EXPANSION OF CONTINUOUS DIFFERENTIABLE FUNCTIONS IN FOURIER LEGENDRE SERIES

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1. Let

$$S_n(f, x) = \sum_{k=0}^{n} a_k \tilde{P}_k(x) \tag{1.1}$$

denote the nth partial sum of the Fourier Legendre series of a function $f(x)$. The references available to us, except **(5)**, prove only that $S_n(f, x)$ converges uniformly to $f(x)$ in $[-1, 1]$ if $f(x)$ has a continuous second derivative on $[-1, 1]$. Very recently Suetin **(5)** has shown by employing a theorem of A. F. Timan **(7)** (which is a stronger form of Jackson's theorem) that $S_n(f, x)$ converges uniformly to $f(x)$ if $f(x)$ belongs to a Lipschitz class of order greater than 1/2 in $[-1, 1]$. More generally he has proved the following theorem.

THEOREM 1 (P. K. Suetin **(5)**). *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \text{Lip}\,\alpha$, then*

$$\left| f(x) - \sum_{k=0}^{n} a_k \tilde{P}_k(x) \right| \leqslant \frac{c_1 \log n}{n^{p+\alpha-1/2}}, \qquad x \in [-1, 1], \tag{1.2}$$

for $p + \alpha \geqslant \frac{1}{2}$.

In the course of his proof it is shown (as is mentioned by him), without using the theorem of Timan, that the uniform convergence of $S_n(f, x)$ to $f(x)$ holds in $[-1, 1]$ if $f'(x)$ is continuous in $[-1, 1]$.

In this paper we shall supplement the above theorem by proving the following theorem.

THEOREM 2. *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \text{Lip}\,\alpha$, then together with* (1.2) *the following inequalities hold:*

$$(1 - x^2)^{\frac{3}{4}} |f'(x) - S'_n(x)| \leqslant c_2 (\log n)/n^{p+\alpha-1} \qquad (0 < \alpha < 1, p \geqslant 1), \tag{1.3}$$

$$(1 - x^2)^{\frac{1}{2}} |f'(x) - S'_n(x)| \leqslant c_3 (\log n)/n^{p+\alpha-3/2} \qquad (\tfrac{1}{2} < \alpha < 1, p \geqslant 1), \tag{1.4}$$

and

$$|f'(x) - S'_n(x)| \leqslant c_4 (\log n)/n^{p+\alpha-5/2} \qquad (\tfrac{1}{2} < \alpha < 1, p \geqslant 2) \tag{1.5}$$

uniformly in $[-1, 1]$.

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2. To prove the above theorem we shall require a number of well-known results on Legendre polynomials.

The orthonormalized Legendre polynomial $\tilde{P}_n(x)$ is given by **(2)**

$$\tilde{P}_n(x) = \sqrt{[\tfrac{1}{2}(n+1)]}\,.\,P_n(x), \tag{2.1}$$

where $P_n(x)$ denotes the nth Legendre polynomial with the normalization $P_n(1) = 1$.

For the $\tilde{P}_n(x)$ we have the uniform estimations **(2, 3, 6)**

$$|\tilde{P}_n(x)| \leqslant c_5\sqrt{n}, \qquad x \in [-1, 1], \tag{2.2}$$

and the inequality

$$(1-x^2)^{\frac{1}{4}}\,|\tilde{P}_n(x)| \leqslant c_6, \qquad x \in [-1, 1]. \tag{2.3}$$

For the derivatives $\tilde{P}'_n(x)$ we have the following Bernstein inequality:

$$(1-x^2)^{\frac{1}{2}}\,|\tilde{P}'_n(x)| \leqslant c_7\, n^{3/2}, \tag{2.4}$$

the Stieltjes inequality

$$(1-x^2)^{\frac{3}{4}}\,|\tilde{P}'_n(x)| \leqslant c_8\, n, \tag{2.5}$$

and Markov's inequality

$$|\tilde{P}'_n(x)| \leqslant c_9\, n^{5/2} \tag{2.6}$$

for $x \in [-1, 1]$.

3. In order to prove Theorem 2 we need the following two lemmas.

LEMMA 3.1. *For* $-1 \leqslant x \leqslant 1$ *we have*

$$(1-x^2)^{\frac{3}{4}} \int_{-1}^{+1} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt \leqslant c_{10}\, n^{3/2}, \tag{3.1}$$

$$(1-x^2)^{\frac{1}{2}} \int_{-1}^{+1} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt \leqslant c_{11}\, n^{2}, \tag{3.2}$$

and

$$\int_{-1}^{+1} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt \leqslant c_{12}\, n^{3}. \tag{3.3}$$

Proof. We give here only the proof for (3.1). In fact we have

$$(1-x^2)^{3/2} \int_{-1}^{+1} \left(\sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right)^2 dt = \sum_{k=1}^{n} |(1-x^2)^{\frac{3}{4}}\, \tilde{P}'_k(x)|^2$$

which, owing to the inequality (2.5), gives (3.1).

LEMMA 3.2. *For* $-1 \leqslant x \leqslant 1$ *we have*

$$(3.4)\qquad (1-x^2)^{\frac{3}{4}} \int_{-1}^{+1} (1-t^2)^{\frac{1}{4}} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt \leqslant c_{13}\, n \log n,$$

$$(3.5)\qquad (1-x^2)^{\frac{1}{2}} \int_{-1}^{+1} (1-t^2)^{\frac{1}{4}} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt \leqslant c_{14}\, n^{3/2} \log n,$$

and

$$(3.6)\qquad \int_{-1}^{+1} (1-t^2)^{\frac{1}{4}} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt \leqslant c_{15}\, n^{5/2} \log n.$$

Proof. We shall confine ourselves to the proof of (3.4). We denote by $\Delta_n(x)$ the part of $[-1, 1]$ on which $|x-t| \leqslant 1/n$ and by $\lambda_n(x)$ the rest of the interval. Thus taking account of (2.3) and (2.5) we have

$$(3.7)\qquad (1-x^2)^{\frac{3}{4}} \int_{\Delta_n(x)} (1-t^2)^{\frac{1}{4}} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt$$
$$\leqslant \int_{\Delta_n(x)} \left[\sum_{k=1}^{n} (1-t^2)^{\frac{1}{4}} |\tilde{P}_k(t)|\, (1-x^2)^{\frac{3}{4}} |\tilde{P}'_k(x)| \right] dt$$
$$\leqslant c_{16} \frac{1}{n} \sum_{k=1}^{n} k \leqslant c_{17}\, n, \qquad x \in [-1, 1].$$

To estimate the integral over λ_n we use the Christoffel–Darboux formula **(6)**,

$$(3.8)\qquad \sum_{k=0}^{n} \tilde{P}_k(t)\tilde{P}_k(x) = \theta_n \frac{\tilde{P}_{n+1}(x)\tilde{P}_n(t) - \tilde{P}_n(x)\tilde{P}_{n+1}(t)}{x-t}, \qquad 0 < \theta_n \leqslant 1.$$

Differentiating the above relation with respect to x we have

$$(3.9)\qquad \sum_{k=0}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) = \theta_n \frac{\tilde{P}'_{n+1}(x)\tilde{P}_n(t) - \tilde{P}'_n(x)\tilde{P}_{n+1}(t)}{x-t}$$
$$- \theta_n \frac{\tilde{P}_{n+1}(x)\tilde{P}_n(t) - \tilde{P}_n(x)\tilde{P}_{n+1}(t)}{(x-t)^2}.$$

Then we have

$$(3.10)\qquad (1-x^2)^{\frac{3}{4}} \int_{\lambda_n(x)} (1-t^2)^{\frac{1}{4}} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt$$
$$\leqslant (1-x^2)^{\frac{3}{4}} \int_{\lambda_n(x)} (1-t^2)^{\frac{1}{4}} \left| \frac{\tilde{P}'_{n+1}(x)\tilde{P}_n(t) - \tilde{P}'_n(x)\tilde{P}_{n+1}(t)}{x-t} \right| dt$$
$$+ (1-x^2)^{\frac{3}{4}} \int_{\lambda_n(x)} (1-t^2)^{\frac{1}{4}} \left| \frac{\tilde{P}_{n+1}(x)\tilde{P}_n(t) - \tilde{P}_n(x)\tilde{P}_{n+1}(t)}{(x-t)^2} \right| dt$$
$$= I_1 + I_2.$$

Since $|x-t| > 1/n$ for $t \in \lambda_n(x)$, we find by using (2.3) and (2.5) that

$$(3.11)\qquad I_1 \leqslant c_8\, n \int_{\lambda_n(x)} (1-t^2)^{\frac{1}{4}} [|\tilde{P}_n(t)| + |\tilde{P}_{n+1}(t)|] \frac{dt}{|x-t|}$$
$$\leqslant c_{18}\, n \int_{\lambda_n(x)} \frac{dt}{|x-t|} \leqslant c_{19}\, n \log n, \qquad x \in [-1, 1].$$

For I_2 we have, on using (2.3),

$$I_2 \leqslant c_5 \int_{\lambda_n(x)} (1-t^2)^{\frac{1}{4}} [|\tilde{P}_n(t)| + |\tilde{P}_{n+1}(t)|] \frac{dt}{(x-t)^2} \tag{3.12}$$

$$\leqslant c_{20}\, n, \qquad x \in [-1, 1].$$

Thus (3.7), (3.10), (3.11), and (3.12) complete the proof of (3.1).

4. Let $Q_n(x)$ be an algebraic polynomial of degree not greater than n; then we have the following theorem of A. F. Timan **(7)** on the order of approximation of the function $f(x)$.

THEOREM 3 (A. F. Timan). *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \text{Lip}\,\alpha$, then there is a sequence of polynomials $\{Q_n(x)\}$ such that*

$$|f(x) - Q_n(x)| \leqslant \frac{c_{21}}{n^{p+\alpha}} \left(\sqrt{1-x^2} + \frac{1}{n}\right)^{p+\alpha}, \qquad x \in [-1, 1]. \tag{4.1}$$

From this theorem, on using the Dzyadyk inequality **(1)**, *we have the following lemma.*

LEMMA 4.1. *Let $f^{(r)}(x) \in \text{Lip}\,\alpha$ $(0 < \alpha < 1, r \geqslant 1)$ in $[-1, 1]$; then there is a polynomial $\rho_n(x)$ of degree at most n possessing the following properties:*

$$|f(x) - \rho_n(x)| \leqslant \frac{c_{22}}{n^{r+\alpha}} \left[(\sqrt{1-x^2})^{r+\alpha} + \frac{1}{n^{r+\alpha}} \right] \tag{4.2}$$

and

$$|f'(x) - \rho'_n(x)| \leqslant \frac{c_{23}}{n^{r+\alpha-1}} \left[(\sqrt{1-x^2})^{r+\alpha-1} + \frac{1}{n^{r+\alpha-1}} \right] \tag{4.3}$$

uniformly in $[-1, 1]$.

The author has proved this lemma for $r = 1$ in **(4)**. For general r the lemma can be proved in the same manner.

We now complete the proof of Theorem 2. We shall confine ourselves to proving (1.3).

We write

$$|f'(x) - S'_n(x)| = |f'(x) - \rho'_n(x) + \rho'_n(x) - S'_n(x)| \tag{4.4}$$

$$\leqslant |f'(x) - \rho'_n(x)| + \int_{-1}^{+1} |\rho_n(t) - f(t)| \left| \sum_{k=1}^{n} \tilde{P}_k(t) \tilde{P}'_k(x) \right| dt.$$

Now using Lemma 4.1 we have

$$|f'(x) - S'_n(x)| \leqslant \frac{c_{23}}{n^{p+\alpha-1}} \left[(\sqrt{1-x^2})^{p+\alpha-1} + \frac{1}{n^{p+\alpha-1}} \right]$$

$$+ \frac{c_{22}}{n^{p+\alpha}} \int_{-1}^{+1} \left\{ (1-t^2)^{p+\alpha/2} + \frac{1}{n^{p+\alpha}} \right\} \left| \sum_{k=1}^{n} \tilde{P}_k(t) \tilde{P}'_k(x) \right| dt$$

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so that

$$(1 - x^2)^{\frac{3}{4}} |f'(x) - S'_n(x)|$$
$$\leqslant \frac{c_{24}}{n^{p+\alpha-1}} + \frac{c_{22}}{n^{p+\alpha}} (1 - x^2)^{\frac{3}{4}} \int_{-1}^{+1} (1 - t^2)^{p+\alpha/2} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt$$
$$+ \frac{c_{22}}{n^{2p+2\alpha}} (1 - x^2)^{\frac{3}{4}} \int_{-1}^{+1} \left| \sum_{k=1}^{n} \tilde{P}_k(t)\tilde{P}'_k(x) \right| dt$$

which, by the help of (3.4) and (3.1), gives

$$(1 - x^2)^{\frac{3}{4}} |f'(x) - S'_n(x)| \leqslant \frac{c_{24}}{n^{p+\alpha-1}} + \frac{c_{22}}{n^{p+\alpha}} c_{13}\, n \log n + \frac{c_{22}}{n^{2p+2\alpha}} c_{10}\, n^{3/2}$$
$$\leqslant c_{25} \frac{\log n}{n^{p+\alpha-1}}, \qquad p \geqslant 1.$$

This completes the proof of (1.3). The proof of (1.4) and (1.5) can be obtained similarly.

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