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A STRATIFICATION GIVEN BY ARTIN-REES ESTIMATES

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0. **Introduction.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let U be an open subset of $\mathbb{K}^n$. Let X be a closed analytic subset of U and let Z be a proper closed analytic subset of X. Let $M(X;Z)$ denote the ring of meromorphic functions on X whose poles lie in Z. Let M be the families of formal power series generated by a finite sequence $f_1, \ldots, f_q \in M(X;Z)[\![y]\!]^p$ (For details, see § 2). We consider the Artin-Rees estimate for M at each point of $X-Z$. By Artin-Rees lemma (see [Bou]), for each point $a \in X - Z$, there exists an integer k_a such that for all $l \in \mathbb{N}$,

$$M_a \cap \hat{m}^{k_a+l}\mathbb{K}[\![y]\!]^p = \hat{m}^l(M_a \cap \hat{m}^{k_a}\mathbb{K}[\![y]\!]^p), \tag{1}$$

or

$$M_a \cap m^{k_a+l}\mathbb{K}\{y\}^p = m^l(M_a \cap m^{k_a}\mathbb{K}\{y\}^p), \tag{1$'$}$$

where $\hat{m}$ (or m) is the maximal ideal of $\mathbb{K}[\![y]\!]$ (or $\mathbb{K}\{y\}$). Therefore, there exists an integer k'_a such that for all $l \in \mathbb{N}$,

$$M_a \cap \hat{m}^{k'_a+l}\mathbb{K}[\![y]\!]^p \subset \hat{m}^l M_a, \tag{2}$$

or

$$M_a \cap m^{k'_a+l}\mathbb{K}\{y\}^p \subset m^l M_a. \tag{2$'$}$$

Equation (1) (or (1′)) is called *a strong Artin-Rees estimate*, and (2) (or (2′)) is called *a weak Artin-Rees estimate*. It is known that the Artin-Rees number, i.e. the k_a and k'_a mentioned above, can be uniformly given for every compact subset of $X - Z$ (see [B-M, 2], Theorem 7.4(3)).

The Artin-Rees estimate is an important subject in algebraic and analytic geometries (see [E-H], [D-O], and [B-M, 2], etc.). In this article, we investigate the properties of the stratification given by the Artin-Rees estimates, namely, on each stratum, the Artin-Rees number is constant. In the case of k_a or k'_a being 0, $\mathbb{K}[\![y]\!]^p/M_a$ is a flat module over $\mathbb{K}[\![y]\!]$. It has been known that the flatness is an open property (see [B-M, 1]). Therefore, naturally, one expects that the Artin-Rees stratification is Zariski semicontinuous (for

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the definition of Zariski semicontinuity, see [B-M, 1]) on $X - Z$. This article gives a discussion on this issue and figures out that it is not the general case.

Since, by Krull's theorem, the formal and the convergent versions of the Artin-Rees estimates are equivalent, we only consider the formal cases (1) and (2) in this paper. In Section 1, we give a criterion for the Artin-Rees estimate (see Theorem 1). All the concepts and the properties of a diagram of the initial exponents of a module we will use have been discussed in [B-M, 3] and [B-M, 2]. In Section 2, with the help of the criterion given in Section 1, we get the constructibility of the stratifications defined by both strong and weak Artin-Rees estimates. In Section 3, we use "minimal-bases" (defined in § 2) to study the openness of the initial strata, which is equivalent to the openness of the flatness of modules. Finally, some examples will show that in general the stratification discussed here is not Zariski semicontinuous.

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1. **Special generators and the Artin-Rees estimate.** Let $T \subset \mathbb{K}[[y]]^p$ be a submodule. By the Artin-Rees lemma, there exists an integer k such that for all $l \in \mathbb{N}$,

$$T \cap \hat{m}^{k+l}\mathbb{K}[[y]]^p = \hat{m}^l(T \cap \hat{m}^k\mathbb{K}[[y]]^p). \tag{3}$$

Also, there exists an integer k' such that for all $l \in \mathbb{N}$,

$$T \cap \hat{m}^{k'+l}\mathbb{K}[[y]]^p \subset \hat{m}^l T. \tag{4}$$

We refer the readers to the articles [B-M, 1] and [B-M, 2] for the notations and properties of a diagram of initial exponents of a module with respect to a given linear form. Take a positive linear form $L\colon \mathbb{N}^n \longrightarrow \mathbb{N}$ defined by $L(\alpha) = |\alpha|$. With respect to this L, the diagram of initial exponents $N_L(T)$ is well-defined and we may assume that $(\alpha_i, j_i), i = 1, \ldots, s$, are all the vertices of $N_L(T)$. Then, we can get a set of special generators of T, say, $h_1(y), \ldots, h_s(y)$, satisfying the following three properties:

$$T = \big(h_1(y), \ldots, h_s(y)\big)\mathbb{K}[[y]]; \tag{5}$$

(6) Let $\nu_L(h_i)$ be the initial exponents of $h_i, i = 1, \ldots, s$, with respect to L. $\{\nu_L(h_i)\}$ are the vertices of $N_L(T)$; and

$$(\alpha_1, j_1) < \cdots < (\alpha_s, j_s). \tag{7}$$

REMARK. Condition (7) implies that

$$|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_s|.$$

A set of generators with the properties (5), (6) and (7) is called a set of *special generators* of T. For the existence of such a set of generators, one is referred to [B-M, 1].

For any given $\lambda \in \mathbb{N}$, and a set of special generators $h_1, \ldots, h_s$, we may define $r = r(\lambda) \in \mathbb{N}$ by

$$|\alpha_1| \leq \lambda, \ldots, |\alpha_r| \leq \lambda, |\alpha_{r+1}| > \lambda, \ldots, |\alpha_s| > \lambda,$$

where $r(\lambda)$ is 0 if all $|\alpha_i| > \lambda$, and it is s if all $|\alpha_i| \leq \lambda$.

THEOREM 1. *For any $\lambda \in \mathbb{N}$:*
[1]: *(3) holds for λ if and only if $r(\lambda) \geq 1$ and for $j = r+1, \ldots, s$,*

$$h_j(y) \in \hat{m}^{|\alpha_j|-|\alpha_1|} h_1 + \cdots + \hat{m}^{|\alpha_j|-|\alpha_r|} h_r.$$

[2]: *(4) holds for λ if and only if $r(\lambda) \geq 1$ and for $j = r+1, \ldots, s$,*

$$h_j(y) \in \hat{m}^{|\alpha_j|-\lambda} h_1 + \cdots + \hat{m}^{|\alpha_j|-\lambda} h_r.$$

PROOF. The proof of [2] is parallel to that of [1]. Therefore, we only give the proof of [1] here.

IF. Assume that $r(\lambda) \geq 1$, and for all $j > r = r(\lambda)$,

$$h_j(y) \in \sum_{i=1}^{r} \hat{m}^{|\alpha_j|-|\alpha_i|} h_i.$$

Let $g \in T \cap \hat{m}^{\lambda+l} \mathbb{K}[\![y]\!]^p$. By the definition and the properties of $\{h_i\}$,

$$g = \sum_{i=1}^{s} t_i h_i,$$

where $t_i \in \hat{m}^{\lambda+l-|\alpha_i|}$, $i = 1, \ldots, s$. We discuss two cases:

1. If $j \leq r(\lambda)$, then

$$t_j h_j \in \hat{m}^l \hat{m}^{\lambda-|\alpha_j|} h_j \subset \hat{m}^l (T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p).$$

2. If $j > r(\lambda)$, then by our assumption,

$$t_j h_j \in t_j \hat{m}^{|\alpha_j|-\lambda} \left(\sum_{i=1}^{r} \hat{m}^{\lambda-|\alpha_i|} h_i\right) \subset \hat{m}^l (T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p).$$

Hence, $g \in \hat{m}^l (T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p)$. This proves

$$T \cap \hat{m}^{\lambda+l} \mathbb{K}[\![y]\!]^p \subset \hat{m}^l (T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p)$$

for all $l \in \mathbb{N}$.

It is trivial to show that $\hat{m}^l (T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p) \subset T \cap \hat{m}^{\lambda+l} \mathbb{K}[\![y]\!]^p$.

ONLY IF. Assume that for all $l \in \mathbb{N}$, $T \cap \hat{m}^{\lambda+l} \mathbb{K}[\![y]\!]^p = \hat{m}^l (T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p)$.
If $r(\lambda) = 0$, i.e. each $|\alpha_i| > \lambda$, then

$$h_1 \in T \cap \hat{m}^{\lambda+|\alpha_1|-\lambda} \mathbb{K}[\![y]\!]^p \subset \hat{m}^{|\alpha_1|-\lambda} (T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p).$$

But $h_1 \notin \hat{m} T$ since h_1 has the minimal initial exponent. Consequently, $r(\lambda) \geq 1$.

For each $h_j(y)$, $\big(j > r(\lambda)\big)$, we have

$$h_j \in T \cap \hat{m}^{\lambda+|\alpha_j|-\lambda} \mathbb{K}[\![y]\!]^p \subset \hat{m}^{|\alpha_j|-\lambda} (T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p).$$

Thus $h_j = \sum_{|\alpha|=|\alpha_j|-\lambda} y^\alpha G_{\alpha j}$, where $G_{\alpha j} \in T \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p$. So we may assume that $G_{\alpha j} = \sum_{i=1}^{s} g^i_{\alpha j} h_i$, where $g^i_{\alpha j} \in \hat{m}^{\lambda-|\alpha_i|}$.

SUBLEMMA. *If (3) holds for λ, then for all $j > r(\lambda)$,*

$$h_j \in \hat{m}^{|\alpha_j|-|\alpha_1|} h_1 + \cdots + \hat{m}^{|\alpha_j|-|\alpha_{j-1}|} h_{j-1}. \tag{8}$$

The "only if" part follows by repeatedly applying the sublemma. ∎

PROOF OF THE SUBLEMMA. By induction on $s - j$:
For $j = s$: $h_s = \sum_{|\alpha|=|\alpha_s|-\lambda} y^\alpha \sum_{i=1}^{s} g^i_{\alpha,s} h_i$, where $g^i_{\alpha,s} \in \hat{m}^{\lambda-|\alpha|}$. Then we have:

$$h_s = \Big(1 - \sum_{|\alpha|=|\alpha_s|-\lambda} g^s_{\alpha,s} y^\alpha\Big)^{-1} \sum_{i=1}^{s-1} \sum_{|\alpha|=|\alpha_s|-\lambda} g^i_{\alpha,s} y^\alpha h_i$$

where $\sum_{|\alpha|=|\alpha_s|-\lambda} g^i_{\alpha,s} y^\alpha \in \hat{m}^{|\alpha_s|-|\alpha_i|}$.

We assume that (8) holds for $k < j < s$. For $j = k - 1 \big(> r(\lambda)\big)$:

$$h_{j-1} \in T \cap \hat{m}^{\lambda+|\alpha_j|-\lambda} \mathbb{K}[[y]]^p = \hat{m}^{|\alpha_{j-1}|-\lambda} (T \cap \hat{m}^\lambda \mathbb{K}[[y]]^p).$$

Our induction hypothesis now implies

$$h_{j-1} \in \sum_{i=1}^{j-2} \hat{m}^{|\alpha_{j-1}|-|\alpha_i|} h_i + \hat{m}^{|\alpha_{j-1}|-\lambda} h_{j-1}.$$

Hence $h_{j-1} \in \sum_{i=1}^{j-2} \hat{m}^{|\alpha_{j-1}|-|\alpha_i|} h_i$. This proves the sublemma. ∎

DEFINITION. For any submodule $T \subset \mathbb{K}[[y]]^p$, define the *strong Artin-Rees number* λ_T and the *weak Artin-Rees number* λ'_T by

$$\lambda_T = \inf\{\, k \in \mathbb{N} \mid (3) \text{ holds for } k \,\};$$
$$\lambda'_T = \inf\{\, k \in \mathbb{N} \mid (4) \text{ holds for } k \,\}.$$

COROLLARY. [1]: *If $\lambda_T \leq \lambda$ (or $\lambda'_T \leq \lambda$), then $h_1(y), \ldots, h_{r(\lambda)}(y)$ generate T.*
[2]:

$$|\alpha_1| \leq \lambda_T \leq |\alpha_s|;$$
$$|\alpha_1| \leq \lambda'_T \leq |\alpha_s|.$$

The proof is trivial. ∎

2. **The stratification.** Let X be a closed analytic subset of an open set $U \subset \mathbb{K}^n$, and let Z be a closed proper analytic subset of X. Let $M(X;Z)$ denote the ring of meromorphic functions on X whose poles lie in Z. Let $M(X;Z)[[y]]^p$ be the families of formal power series with coefficients in $M(X;Z)$, and let $f_1, \ldots, f_q \in M(X;Z)[[y]]^p$. For each point $a \in X - Z$, let M_a be the submodule of $\mathbb{K}[[y]]^p$ which is generated by $\{f_i(a;y)\}$ over $\mathbb{K}[[y]]$. Let N_a denote the diagram of initial exponents of M_a. For this parametrized families of power series $M(= \cup_{a \in X-Z} M_a)$, we have (see [B-M, 1], Theorem 5.5 and Corollary 5.9):

THEOREM 2. *Let M be defined as above. Then we have:*

[1]: *Let $a_0 \in X$. Then there is a neighborhood V of a_0 in U and a filtration of $X \cap V$ by closed analytic subsets:*

$$X \cap V = X_0 \supset X_1 \supset \cdots \supset X_{\omega+1} = Z \cap V$$

satisfying: for each $\mu = 0, \ldots, \omega$, N_a is constant on $X_\mu - X_{\mu+1}$.

[2]: *For each $\mu = 0, \ldots, \omega$, there exists $\xi^1(x;y), \ldots, \xi^{t_\mu}(x;y) \in M(X_\mu; X_{\mu+1})[\![y]\!]^p$ such that for all points $a \in X_\mu - X_{\mu+1}$, $\xi^1(a;y), \ldots, \xi^{t_\mu}(a;y)$ generate M_a, and moreover, they are a set of special generators of M_a for each $a \in X_\mu - X_{\mu+1}$.*

[3]: *Let $N_\mu(M) = N_a$ if $a \in X_\mu - X_{\mu+1}$. Then $N_\mu(M) < N_{\mu+1}(M)$ for all $\mu = 0, \ldots, \omega$.* ■

Let $R_a \subset \mathbb{K}[\![y]\!]^q$ be the relations among $\{f_i(a;y), i = 1, \ldots, q\}$ over $\mathbb{K}[\![y]\!]^q$, and let $R(M)$ be the parametrized families of the relations R_a. Then we have (see [B-M, 1] Theorem 6.3):

THEOREM 3. *Let R be defined as above. It follows that:*

[1]: *Let $a_0 \in X$. Then there is a neighborhood V of $a_0 \in U$ and a filtration of $X \cap V$ by analytic subsets:*

$$X \cap V = X_0 \supset X_1 \supset \cdots \supset X_{\mu+1} = Z \cap V$$

satisfying: for each $\mu = 0, \ldots, \nu$, N_a is constant on $X_\mu - X_{\mu+1}$.

[2]: *For each $\mu = 0, \ldots, \nu$, there exist $\zeta^1(x;y), \ldots, \zeta^{t'_\mu}(x;y) \in M(X_\mu; X_{\mu+1})[\![y]\!]^q$ such that for all $a \in X_\mu - X_{\mu+1}$, $\zeta^1(a;y), \ldots, \zeta^{t'_\mu}(a;y)$ generate R_a, and they are a set of special generators of R_a for each $a \in X_\mu - X_{\mu+1}$.*

[3]: *Let $N_\mu(R) = N_\mu(R_a)$ if $a \in X_\mu - X_{\mu+1}$. Then $N_\mu(R) < N_{\mu+1}(R)$ for all $\mu = 0, \ldots, \nu$.* ■

We now discuss the Artin-Rees estimates related to M, namely,

$$M_a \cap \hat{m}^{\lambda+l}\mathbb{K}[\![y]\!]^p = \hat{m}^l(M_a \cap \hat{m}^\lambda \mathbb{K}[\![y]\!]^p) \tag{9}$$

and

$$M_a \cap \hat{m}^{\lambda+l}\mathbb{K}[\![y]\!]^p \subset \hat{m}^l M_a. \tag{10}$$

DEFINITION. For each $\lambda \in \mathbb{N}$, define

$$\Gamma^\lambda = \{a \in X - Z \mid (9) \text{ holds for all } l \in \mathbb{N} \text{ at } a\};$$

$$\Gamma_\lambda = \{a \in X - Z \mid (10) \text{ holds for all } l \in \mathbb{N} \text{ at } a\}.$$

It is easy to see the following properties:

$$\Gamma^\lambda \subset \Gamma_\lambda;\ \forall \lambda \in \mathbb{N};\quad \Gamma^0 = \Gamma_0. \tag{11}$$

$$\Gamma^\lambda \subset \Gamma^{\lambda+1},\quad \Gamma_\lambda \subset \Gamma_{\lambda+1};\ \forall \lambda \in \mathbb{N}. \tag{12}$$

$$X - Z = \cup_{\lambda=0}^\infty \Gamma^\lambda;\quad X - Z = \cup_{\lambda=0}^\infty \Gamma_\lambda. \tag{13}$$

DEFINITION. A set $Y \subset U$ is a *constructible set* if it is a union of differences of analytic subsets.

THEOREM 4. *Both* $\{U-\Gamma^{\lambda}\}$ *and* $\{U-\Gamma_{\lambda}\}$ *are constructible sets.*

PROOF. The proof for $U-\Gamma_{\lambda}$ is similar to that of $U-\Gamma^{\lambda}$. Therefore we only discuss the constructibility of $U-\Gamma^{\lambda}$ here. The proof is local. So, we consider the question at a point $a \in X-Z$:

By Theorem 2, there exists a neighborhood V of a such that

$$X \cap V = X_0 \supset X_1 \supset \cdots \supset X_{\omega+1} = Z \cap V$$

with the properties:

(i) $N(M)$ is constant on each $X_{\mu}-X_{\mu+1}$, $\mu = 0, \ldots, \omega$.

(ii) For each μ, there exist $\xi^{1}, \ldots, \xi^{t_{\mu}} \in M(X_{\mu}; X_{\mu+1})[\![y]\!]^{p}$ which are a set of special generators of M_a for each point $a \in X_{\mu}-X_{\mu+1}$.

For a fixed $\lambda \in \mathbb{N}$, and for each $\mu = 0, \ldots, \omega$, we have a well-defined $r_{\mu} = r_{\mu}(\lambda)$. For each $j > r_{\mu}$, let us consider the relations among ξ^{j} and $\{y^{|\alpha_j|-|\alpha_i|}\xi^{i}\}$ $(i = 1, \ldots, r_{\mu})$ which is denoted by $R_j = R(\xi^{j}, y^{|\alpha^{j}|-|\alpha^{1}|}\xi^{1}, \ldots, y^{|\alpha^{j}|-|\alpha^{r_{\mu}}|}\xi^{r_{\mu}})$ on each $X_{\mu}-X_{\mu+1}$. By Theorem 3, without loss of generality, we may assume that on $X_{\mu}-X_{\mu+1}$, R_j has a constant diagram for each $j > r_{\mu}$, and $\zeta^{1}, \ldots, \zeta^{s_j}$ are a set of special generators of R_j at each point $a \in X_{\mu}-X_{\mu+1}$.

Each ζ^{l} is a vector $(\zeta_1^{l}, \ldots, \zeta_{r_{\mu}+1}^{l})$. It is easy to see that at one point $b \in X_{\mu}-X_{\mu+1}$, $\xi^{j} \notin \sum_{i=1}^{r_{\mu}} \hat{m}^{|\alpha_j|-|\alpha_i|}$ if and only if $\zeta_1^{1}(b; 0) \equiv 0$. This is a closed condition. By Theorem 1, now we conclude that $(U-\Gamma^{\lambda}) \cap (X_{\mu}-X_{\mu+1})$ is constructible. As a result, $U-\Gamma^{\lambda}$ is constructible. ∎

COROLLARY. *By Theorem 1, we have the following results:*

[1]: *In a particular case, the sets* $U-\Gamma^{\lambda}$ *and* $U-\Gamma_{\lambda}$ *defined by a coherent sheaf of* $\hat{O}$*- modules are also constructible.*

[2]: *For any compact subset of* U *(or* $X-Z$*), we have a uniform Artin-Rees estimate for a parametrized families.* ∎

3. **Minimal-bases and the openness of Γ^{0}.** For a submodule $M \subset \mathbb{K}[\![y]\!]^{p}$ (or $\mathbb{K}\{y\}^{p}$), a set of generators $h_1, \ldots, h_q$ is called a *minimal-basis* of M if q is the minimal size of all generators of M. We denote this integer by n_M. It is also known that:

$$n_M = \dim_{\mathbb{K}} M \otimes_{\mathbb{K}[\![y]\!]} \mathbb{K}.$$

Let $Z \subset X \subset U \subset \mathbb{K}^{n}$, where U is an open subset and X, Z are closed analytic subsets. Let $f_1, \ldots, f_q \in M(X; Z)[\![y]\!]^{p}$ and let M be the parametrized families of submodules generated by $f_1(a; y), \ldots, f_q(a; y)$ at each point $a \in X-Z$. For M_a, abbreviate n_{M_a} by n_a. This procedure defines a function:

$$\mathbf{n}: X-Z \longrightarrow \mathbb{N}.$$

In general, $\mathbf{n}$ is not upper semicontinuous. Consider the following counterexample,

EXAMPLE 1. Let $f_1, f_2 \in (\mathbb{K}[[x]])[[y]]^2$ be defined by:

$$f_1(x;y) = \begin{pmatrix} xy \\ 1 \end{pmatrix}; \quad f_2(x;y) = \begin{pmatrix} 0 \\ xy \end{pmatrix},$$

where $x \in X = \mathbb{K}$. Then at $x = 0$,

$$M_0 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

and it has only one generator. If $x \neq 0$,

$$M_x = \left(\begin{pmatrix} xy \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix} \right),$$

so it has at least two generators. ∎

However, in the case of coherent sheaves, where we assume that $X = U$ and M is a coherent sheaf of $\hat{O}$-modules, we have:

PROPOSITION 5. *If M is a coherent sheaf of $\hat{O}_X$-modules, then* $\mathbf{n}$ *is a Zariski upper semicontinuous function.*

PROOF. Let $a \in X$, and let M_a be generated by $s_1, \dots, s_q$, where $q = n_a$. Then, in a neighborhood V of a, M is generated by $s_1, \dots, s_q$ as well. Thus $n_b \leq n_a$ for all $b \in V \cap X$.

Let M_j be the coherent sheaf generated by $\{s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_q\}, j = 1, \dots, q$. For a point $b \in X$, let $M_{j,b}$ and $s_{j,b}$ denote the germ of M_j and the germ of s_j at the point b respectively. We have:

CLAIM. *The subset $\{b \in X \mid s_{j,b} \notin M_{j,b}\}$ is a closed analytic subset.*

PROOF OF THE CLAIM. The relations among $s_1, \dots, s_q$ is a coherent sheaf of $\hat{O}_X$-modules. The condition of $s_{j,b} \notin M_{j,b}$ is equivalent to all the j-th components of the generators of the relations among $\{s_1, \dots, s_q\}$ vanishing at b. This is a closed condition. ∎

Let X_i ($\subset X \cap V$) denote the subset consisting of all b such that $n_b = i$. Then,

$$X_q = X_{n_a} = \cap_{j=1}^{q} \{b \in X \mid s_{j,b} \notin M_{j,b}\}$$

is a closed analytic subset by the above claim. This proves Proposition 5. ∎

LEMMA 6. *The following statements are equivalent:*

$$M_a \cap \hat{m}^l \mathbb{K}[[y]]^p = \hat{m}^l M_a \textit{ for all } l \in \mathbb{N}; \tag{14}$$

$$M_a \cap \hat{m} \mathbb{K}[[y]]^p = \hat{m} M_a. \tag{15}$$

For the proof, see [Bou] or [Mat]. ∎

For M_a, we may have such a minimal-basis $h_1, \dots, h_t$ with the following property:

(∗) If $\sum_{i=1}^{t} \xi_i h_i \in \hat{m}\mathbb{K}[[y]]^p$, and if for some $k, \xi_k \notin \hat{m}$, then h_k must be in $\hat{m}\mathbb{K}[[y]]^p$.

In fact, to get a minimal-basis with the property $(*)$, one needs only to repeat the following procedure on a given minimal-basis:

For a given minimal-basis, say $h_1, \ldots, h_t$, if there exists a combination $g = \sum_{i=1}^t \xi_i h_i \in \hat{m}\mathbb{K}[\![y]\!]^p$, with some $\xi_k \notin \hat{m}$, then $\xi_k h_k = g - \sum_{i\neq k} \xi_i h_i$, or $h_k = \xi_k^{-1} g - \xi_k^{-1} \sum_{i\neq k} \xi_i h_i$. By replacing h_k by $g \in \hat{m}\mathbb{K}[\![y]\!]^p$, we get a new minimal-basis. After finitely many replacements, we will get such a minimal-basis with the property $(*)$.

PROPOSITION 7. *That a point a belongs to Γ^0 is equivalent to the statement:*

($\ddagger$) For any minimal-basis of M_a, say $h_1(y), \ldots, h_t(y)$, each $h_i \notin \hat{m}\mathbb{K}[\![y]\!]^p, i = 1, \ldots, t$.

PROOF. We will prove the following statement:

$$M_a \cap \hat{m}\mathbb{K}[\![y]\!]^p = \hat{m}M_a \text{ is equivalent to } (\ddagger).$$

Then by applying Lemma 6, Proposition 7 follows.

IF. We assume that $M_a \cap \hat{m}\mathbb{K}[\![y]\!]^p = \hat{m}M_a$. If there exists a minimal-basis $h_1, \ldots, h_t$ with some $h_k \in \hat{m}\mathbb{K}[\![y]\!]^p$, then $h_k \in \hat{m}M_a$. Thus

$$h_k = \sum_{i=1}^t \xi_i h_i, \text{ for some } \xi_i \in \hat{m}.$$

Hence we have $h_k = (1 - \xi_k)^{-1} \sum_{i\neq k} \xi_i h_i$ which contradicts the fact that $h_1, \ldots, h_t$ is a minimal-basis.

ONLY IF. If all minimal-bases of M_a satisfy ($\ddagger$), then take a minimal-basis of M_a with the property $(*)$, say $h_1, \ldots, h_t$. For any $g \in M_a \cap \hat{m}\mathbb{K}[\![y]\!]^p$, $g = \sum_{i=1}^t \xi_i h_i$ with each $\xi_i \in \mathbb{K}[\![y]\!]$. By the property ($\ddagger$), it follows that all $\xi_i \in \hat{m}$. This proves that $g \in \hat{m}M_a$. ∎

REMARK. Applying Theorem 1 and the *if* part of the proof of Proposition 7, we obtain a new proof of Lemma 6:

Take a set of special generators of M_a, say $h_1, \ldots, h_t$. If $M_a \cap \hat{m}\mathbb{K}[\![y]\!]^p = \hat{m}M_a$, then, by the proof of Proposition 7, there is a minimal-basis of M_a which can be obtained from $h_1, \ldots, h_t$, namely, $h_1, \ldots, h_r$, and each $h_i \notin \hat{m}\mathbb{K}[\![y]\!]^p$ $(i = 1, \ldots, r)$. Without loss of generality, we may assume that

$$h_i = \begin{pmatrix} f_1 \\ \vdots \\ f_i \\ \vdots \\ f_p \end{pmatrix}, \text{ where } f_j \in \hat{m}, j = 1, \ldots, i-1, \text{ and } f_i \notin \hat{m}.$$

It is easy to see that the following relations must hold:

$$h_j \in \sum_{i=1}^r \hat{m}^{|\alpha_j|} h_i, \text{ for all } j > r.$$

By Theorem 1, Lemma 6 now follows. ∎

PROPOSITION 8. *For a coherent sheaf of $\hat{O}$-modules M, Γ^0 is open.*

PROOF. Assume that $a \in \Gamma^0$. Let $h_1, \ldots, h_q$ be a minimal-basis of M_a. By Proposition 7, each $|\nu_L(h_i)| = 0$. After a linear transformation, we may assume that

$$h_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h_{i,i}(x) \\ \vdots \\ h_{p,i}(x) \end{pmatrix}, \text{ where } h_{i,i}(a) \neq 0. \tag{16}$$

Then in a neighborhood of a, $h_1, \ldots, h_q$ are in the same form as (16). This implies that $h_1, \ldots, h_q$ are still a minimal-basis for M_b at any nearby point b. From the form (16), all minimal-bases of M_a must satisfy the condition of Proposition 7. Thus Γ^0 is open. ∎

REMARK. Proposition 8 is false if M is not a coherent sheaf. A typical counterexample is Example 1, where $\Gamma^0 = \{0\}$.

4. **Some examples.** In the last two sections, we have seen that $\Gamma^0 = \Gamma_0$ is open if M is a coherent sheaf, and $U - \Gamma^\lambda$ and $U - \Gamma_\lambda$ are constructible sets if M is a parametrized family. Here we give more examples to show all the possibilities that $U - \Gamma^\lambda$ (or $U - \Gamma_\lambda$) may have.

EXAMPLE 2. For any coherent sheaf of O-modules generated by one function, all Γ^λ and Γ_λ are open sets. ∎

EXAMPLE 3. Let $n = 3$, $p = 2$. Consider the coherent sheaf of O-modules M generated by

$$f_1(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \quad f_2(x_1, x_2, x_3) = \begin{pmatrix} x_2 \\ x_3^\lambda \end{pmatrix},$$

where $\lambda > 0$ is a fixed integer.

CLAIM 1. *At $0 = (0,0,0)$, M_0 has a set of special generators as follows:*

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ x_3^\lambda \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ x_1 x_3^\lambda \end{pmatrix}.$$

PROOF OF CLAIM 1. Consider the diagram N generated by $\big((1,0,0);1\big)$, $\big((0,1,0);1\big)$ and $\big((1,0,\lambda);2\big)$.

Let $g = \xi_1 f_1 + \xi_2 f_2$, where $\xi_1, \xi_2 \in \mathbb{K}\{x_1, x_2, x_3\}$. We may assume that $\xi_2(x_1, x_2, x_3) = \xi_3(x_2, x_3) + x_1 \xi_4(x_1, x_2, x_3)$. Then

$$g = \xi_1 \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} y \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} 0 \\ x_3^\lambda \end{pmatrix} + x_1 \xi_4 \begin{pmatrix} 0 \\ x_4^\lambda \end{pmatrix}.$$

It is easy to see that

$$\operatorname{support}(\xi_1\begin{pmatrix}x_1\\0\end{pmatrix}) \subset N,\ \operatorname{support}(\xi_2\begin{pmatrix}x_2\\0\end{pmatrix}) \subset N,\ \text{and } \operatorname{support}(x_1\xi_4\begin{pmatrix}0\\x_3^\lambda\end{pmatrix}) \subset N.$$

If the initial form of g is a term of $\xi_3\binom{0}{x_3^\lambda}$, i.e.

$$\operatorname{int}(g) = kx_2^{t_1}x_3^{t_2}\begin{pmatrix}0\\x_3^\lambda\end{pmatrix},$$

which implies that the term

$$kx_2^{t_1}x_3^{t_2}\begin{pmatrix}x_2\\0\end{pmatrix}$$

has been eliminated since

$$\nu(\begin{pmatrix}x_2\\0\end{pmatrix}) < \nu(\begin{pmatrix}0\\x_3^\lambda\end{pmatrix}).$$

But this is impossible because each term of

$$\xi_1\begin{pmatrix}x_1\\0\end{pmatrix}$$

is divisible by x_1. Therefore $\operatorname{int}(g)$ must be a term of

$$\xi_1\begin{pmatrix}x_1\\0\end{pmatrix} + \xi_2\begin{pmatrix}x_2\\0\end{pmatrix} + \xi_4\begin{pmatrix}0\\x_1x_3^\lambda\end{pmatrix}.$$

i.e. $\nu(g) \in N$. This proves Claim 1.

By Theorem 1, it follows that $0 \in \Gamma_\lambda$, since

$$\begin{pmatrix}0\\x_1x_3^\lambda\end{pmatrix} = x_1\begin{pmatrix}x_2\\x_3^\lambda\end{pmatrix} + (-x_2)\begin{pmatrix}x_1\\0\end{pmatrix}.$$

CLAIM 2. *At any point $a = (0, y_0, 0)$, where $y_0 \neq 0$, a set of special generators of M_a can be given as the following:*

$$\begin{pmatrix}y_0\\x_3^\lambda\end{pmatrix}, \begin{pmatrix}0\\x_1x_3^\lambda\end{pmatrix}.$$

PROOF OF CLAIM 2. Let

$$g = \xi_1\begin{pmatrix}y_0\\x_3^\lambda\end{pmatrix} + \xi_2\begin{pmatrix}0\\x_1x_3^\lambda\end{pmatrix}.$$

We may assume that $\xi_1 = \xi_3(x_2, x_3) + x_1\xi_4(x_1, x_2, x_3)$.

Then

$$g = \xi_3 \begin{pmatrix} y_0 \\ 0 \end{pmatrix} + \xi_4 \begin{pmatrix} 0 \\ x_1 x_3^\lambda \end{pmatrix} + \xi_2 \begin{pmatrix} 0 \\ x_1 x_3^\lambda \end{pmatrix} = \xi_3 \begin{pmatrix} y_0 \\ 0 \end{pmatrix} + (\xi_2 + \xi_4) \begin{pmatrix} 0 \\ x_1 x_3^\lambda \end{pmatrix}.$$

We may prove directly that $\nu(g)$ is inside the diagram generated by the initial forms of

$$\begin{pmatrix} y_0 \\ x_3^\lambda \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ x_1 x_3^\lambda \end{pmatrix}.$$

Therefore, Claim 2 is true.

Then by Theorem 1, $(0, y_0, 0) \in \Gamma_\lambda$ since

$$\begin{pmatrix} 0 \\ x_1 x_3^\lambda \end{pmatrix} \notin \begin{pmatrix} y_0 \\ x_3^\lambda \end{pmatrix} \mathbb{K}\{x_1, x_2, x_3\}.$$

Consequently, for each $\lambda \in \mathbb{N}$, there is an example that Γ_λ is not open. ∎

EXAMPLE 4. Let $n = 3, p = 2$. Let M be a coherent sheaf of $\hat{O}$-modules generated by

$$f_1(x_1, x_2, x_3) = \begin{pmatrix} x_1^\lambda \\ 0 \end{pmatrix}, \quad f_2(x_1, x_2, x_3) = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix},$$

where $\lambda > 0$ is a fixed integer.

As we discussed in Example 3, we may prove the following:

At 0=(0,0,0), a set of special generators of M_0 can be given as

$$\begin{pmatrix} x_1^\lambda \\ 0 \end{pmatrix}; \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ x_1^\lambda x_3 \end{pmatrix}.$$

At any point $a = (0, y_0, 0)$, where $y_0 \neq 0$, a set of special generators of M_a can be given as

$$\begin{pmatrix} y_0 \\ x_3 \end{pmatrix} \text{and} \begin{pmatrix} 0 \\ x_1^\lambda x_3 \end{pmatrix}.$$

By Theorem 1,

$$\begin{pmatrix} 0 \\ x_1^\lambda x_3 \end{pmatrix} = (-x_2) \begin{pmatrix} x_1^\lambda \\ 0 \end{pmatrix} + x_1^\lambda \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

implies that $0 \in \Gamma^\lambda$. Finally, if $y_0 \neq 0$,we have

$$\begin{pmatrix} 0 \\ x_1^\lambda x_3 \end{pmatrix} \notin \begin{pmatrix} y_0 \\ x_3 \end{pmatrix} \mathbb{K}\{x_1, x_2, x_3\}$$

which implies $(0, y_0, 0) \notin \Gamma^\lambda$. So for each $\lambda \in \mathbb{N}$, there is an example that Γ^λ is not open. ∎

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