i_\varsigma}(\varsigma)e^{-(S^+ +\varepsilon)} - \lambda]|U_{i_\varsigma}(\varsigma)| + \sum_{j=1, j\neq i_\varsigma}^{n} b_{i_\varsigma j}(\varsigma)e^{-(S_- -\varepsilon)}|U_j(\varsigma)| \\
& + \sum_{j=1}^{m} \beta_{i_\varsigma j}(\varsigma) \max\{\frac{1}{e^2}, \frac{1-\gamma_{i_\varsigma j}^{inf}(S_- -\varepsilon)}{e^{\gamma_{i_\varsigma j}^{inf}(S_- -\varepsilon)}}\} e^{\lambda \tau_{i_\varsigma j}(\varsigma)} |U_{i_\varsigma}(\varsigma-\tau_{i_\varsigma j}(\varsigma))| + e^{\lambda\varsigma}|A_{i_\varsigma}(\delta, \varsigma)| \\
\leq \; & \{-[b_{i_\varsigma i_\varsigma}(\varsigma)e^{-(S^+ +\varepsilon)} - \lambda] + \sum_{j=1, j\neq i_\varsigma}^{n} b_{i_\varsigma j}(\varsigma)e^{-(S_- -\varepsilon)} \\
& + \sum_{j=1}^{m} \beta_{i_\varsigma j}(\varsigma) \max\{\frac{1}{e^2}, \frac{1-\gamma_{i_\varsigma j}^{inf}(S_- -\varepsilon)}{e^{\gamma_{i_\varsigma j}^{inf}(S_- -\varepsilon)}}\} e^{\lambda \tau_{i_\varsigma j}(\varsigma)}\} E(\varsigma) + \frac{1}{2}B\varepsilon e^{\lambda\varsigma} \\
< \; & -BE(\varsigma) + \frac{1}{2}B\varepsilon e^{\lambda\varsigma}, \qquad (2.21)
\end{aligned}
$$

which leads to

$$e^{\lambda\varsigma}\|u(\varsigma)\| = E(\varsigma) < \frac{\varepsilon}{2}e^{\lambda\varsigma}, \quad \text{and} \quad \|u(\varsigma)\| < \frac{\varepsilon}{2}. \tag{2.22}$$

For any $t > \varsigma$ satisfying $E(t) = e^{\lambda t}\|u(t)\|$, by the same method as that in the derivation of (2.22), we can show

$$e^{\lambda t}\|u(t)\| < \frac{\varepsilon}{2}e^{\lambda t}, \quad \text{and} \quad \|u(t)\| < \frac{\varepsilon}{2}. \tag{2.23}$$

Furthermore, if $E(t) > e^{\lambda t}\|u(t)\|$ and $t > \varsigma$, one can pick $\Lambda_3 \in [\varsigma,\ t)$ such that

$$E(\Lambda_3) = e^{\lambda\Lambda_3}\ |u(\Lambda_3)\| \quad \text{and} \quad E(s) > e^{\lambda s}\|u(s)\| \quad \text{for all} \quad s \in (\Lambda_3,\ t],$$

which, together with (2.22) and (2.23), suggests that

$$\|u(\Lambda_3)\| < \frac{\varepsilon}{2}. \tag{2.24}$$

With a similar reasoning as that in the proof of Step one, we can entail that

$$E(s) \equiv E(\Lambda_3) \quad \text{is a constant for all} \quad s \in (\Lambda_3,\ t],$$

which, together with (2.24), follows that

$$\|u(t)\| < e^{-\lambda t}E(t) = e^{-\lambda t}E(\Lambda_3) = \|u(\Lambda_3)\|e^{-\lambda(t-\Lambda_3)} < \frac{\varepsilon}{2}.$$

Finally, the above discussion infers that there exists $\hat{\Lambda} > \max\{\varsigma,\ \Lambda_0,\ \Lambda_2\}$ obeying that

$$\|u(t)\| \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{for all} \quad t > \hat{\Lambda}, \tag{2.25}$$

which finishes the proof of Lemma 2.3.

3 Global exponential stability of almost periodic solutions

Combining Lemma 2.1 with Lemma 2.2, we have the following theorem:

Theorem 3.1. Assume that all assumptions of Lemma 2.3 are satisfied. Then, (1.2) has a globally exponentially stable positive almost periodic solution $x^*(t)$. Moreover, there exist constants K_{φ,x^*} and t_{φ,x^*} such that

$$\|x(t;t_0,\varphi) - x^*(t)\| < K_{\varphi,x^*}e^{-\lambda t} \quad \text{for all} \quad t > t_{\varphi,x^*}.$$

Proof. Let $v(t) = v(t; t_0, \varphi^v)$ be a solution of equation (1.2) with initial conditions satisfying the assumptions in Lemma 2.3. We also denote $v_i(t) \equiv v_i(t_0 - \sigma_i)$ for all $t \in (-\infty, t_0 - \sigma_i]$, $i \in Q$. Define

$$\begin{aligned}
&\Pi_{i,k}(t)\\
= \; & [b_{ii}(t+t_k) - b_{ii}(t)]e^{-v_i(t+t_k)} - \sum_{j=1, j\neq i}^{n} [b_{ij}(t+t_k) - b_{ij}(t)]e^{-v_j(t+t_k)}\\
& + \sum_{j=1}^{m} [\beta_{ij}(t+t_k) - \beta_{ij}(t)] v_i(t+t_k - \tau_{ij}(t+t_k)) e^{-\gamma_{ij}(t+t_k)v_i(t+t_k-\tau_{ij}(t+t_k))}\\
& + \sum_{j=1}^{m} \beta_{ij}(t)[v_i(t+t_k-\tau_{ij}(t+t_k))e^{-\gamma_{ij}(t+t_k)v_i(t+t_k-\tau_{ij}(t+t_k))}\\
& - v_i(t-\tau_{ij}(t)+t_k)e^{-\gamma_{ij}(t+t_k)v_i(t-\tau_{ij}(t)+t_k)}]\\
& + \sum_{j=1}^{m} \beta_{ij}(t)[v_i(t-\tau_{ij}(t)+t_k)e^{-\gamma_{ij}(t+t_k)v_i(t-\tau_{ij}(t)+t_k)}\\
& - v_i(t-\tau_{ij}(t)+t_k)e^{-\gamma_{ij}(t)v_i(t-\tau_{ij}(t)+t_k)}] - [a_{ii}(t+t_k) - a_{ii}(t)]\\
& + \sum_{j=1, j\neq i}^{n} [a_{ij}(t+t_k) - a_{ij}(t)], \quad t \in \mathbb{R}, \; i \in Q. \qquad (3.1)
\end{aligned}$$

where $\{t_k\}$ is any sequence of real numbers. For any $\varepsilon \in (0, \min\{\eta, S_-\})$, by Lemma 2.2, we can choose $t_{\varphi^v} > t_0$ such that

$$S_- - \varepsilon < v_i(t) < S^+ + \varepsilon, \quad \text{for all} \quad t \geq t_{\varphi^v}, \; i \in Q, \qquad (3.2)$$

which, together with the boundedness of $v_i'(t)$ and the fact that $v_i(t) \equiv v_i(t_0 - \sigma_i)$ for $t \in (-\infty, t_0 - \sigma_i]$, entails that $v(t)$ is uniformly continuous on $\mathbb{R}$. Then, from the almost periodicity of $a_{ij}, b_{ij}, \tau_{ij}, \gamma_{ij}$ and β_{ij}, we can select a sequence $\{t_k\} \to +\infty$ such that

$$\left.\begin{array}{l}
|a_{ij}(t+t_k) - a_{ij}(t)| \leq \frac{1}{k}, \; |b_{ij}(t+t_k) - b_{ij}(t)| \leq \frac{1}{k}, \; |\tau_{ij}(t+t_k) - \tau_{ij}(t)| \leq \frac{1}{k}\\
|\beta_{ij}(t+t_k) - \beta_{ij}(t)| \leq \frac{1}{k}, \; |\gamma_{ij}(t+t_k) - \gamma_{ij}(t)| \leq \frac{1}{k}, \; |\epsilon(k,t)| \leq \frac{1}{k}
\end{array}\right\}, \qquad (3.3)$$

for all i, j, t.

Since $\{v(t+t_k)\}_{k=1}^{+\infty}$ is uniformly bounded and equiuniformly continuous, from Arzala-Ascoli Lemma and diagonal selection principle, we can select a subsequence $\{t_{k_q}\}$ of $\{t_k\}$, such that $v(t+t_{k_q})$(for convenience, we still designate by $v(t+t_k)$) uniformly converges to a continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))$ on any compact set of $\mathbb{R}$, and

$$S_- - \varepsilon \leq x_i^*(t) \leq S^+ + \varepsilon, \quad \text{for all} \quad t \in \mathbb{R}, \; i \in Q. \qquad (3.4)$$

Now, we prove that $x^*(t)$ is a solution of (1.2). In fact, for any $t \geq t_0$ and $\Delta t \in \mathbb{R}$, from (3.3), we have

$$\begin{aligned}
& x_i^*(t+\Delta t) - x_i^*(t) \\
= & \lim_{k\to+\infty} [v_i(t+\Delta t+t_k) - v_i(t+t_k)] \\
= & \lim_{k\to+\infty} \int_t^{t+\Delta t} [-a_{ii}(s) + b_{ii}(s)e^{-v_i(s+t_k)} + \sum_{j=1,j\neq i}^{n} (a_{ij}(s) - b_{ij}(s)e^{-v_j(s+t_k)}) \\
& + \sum_{j=1}^{m} \beta_{ij}(s) v_i(s+t_k-\tau_{ij}(s)) e^{-\gamma_{ij}(s)v_i(s+t_k-\tau_{ij}(s))} + \Pi_{i,k}(s)]ds \\
= & \int_t^{t+\Delta t} [-a_{ii}(s) + b_{ii}(s)e^{-x_i^*(s)} + \sum_{j=1,j\neq i}^{n} (a_{ij}(s) - b_{ij}(s)e^{-x_j^*(s)}) \\
& + \sum_{j=1}^{m} \beta_{ij}(s) x_i^*(s-\tau_{ij}(s)) e^{-\gamma_{ij}(s)x_i^*(s-\tau_{ij}(s))}]ds, \quad \text{where } t, \ t+\Delta t \geq t_0, \ i \in Q. \qquad (3.5)
\end{aligned}$$

Consequently, (3.5) suggests that

$$\begin{aligned}
\frac{d}{dt}\{x_i^*(t)\} \ = \ & -a_{ii}(t) + b_{ii}(t)e^{-x_i^*(t)} + \sum_{j=1,j\neq i}^{n} (a_{ij}(t) - b_{ij}(t)e^{-x_j^*(t)}) \\
+ & \sum_{j=1}^{m} \beta_{ij}(t) x_i^*(t-\tau_{ij}(t)) e^{-\gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))}, \ i \in Q. \qquad (3.6)
\end{aligned}$$

Hence, $x^*(t)$ is a solution of (1.2).

Furthermore, from Lemma 2.3 and (2.25), for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $\hat{\Lambda} > 0$ obeying

$$||v(t+\delta) - v(t)|| \leq \frac{\varepsilon}{2} < \varepsilon, \quad \text{for all} \quad t > \hat{\Lambda}. \qquad (3.7)$$

Given $s \in \mathbb{R}$, one can pick a sufficiently large positive integer $N_1 > \hat{\Lambda}$ such that, for any $k > N_1$,

$$s + t_k > \hat{\Lambda}, \ ||v(s+t_k+\delta) - v(s+t_k)|| \leq \frac{\varepsilon}{2} < \varepsilon. \qquad (3.8)$$

Letting $k \to +\infty$ gives us

$$||x^*(s+\delta) - x^*(s)|| < \varepsilon,$$

which suggests that $x^*(t)$ is a positive almost periodic solution of (1.2).

Next, we validate that $x^*(t)$ is globally exponentially stable. Let $x(t) = x(t; t_0, \varphi)$, and

$$z_i(t) = x_i(t) - x_i^*(t), \quad W_i(t) = |z_i(t)|e^{\lambda t} \quad \text{for all} \quad t \in [t_0 - \sigma_i, +\infty). \qquad (3.9)$$

Clearly,

$$z_i'(t) = b_{ii}(t)[e^{-x_i(t)} - e^{-x_i^*(t)}] - \sum_{j=1,j\neq i}^{n} b_{ij}(t)[e^{-x_j(t)} - e^{-x_j^*(t)}]$$
$$+\sum_{j=1}^{m} \beta_{ij}(t)[x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))} - x_i^*(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))}]. \quad (3.10)$$

For any $\varepsilon \in (0, \min\{\eta, S_-\})$, it follows from Lemma 2.3 that there exists $T_{\varphi,x^*} > t_0$ such that

$$S_- - \varepsilon \leq x_i(t) \leq S^+ + \varepsilon, \quad \text{for all} \quad t \in [T_{\varphi,x^*} - \sigma_i,\ +\infty),\ i \in Q. \quad (3.11)$$

By (3.10) and calculating the upper-right Dini derivative of $W_i(t)$, we obtain

$$D^-(W_i(t))$$
$$\leq b_{ii}(t)[e^{-x_i(t)} - e^{-x_i^*(t)}]\mathrm{sgn}(x_i(t) - x_i^*(t))e^{\lambda t} + \sum_{j=1,j\neq i}^{n} b_{ij}(t)|e^{-x_j(t)} - e^{-x_j^*(t)}|e^{\lambda t}$$
$$+\sum_{j=1}^{m} \beta_{ij}(t)|x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))} - x_i^*(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))}|e^{\lambda t}$$
$$+\lambda|z_i(t)|e^{\lambda t}, \text{ for all } t > T_{\varphi,\varphi^*},\ i \in Q. \quad (3.12)$$

Now, we assert that

$$W_i(t) < e^{\lambda T_{\varphi,x^*}}(\max_{j\in Q}\{\max_{t\in[t_0-\sigma_j,\ T_{\varphi,x^*}]}|x_j(t) - x_j^*(t)|\} + 1) := M_{\varphi,x^*} \quad \text{for all} \quad t > T_{\varphi,\varphi^*},\ i \in Q.$$

Otherwise, we can choose $\bar{i} \in Q$ and $T_*^{\bar{i}} > T_{\varphi,x^*}$ such that

$$W_{\bar{i}}(T_*^{\bar{i}}) = M_{\varphi,x^*} \quad \text{and} \quad W_j(t) < M_{\varphi,x^*} \quad \text{for all} \quad t \in [t_0 - \sigma_j,\ T_*^{\bar{i}}),\ j \in Q. \quad (3.13)$$

With the help of (3.4), (3.11) and Lemma 2.1, one can show the following inequalities:

$$\gamma_{\bar{i}j}^{inf}(S_- - \varepsilon) \leq \gamma_{\bar{i}j}(T_*^{\bar{i}})x(T_*^{\bar{i}} - \tau_{\bar{i}j}(T_*^{\bar{i}})),\ \gamma_{\bar{i}j}(T_*^{\bar{i}})x_{\bar{i}}^*(T_*^{\bar{i}} - \tau_{\bar{i}j}(T_*^{\bar{i}})),\ j \in I,$$

$(e^{-s}-e^{-t})\mathrm{sgn}(s-t) \leq -e^{-(S^+ +\varepsilon)}|s-t|,\ |e^{-s}-e^{-t}| \leq e^{-(S_- -\varepsilon)}|s-t|$ where $s,t \in [S_- -\varepsilon, S^+ +\varepsilon]$,

and

$$|se^{-s} - te^{-t}| \leq \max\{\frac{1}{e^2}, \frac{1-\gamma_{\bar{i}j}^{inf}(S_- - \varepsilon)}{e^{\gamma_{\bar{i}j}^{inf}(S_- -\varepsilon)}}\}|s-t| \quad \text{where} \quad s,t \in [\gamma_{\bar{i}j}^{inf}(S_- - \varepsilon),\ +\infty), j \in I.$$

This, together with (3.12) and (3.13), follows that

$$\begin{aligned}
0 &\le D^-(W_{\bar i}(T_*^{\bar i})) \\
&\le b_{\bar i\,\bar i}(T_*^{\bar i})[e^{-x_{\bar i}(T_*^{\bar i})} - e^{-x^*_{\bar i}(T_*^{\bar i})}]\mathrm{sgn}(x_{\bar i}(T_*^{\bar i}) - x^*_{\bar i}(T_*^{\bar i}))e^{\lambda T_*^{\bar i}} \\
&\quad + \sum_{j=1, j\neq \bar i}^{n} b_{\bar i j}(T_*^{\bar i})|e^{-x_j(T_*^{\bar i})} - e^{-x^*_j(T_*^{\bar i})}|e^{\lambda T_*^{\bar i}} \\
&\quad + \sum_{j=1}^{m} \beta_{\bar i j}(T_*^{\bar i})|x_{\bar i}(T_*^{\bar i} - \tau_{\bar i j}(T_*^{\bar i}))e^{-\gamma_{\bar i j}(T_*^{\bar i})x_{\bar i}(T_*^{\bar i}-\tau_{\bar i j}(T_*^{\bar i}))} \\
&\quad - x^*_{\bar i}(T_*^{\bar i} - \tau_{\bar i j}(T_*^{\bar i}))e^{-\gamma_{\bar i j}(T_*^{\bar i})x^*_{\bar i}(T_*^{\bar i}-\tau_{\bar i j}(T_*^{\bar i}))}|e^{\lambda T_*^{\bar i}} + \lambda|z_{\bar i}(T_*^{\bar i})|e^{\lambda T_*^{\bar i}} \\
&\le -[b_{\bar i\,\bar i}(T_*^{\bar i})e^{-(S^+ + \varepsilon)} - \lambda]|z_{\bar i}(T_*^{\bar i})|e^{\lambda T_*^{\bar i}} + \sum_{j=1, j\neq \bar i}^{n} b_{\bar i j}(T_*^{\bar i})e^{-(S_- - \varepsilon)}|z_j(T_*^{\bar i})|e^{\lambda T_*^{\bar i}} \\
&\quad + \sum_{j=1}^{m} \frac{\beta_{\bar i j}(T_*^{\bar i})}{\gamma_{\bar i j}(T_*^{\bar i})}|\gamma_{\bar i j}(T_*^{\bar i})x_{\bar i}(T_*^{\bar i} - \tau_{\bar i j}(T_*^{\bar i}))e^{-\gamma_{\bar i j}(T_*^{\bar i})x_{\bar i}(T_*^{\bar i}-\tau_{\bar i j}(T_*^{\bar i}))} \\
&\quad - \gamma_{\bar i j}(T_*^{\bar i})x^*_{\bar i}(T_*^{\bar i} - \tau_{\bar i j}(T_*^{\bar i}))e^{-\gamma_{\bar i j}(T_*^{i})x^*_{\bar i}(T_*^{\bar i}-\tau_{\bar i j}(T_*^{\bar i}))}|e^{\lambda T_*^{\bar i}} \\
&\le -[b_{\bar i\,\bar i}(T_*^{\bar i})e^{-(S^+ + \varepsilon)} - \lambda]|z_{\bar i}(T_*^{\bar i})|e^{\lambda T_*^{\bar i}} + \sum_{j=1, j\neq \bar i}^{n} b_{\bar i j}(T_*^{\bar i})e^{-(S_- - \varepsilon)}|z_j(T_*^{\bar i})|e^{\lambda T_*^{\bar i}} \\
&\quad + \sum_{j=1}^{m} \beta_{\bar i j}(T_*^{\bar i})\max\{\frac{1}{e^2}, \frac{1-\gamma_{\bar i j}^{inf}(S_- - \varepsilon)}{e^{\gamma_{\bar i j}^{inf}(S_- - \varepsilon)}}\}|z_{\bar i}(T_* - \tau_{\bar i j}(T_*^{\bar i}))|e^{\lambda(T_*^{\bar i} - \tau_{\bar i j}(T_*^{\bar i}))}e^{\lambda \tau_{\bar i j}(T_*^{\bar i})} \\
&\le \{-[b_{\bar i\,\bar i}(T_*^{\bar i})e^{-(S^+ + \varepsilon)} - \lambda] + \sum_{j=1, j\neq \bar i}^{n} b_{\bar i j}(T_*^{\bar i})e^{-(S_- - \varepsilon)} \\
&\quad + \sum_{j=1}^{m} \beta_{\bar i j}(T_*^{\bar i})\max\{\frac{1}{e^2}, \frac{1-\gamma_{\bar i j}^{inf}(S_- - \varepsilon)}{e^{\gamma_{\bar i j}^{inf}(S_- - \varepsilon)}}\}e^{\lambda \sigma_{\bar i}}\}M_{\varphi,\varphi^*},
\end{aligned}$$

which, together with (2.12), derives that

$$\begin{aligned}
0 \le -[b_{\bar i\,\bar i}(T_*^{\bar i})e^{-(S^+ + \varepsilon)} - \lambda] + \sum_{j=1, j\neq \bar i}^{n} b_{\bar i j}(T_*^{\bar i})e^{-(S_- - \varepsilon)} \\
+ \sum_{j=1}^{m} \beta_{\bar i j}(T_*^{\bar i})\max\{\frac{1}{e^2}, \frac{1-\gamma_{\bar i j}^{inf}(S_- - \varepsilon)}{e^{\gamma_{\bar i j}^{inf}(S_- - \varepsilon)}}\}e^{\lambda \sigma_{\bar i}} < 0.
\end{aligned}$$

This is a clear contradiction and proves the above assertion. Hence,

$$|z_i(t)| < M_{\varphi,x^*}e^{-\lambda t} \quad \text{for all} \quad t > T_{\varphi,x^*}, \ i \in Q,$$

which finishes the proof of Theorem 3.1.

4 A numerical example

Example 4.1. Let us consider system (1.2) involving the following parameters:

$$\left.\begin{array}{l} n=m=2, a_{11}(t)=e^{-(2+|\cos\sqrt{2}t|)},\ b_{11}(t)=10.1+10.1\cos^2 t, \\ a_{12}(t)=(0.2+0.2\cos t)e^{-(2+|\cos t|)},\ b_{12}(t)=0.01+0.01\cos^2 t, \\ \beta_{11}(t)=\frac{1+\cos t}{1000}, \beta_{12}(t)=\frac{1+\sin t}{2000},\ \gamma_{11}(t)=\gamma_{12}(t)=0.5, \\ \tau_{11}(t)=2|\sin\sqrt{5}t|, \tau_{12}(t)=3|\sin\sqrt{5}t|, \\ a_{22}(t)=e^{-(2+|\sin\sqrt{3}t|)},\ b_{22}(t)=20.2+20.2\sin^2 t, \\ a_{21}(t)=(0.2+0.2\sin t)e^{-(2+|\sin t|)},\ b_{21}(t)=0.02+0.02\sin^2 t, \\ \gamma_{21}(t)=\gamma_{22}(t)=0.5, \beta_{21}(t)=\frac{1+\cos t}{2000}, \beta_{22}(t)=\frac{1+\sin t}{3000}, \\ \tau_{21}(t)=2|\cos\sqrt{7}t|, \tau_{22}(t)=3|\cos\sqrt{7}t|. \end{array}\right\} \quad (4.1)$$

Obviously, it is observed that

$$S_- = \min_{1\leq i\leq 2}\{\liminf_{t\to+\infty}\ln(\frac{b_{ii}(t)-\sum\limits_{j=1,j\neq i}^{n} b_{ij}(t)}{a_{ii}(t)-\sum\limits_{j=1,j\neq i}^{n} a_{ij}(t)})\}\approx 4.1,$$

$$S^+ = \max_{1\leq i\leq 2}\{\limsup_{t\to+\infty}\ln(\frac{b_{ii}(t)-\sum\limits_{j=1,j\neq i}^{n} b_{ij}(t)}{a_{ii}(t)-\sum\limits_{j=1,j\neq i}^{n} (a_{ij}(t)+\sum\limits_{j=1}^{m}\frac{1}{e}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)})})\}\approx 7.1,$$

and

$$\max_{t\in\mathbb{R}}\{-b_{ii}(t)e^{-S^+}+\sum_{j=1,j\neq i}^{2} b_{ij}(t)e^{-S_-}+\sum_{j=1}^{2}\beta_{ij}(t)\max\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{inf}S_-}{e^{\gamma_{ij}^{inf}S_-}}\}\approx -0.05<0,\ i=1,2,$$

which suggest that (4.1) satisfies all assumptions adopted in Theorem 3.1. Thus, by Theorem 3.1, we know that system (1.2) with parameters (4.1) has a unique almost periodic solution $x^*(t)=(x_1^*(t),x_2^*(t))$, whic is globally exponentially stable (see Figure 1), and $x_i^*(t)\geq S_- > 4$ for all $t\in\mathbb{R}$ and $i=1,2$.

Remark 4.1. It should be mentioned that, the following assumptions:

$$\gamma_{ij}(t)\geq 1,\ \text{for all } t\in\mathbb{R}, i\in Q, j\in I,$$

and

$$\inf_{t\geq 0}\{1-\tau_{ij}'(t)\}=\mu>0,\ \text{for all } t\in\mathbb{R}, i\in Q, j\in I,$$

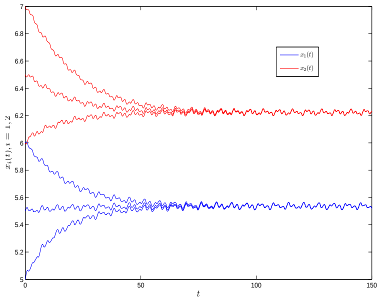

Figure 1: Numerical solutions of (4.1) for initial value $(\varphi_1(s), \varphi_2(s)) = (5,6),(6,7),(5.5,6.5), s \in [-3,0]$.

have been adopted as fundamental to show the stability of periodic and almost periodic solutions for Nicholson's blowflies models in [11, 14, 18, 19, 22, 25] and [24] respectively. In particular, the results on periodic scalar Nicholson's blowflies model in [23] give no opinions about the problem of almost periodic solutions of Nicholson's blowflies systems involving patch structure and nonlinear density-dependent mortality terms. Clearly, the parameters that $\gamma_{ij}(t) = \frac{1}{2},\ i,j=1,2$ and $\tau_{11}(t) = 2|\sin\sqrt{5}t|, \tau_{12}(t) = 3|\sin\sqrt{5}t|, \tau_{21}(t) = 2|\cos\sqrt{7}t|, \tau_{22}(t) = 3|\cos\sqrt{7}t|$ do not satisfy the above assumptions. Moreover, the fact that

$$x_i^*(t) \geq S_- > 4 > \widetilde{\kappa}, \quad \text{for all} \quad t \in \mathbb{R}, i=1,2,$$

entails that $x^*(t)$ is out of $[\kappa, \widetilde{\kappa}] \times [\kappa, \widetilde{\kappa}]$. Hence, all the results in [10, 11, 14, 18, 19, 20, 21, 22, 23, 24, 25] cannot be used to show the global exponential stability on the positive almost periodic solution of system (1.1) involving parameters (4.1).

5 Conclusions

In this paper, we combine the Lyapunov function method with the differential inequality method to establish some new criteria ensuring the existence and exponential stability of positive almost periodic solutions for a class of a class of delayed Nicholson's blowflies systems with patch structure and nonlinear density-dependent mortality terms. These criteria are obtained without assuming that

$$\underbrace{[\kappa, \widetilde{\kappa}] \times \cdots \times [\kappa, \widetilde{\kappa}]}_{n} \approx \underbrace{[0.7215355, 1.342276] \times \cdots \times [0.7215355, 1.342276]}_{n}$$

is the existence region of almost periodic solutions, and the homologous results in the recently published literature are summarized and refined. The approach presented in this article can be used as a possible way to study the patch structure population models with nonlinear density-dependent mortality terms, for example, neoclassical growth model, Mackey-Glass model, epidemical system or age-structured population model and so on.

Acknowledgement

The authors are extremely grateful to one anonymous reviewer and editor for their valuable comments and suggestions, which have contributed a lot to the improved presentation of this paper.

Conflict of Interests statement

The authors confirm that there is no conflict of interest.

References

[1] Y. Tan, M. Zhang, Global exponential stability of periodic solutions in a nonsmooth model of hematopoiesis with time-varying delays, Mathematical Methods in the Applied Sciences, 40 (16) (2017) 5986-5995.

[2] C. Huang, Y. Qiao, L. Huang, R. Agarwal, Dynamical behaviors of a food-chain model with stage structure and time delays, Advances in Difference Equations, 2018 (2018) 186. https://doi.org/10.1186/s13662-018-1589-8

[3] H. Hu, T. Yi, X. Zou, On spatial-temporal dynamics of Fisher-KPP equation with a shifting environment, Proceedings of the American Mathematical Society, 2019, hppts://doi.org/10.1090/proc/14659.

[4] Z. Yang, C. Huang, X. Zou, Effect of impulsive controls in a model system for age-structured population over a patchy environment, Journal of Mathematical Biology, 76 (6) (2018) 1387-1419.

[5] C. Huang, Z. Yang, T. Yi, X. Zou, On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities, Journal of Differential Equations, 256(2014) 2101-2114.

[6] Y. Li, T. Zhang, Y. Ye, On the existence and stability of a unique almost periodic sequence solution in discrete predatorCprey models with time delays, Applied Mathematical Modelling, 35 (2011) 5448-5459.

[7] C. Huang, H. Zhang, J. Cao, H. Hu, Stability and Hopf bifurcation of a delayed prey-predator model with disease in the predator, International Journal of Bifurcation and Chaos, 29 (2019)1950091, 23 Pages.

[8] A. M. Fink, Almost periodic differential equations, Lecture Notes in Mathematics, Vol. 377, Springer, Berlin, 1974.

[9] C. Zhang, Almost Periodic Type Functions and Ergodicity. Kluwer Academic/Science Press, Beijing, 2003.

[10] T. Diagana, Pseudo almost periodic functions in Banach space, Nova Science Publishers, New York, 2007.

[11] L. Yao, Dynamics of Nicholson's blowflies models with a nonlinear density-dependent mortality, Appl Math Modelling. 64 (2018) 185-195.

[12] C. Huang, B. Liu, X. Tian, L. Yang, X. Zhang, Global convergence on asymptotically almost periodic SICNNs with nonlinear decay functions, Neural Processing Letters, 49 (2) (2019) 625-641.

[13] C. Huang, H. Zhang, L. Huang. Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term. Communications on Pure and Applied Analysis, 18 (6) (2019) 3337-3349.

[14] B. Liu, Almost periodic solutions for a delayed Nicholsons blowflies model with a non-linear density-dependent mortality term, Adv. Difference Equ. 72 (2014) 1-16.

[15] L. Duan, X. Fang, C. Huang, Global exponential convergence in a delayed almost periodic nicholsons blowflies model with discontinuous harvesting, Mathematical Methods in the Applied Sciences, 41 (5) (2017) 1954-1965.

[16] L. Duan, L. Huang, Z. Guo, X. Fang, Periodic attractor for reactiondiffusion high-order hopfield neural networks with time-varying delays, Computers & Mathematics with Applications, 73 (2) (2017) 233-245.

[17] X. Long, S. Gong, New results on stability of Nicholson's blowflies equation with multiple pairs of time-varying delays, Applied Mathematics Letters, 2019, in press.

[18] Y. Tang, S. Xie, Global attractivity of asymptotically almost periodic Nicholsons blowflies models with a nonlinear density-dependent mortality term, International Journal of Biomathematics, 11(6) (2018) 1850079 (15 pages) DOI: 10.1142/S1793524518500791.

[19] W. Chen , W. Wang, Almost periodic solutions for a delayed Nicholsons blowflies system with nonlinear density-dependent mortality terms and patch structure, Adv. Difference Equ. 205 (2014) 1-19.

[20] L. Berezansky, E. Braverman, L. Idels, Nicholson's blowflies differential equations revisited: Main results and open problems, Appl. Math. Model. 34 (2010) 1405-1417.

[21] W. Chen Permanence for Nicholson-type delay systems with patch structure and nonlinear density-dependent mortality terms. Electron J. Qual. Theory Differ. Equ. 73 (2012)1-14.

[22] Y. Xu, Existence and global exponential stability of positive almost periodic solutions for a delayed Nicholson's blowflies model. J. Korean Math. Soc. 51(3) (2014) 473-493.

[23] Y. Xu, New stability theorem for periodic Nicholsons model with mortality term, Appl. Math. Lett. 94 (2019) 59-65.

[24] Doan Thai Son, Le Van HienB and Trinh Tuan Anh, Global attractivity of positive periodic solution of a delayed Nicholson model with nonlinear density-dependent mortality term, J. Qual. Theory Differ. Equ. 8 (2019) 1-21.

[25] B. Liu, Global exponential stability of positive periodic solutions for a delayed Nicholsons blowflies model, J. Math. Anal. Appl. 412(1) (2014) 212-221.

[26] J. K. Hale, S. M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.

[27] H. L. Smith, An Introduction to Delay Differential Equations with Applications to the Life Sciences, Springer New York, 2011.