TOPOLOGICAL INVARIANTS OF GERMS OF REAL ANALYTIC FUNCTIONS

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(Received 10 August, 1995)

Let $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be a germ of a real analytic function. Let L and $F(f)$ denote the link of f and the Milnor fibre of $f_{\mathbb{C}}$ respectively, i.e., $L=\{x\in S^{n-1}\mid f(x)=0\}$, $F(f)=f_{\mathbb{C}}^{-1}(\xi)\cap B_r^{2n}$, where $0<\xi\ll r\ll 1$, $B_r^{2n}=\{z\in\mathbb{C}^n \mid \|z\|<r\}$. In [2] Szafraniec introduced the notion of an $\mathcal{A}_d$-germ as a generalization of a germ defined by a weighted homogeneous polynomial satisfying some condition concerning the relation between its degree and weights (definition 1). He also proved that if f is an $\mathcal{A}_d$-germ (presumably with nonisolated singularity) then the number $\chi(F(f))/d \bmod 2$ is a topological invariant of f, where $\chi(F(f))$ is the Euler characteristic of $F(f)$, and gave the formula for $\chi(L)/2 \bmod 2$ (it is a well-known fact that $\chi(L)$ is an even number). As a simple consequence he got the fact that $\chi(F(f)) \bmod 2$ is a topological invariant for any f, which is a generalization of Wall's result [3] (he considered only germs with an isolated singularity).

The aim of this paper is to obtain similar results for a larger class of germs. For this purpose, we shall generalize the notion of an $\mathcal{A}_d$-germ (see Definition 3). For such germs we shall give the formula that relates numbers $\chi(L)/2$ and $\chi(F(f))$. As in [2], we do not assume that f has an isolated singularity (hence L does not have to be smooth).

The author wants to thank Zbigniew Szafraniec for several comments concerning this paper.

Let us first recall results from [2].

Let $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be a germ of a real analytic function.

DEFINITION 1. Let $d\geq 2$ be an integer. We shall say that f is an $\mathcal{A}_d$-germ if there are positive integers $w_1,\ldots,w_n$ all prime to d such that if $f(x)=\sum_\alpha a_\alpha x^\alpha$ and $a_\alpha\neq 0$ then $\alpha_1w_1+\ldots+\alpha_nw_n\equiv d \bmod 2d$.

THEOREM 2. *If f is an $\mathcal{A}_d$-germ then*

(i) $\chi(F(f))\equiv 0 \bmod d$ *and the number* $\chi(F(f))/d \bmod 2$ *is a topological invariant of* f,

(ii) $\chi(L)/2\equiv\chi(F(f))/d+\chi(S^{n-1})/2 \bmod 2$.

EXAMPLE. Let $f(x_1,\ldots,x_n)=x_1^d$, $g(x_1,\ldots,x_n)=x_1^{d-1}x_2$, $d\geq 2$. Then both f and g are $\mathcal{A}_d$-germs and $\chi(F(f))/d\equiv 1 \bmod 2$, $\chi(F(g))/d\equiv 0 \bmod 2$, but for even d $\chi(F(f))\equiv\chi(F(f))\equiv 0 \bmod 2$, hence the number $\chi(F(f))/d \bmod 2$ is more precise invariant then $\chi(F(f)) \bmod 2$.

We shall prove an analogous theorem in the general case, i.e. for arbitrary $w_1,\ldots,w_n$, d, from which one can obtain Theorem 2.

DEFINITION 3. Let $d\geq 2$ be an integer. We shall say that f is a generalized $\mathcal{A}_d$-germ if

† Supported by grant KBN 610/P3/94/07

there are positive integers $w_1, \ldots, w_n$ such that if $f(x) = \sum_\alpha a_\alpha x^\alpha$ and $a_\alpha \neq 0$ then $\alpha_1 w_1 + \ldots + \alpha_n w_n \equiv d \bmod 2d$.

EXAMPLES.

(i) Each germ defined by a weighted homogeneous polynomial of degree d is a generalized $\mathscr{A}_d$-germ.

(ii) The germ defined by the polynomial $f(x, y, z, t) = x^4 + x^{12} + y^2 + z^3 t + z^4 t^4$ is not an $\mathscr{A}_d$-germ for any d, but it is a generalized $\mathscr{A}_8$-germ, where $w_1 = 2$ $w_2 = 4$, $w_3 = 1$, $w_4 = 5$.

(iii) The germ $f(x, y) = x + x^2 + y^2$ is not a generalized $\mathscr{A}_d$-germ for any d.

It turns out that Theorem 2 does not hold in the case of generalized $\mathscr{A}_d$-germs.

EXAMPLE. Let $f(x, y) = x^4 + y^2$. Then f is a generalized $\mathscr{A}_4$-germ for $w_1 = 1$ $w_2 = 2$ (f is a weighted homogeneous polynomial) and f is not an $\mathscr{A}_d$-germ for any d. It is easy to see that $\chi(F(f)) = -2 \not\equiv 0 \bmod 4$.

From now on we shall assume that f is a generalized $\mathscr{A}_d$-germ.

Write $d = p^u v$, where p, u, v are positive integers such that p is prime, v is odd and prime to p (hence if d is even then $p = 2$). Renumbering the variables, if necessary, we may assume that $w_k \equiv 0 \bmod p$ if and only if $k \leq m = m(p)$, where $1 \leq m < n$. For an arbitrary function $\phi : \mathbb{R}^n \to \mathbb{R}$ we will write $\tilde{\phi}$ to denote its restriction to $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$. It is clear that $\tilde{f}$ is a generlized $\mathscr{A}_d$-germ. Moreover, if $f_\mathbb{C}$ has an isolated singularity then also $\tilde{f}_\mathbb{C}$ has one. To see this, consider the linear transformation $H : \mathbb{C}^n \to \mathbb{C}^n$,

$$H(z_1, \ldots, z_n) = (z_1, \ldots, z_m, \exp(2\pi w_{m+1} i/p) z_{m+1}, \ldots, \exp(2\pi w_n i/p) z_n).$$

The subspace fixed by H is $\mathbb{C}^m \times \{0\}$. It is easy to check that for any z fixed by H the vector $\operatorname{grad}(f_\mathbb{C}(z))$ is also fixed by H, hence $\operatorname{grad}(\tilde{f}_\mathbb{C}(z)) = \operatorname{grad}(f_\mathbb{C}(z))$, which proves our claim.

Denote $L = f^{-1}(0) \cap S_r^{n-1}$, $\tilde{L} = \tilde{f}^{-1}(0) \cap S_r^{m-1}$, where S_r^{n-1} (resp. S_r^{m-1}) is a sphere of a small radius r centred at the origin in $\mathbb{R}^n$ (resp. in $\mathbb{R}^m$).

THEOREM 4. *Assume that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is a generalized $\mathscr{A}_d$-germ. Then*

$$\chi(L)/2 + \chi(\tilde{L})/2 \equiv (\chi(F(f)) - \chi(F(\tilde{f})))/p^u + \chi(S^{n-1})/2 + \chi(S^{m-1})/2 \bmod 2.$$

Proof. Set $a = p^u$, $\eta = \exp(\pi i/a) \in \mathbb{C}$ and $\varepsilon = \eta^2$. For $h = 0, 1, \ldots, a-1$ and $z \in \mathbb{C}^n$ we define

$$h(z) = \varepsilon^h . z = (\varepsilon^{hw_1} z_1, \ldots, \varepsilon^{hw_n} z_n).$$

For any $h = 0, \ldots, a-1$ we have $f_\mathbb{C}(h(z)) = f_\mathbb{C}(z)$, hence the group $\mathbb{Z}_a$ acts on $F(f)$. Since f is a real analytic germ, the complex conjugation also acts on $F(f)$. Let G be the dihedral group of order $2a$, i.e. the group generated by elements α, β with the relations $\gamma^2 = 1$, $\beta^a = 1$, $\gamma\beta^h = \beta^{-h}\gamma$, $h \in \mathbb{Z}$. From the above, there is an action of G on the $F(f)$ given by: $\gamma(z) = \bar{z}$, $\beta(z) = \varepsilon . z$.

Suppose that $z \in F(f)$ and that $z = h(z)$ for some $1 \leq h \leq a-1$. Then $z_k = \varepsilon^{hw_k} z_k$ for $k = 1, \ldots, n$. Assume that $z_k \neq 0$ for some k. Then $hw_k \equiv 0 \bmod a$. Since $h \not\equiv 0 \bmod p^r$, it follows that $w_k \equiv 0 \bmod p$. Thus we obtain $z_k = 0$ for $k > m$. We shall write $A_h = \{z \in F(f) \mid h(z) = z\}$. If $h = a/p = p^{u-1}$ then $\varepsilon^{hw_k} z_k = z_k \exp(2\pi w_k i/p)$, hence

$$A_{a/p} = \{z \in F(f) \mid z_k = 0 \text{ for } k > m\} = F(f) \cap (\mathbb{C}^m \times \{0\}).$$

https://doi.org/10.1017/S0017089500031943 Published online by Cambridge University Press

Thus, for $1 \le h \le a-1$ we have $A_h \subset A_{a/p}$. It follows that

$$\bigcup_{h=1}^{a-1} A_h = A_{a/p}. \tag{1}$$

Assume that $h(\bar{z}) = z$ and $0 \le h \le a-1$. Then $z_k = \varepsilon^{hw_k}\bar{z}_k$, $1 \le k \le n$, and consequently $z_k = \eta^{hw_k}x_k$, where $x_k \in \mathbb{R}$ (see [**2**]). Set $x = (x_1, \ldots, x_n)$. It is easy to check that $f_{\mathbb{C}}(z) = (-1)^h f(x)$ (it follows from the fact that v is odd). Let $B_h = \{z \in F(f) \mid h(\bar{z}) = z\}$, $C_h = \{z \in F(-f) \mid h(\bar{z}) = z\}$. Then $B_0 = F(f) \cap \mathbb{R}^n$ and $C_0 = F(-f) \cap \mathbb{R}^n$. Hence

$$\chi(B_h) = \begin{cases} \chi(B_0) & \text{if } h \text{ is even} \\ \chi(C_0) & \text{if } h \text{ is odd.} \end{cases} \tag{2}$$

We next claim that if $0 \le h < h' \le a-1$ (*) then

$$B_h \cap B_{h'} = A_{a/p} \cap B_h \cap B_{h'}. \tag{3}$$

Let $z \in B_h \cap B_{h'}$. Choose $k > m$. Then $z_k = \eta^{hw_k}x_k = \eta^{h'w_k}x_k'$ where $x_k, x_k' \in \mathbb{R}$. This implies that $|x| = |x_k'|$. Suppose that $x_k \ne 0$. We thus get $w_k(h'-h) \equiv 0 \bmod p^u$. Since $w_k \not\equiv 0 \bmod p$, it follows that $h' - h \equiv 0 \bmod p^u$, which contradicts assumption (*).

Our next goal is to calculate $\chi\left(\bigcup_{h=1}^{a-1} A_h \cup \bigcup_{j=0}^{a-1} B_j\right)$. By (1),

$$\bigcup_{h=1}^{a-1} A_h \cup \bigcup_{j=0}^{a-1} B_j = A_{a/p} \cup \bigcup_{j=0}^{a-1} B_j.$$

To simplify notation, we write B_a instead of $A_{a/p}$. Clearly,

$$\chi\left(\bigcup_{j=0}^{a} B_j\right) = \sum_{q=1}^{a+1} (-1)^{q-1} S_q,$$

where $S_q = \sum_J T_J$, $J = (j_1, \ldots, j_q)$, $0 \le j_1 < \ldots < j_q \le a$, and $T_J = \chi(B_{j_1} \cap \ldots \cap B_{j_q})$. We may write $S_q = \sum_H T_H + \sum_I T_I$, where $H = (h_1, \ldots, h_q)$, $0 \le h_1 < \ldots < h_q = a$, $I = (i_1, \ldots, i_q)$, $0 \le i_1 < \ldots < i_q < a$. Thus

$$S_{q+1} = \sum_{I_a} T_{I_a} + \sum_{I_1} T_{I_1},$$

where $I_a = (i_1, \ldots, i_q, a)$, $I_1 = (i_1', \ldots, i_{q+1}')$, $0 \le i_1' < \ldots < i_{q+1}' < a$. If $q = a$ then

$$S_{q+1} = S_{a+1} = \chi\left(\bigcap_{j=0}^{a} B_j\right).$$

By (3), $\sum_I T_I = \sum_{I_a} T_{I_a}$ and consequently

$$\chi\left(\bigcup_{j=0}^{a} B_j\right) = \sum_{j=0}^{a} \chi(B_j) - \sum_{h=0}^{a-1} \chi(B_h \cap B_a).$$

Let a_+ (resp. a_-) denote the number of integers h such that $0 \le h \le a-1$ and h is even (resp. odd). Then applying (2) we obtain

$$\chi\left(\bigcup_{j=0}^{a} B_j\right) = a_+\chi(B_0) + a_-\chi(C_0) + \chi(B_a) - \sum_{h=0}^{a-1} \chi(B_h \cap B_a)$$

G acts on $F(f) - \bigcup_{j=0}^{a} B_j$ freely and has 2a elements, hence

$$\chi(F(f)) \equiv \chi\left(\bigcup_{j=0}^{a} B_j\right) \bmod 2a.$$

It is easy to check that $B_h \cap B_a = B_{h'} \cap B_a$, where $h = 0, 1, \ldots, a/p - 1$ and $h' = h + a/p$. Thus

$$\chi(F(f)) \equiv a_+\chi(B_0) + a_-\chi(C_0) + \chi(B_a) - p \sum_{h=0}^{a/p-1} \chi(B_h \cap B_a) \bmod 2a.$$

By definition $\tilde{f} : \mathbb{R}^m \to \mathbb{R}$. For $z' = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and $h = 0, 1, \ldots, a/p - 1$, we define $h(z') = (\varepsilon^{hw_1} z_1, \ldots, \varepsilon^{hw_m} z_m)$, $\tilde{B}_h = \{z' \in F(\tilde{f}) \mid h(\bar{z}') = z'\}$, $\tilde{C}_h = \{z' \in F(-\tilde{f}) \mid h(\bar{z}') = z'\}$. Using the same arguments as above one can prove that

$$\chi(\tilde{B}_h) = \begin{cases} \chi(\tilde{B}_0) & \text{if } h \text{ is even} \\ \chi(\tilde{C}_0) & \text{if } h \text{ is odd.} \end{cases}$$

Clearly $\chi(B_a) = \chi(F(\tilde{f}))$ and $\chi(\tilde{B}_h) = \chi(B_h \cap B_a)$ for $h = 0, \ldots, a/p - 1$. Let $\tilde{a}_+$ (resp. $\tilde{a}_-$) denote the number of integers h such that $0 \le h \le a/p - 1$ and h is even (resp. odd). Thus we can write

$$\chi(F(f)) \equiv a_+\chi(B_0) + a_-\chi(C_0) + \chi(F(\tilde{f})) - p(\tilde{a}_+\chi(\tilde{B}_0) + \tilde{a}_-\chi(\tilde{C}_0)) \bmod 2a.$$

If d is even then $p = 2$ and $a_+ = a_- = a/2$. If d is odd then $(x_1, \ldots, x_n) \mapsto ((-1)^{w_1}x_1, \ldots, (-1)^{w_n}x_n)$ maps B_0 homeomorphically onto C_0. Then $a\chi(B_0) = a\chi(C_0) = a(\chi(B_0) + \chi(C_0))/2$ (similarly for $\tilde{f}$). Hence in both cases we obtain

$$\chi(F(f)) \equiv a(\chi(B_0) + \chi(C_0))/2 + \chi(F(\tilde{f})) - a(\chi(\tilde{B}_0) + \chi(\tilde{C}_0))/2 \bmod 2a. \tag{4}$$

From the Alexander duality theorem we have

$$\chi(L) = \chi(S_r^{n-1}) + (-1)^n \chi(S_r^{n-1} - L)$$

Clearly,

$$\chi(S_r^{n-1} - L) = \chi(\{f > 0\} \cap S_r^{n-1}) + \chi(\{f < 0\} \cap S_r^{n-1}).$$

It is a well-known fact that the set $\{f > 0\} \cap S_r^{n-1}$ (resp. $\{f < 0\} \cap S_r^{n-1}$) is homeomorphic to B_0 (resp. C_0). Thus

$$\chi(S_r^{n-1} - L) = \chi(B_0) + \chi(C_0),$$

and consequently

$$\chi(B_0) + \chi(C_0) = (-1)^n(\chi(L) - \chi(S_r^{n-1}))$$

(the same reasoning applied to $\tilde{f}$). Hence we may rewrite (4) in the following form:

$$a(-1)^n\chi(L)/2 + a(-1)^m\chi(\tilde{L})/2 \equiv \chi(F(f)) - \chi(F(\tilde{f})) + a((-1)^n\chi(S_r^{n-1}) + a(-1)^m\chi(S_r^{m-1}))/2 \bmod 2a.$$

Dividing by a we obtain

$$\chi(L)/2 + \chi(\tilde{L})/2 \equiv (\chi(F(f)) - \chi(F(\tilde{f})))/a + \chi(S^{n-1})/2 + \chi(S^{m-1})/2 \bmod 2.$$

which proves the theorem.

Let us recall that $d = p^u v$, where v is odd and prime to d. If each w_k is prime to d, then the group $\mathbb{Z}_d$ acts freely on $F(f)$, hence $\chi(F(f)) \equiv 0 \bmod d$. For v is odd, $\chi(F(f))/p^u \equiv \chi(F(f))/p^u v \bmod 2$. Clearly, $\chi(F(\tilde{f})) = \chi(\tilde{L}) = 0$. This follows that $\chi(L)/2 \equiv \chi(F(f))/d + \chi(S^{n-1})/2 \bmod 2$, hence we have proved Theorem 2 as a consequence of Theorem 4, as claimed.

Clearly, repeated application of Theorem 3 enables us to express $\chi(L)/2 \bmod 2$ only in terms of Euler characteristics of Milnor fibers of appropriate restrictions of $f_{\mathbb{C}}$. It is possible to write an explicit formula for $\chi(L)/2 \bmod 2$, however it requires to introduce some further notation.

REFERENCES

1. J. Milnor, *Singular points of complex hypersurfaces* (Princeton University Press 1968).
2. Z. Szafraniec, On the topological invariants of germs of analytic functions *Topology* **26** (1987), 235–238.
3. C. T. C. Wall, Topological invariance of the Milnor number mod 2, *Topology* **22** (1983), 345–350.

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https://doi.org/10.1017/S0017089500031943 Published online by Cambridge University Press