ON A CONJECTURE BY J. H. CHUNG

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1. Introduction. The present paper is a sequel to that of J. H. Chung (**2**) and contains a proof of a conjecture made by him, namely, that *the number of ordinary (modular) irreducible representations contained in a given p-block of S_n is independent of the p-core.* A summary of the results contained herein appeared in the Proceedings of the National Academy of Sciences (**9**).

The Main Theorem on which the proof of the conjecture depends is of some interest. It was stated without proof by Nakayama and Osima (**6**, p. 115) and obtained independently by the author (**9**). It has been known for some time that the p-hook structure of a Young diagram $[\alpha]$ is given by a certain star diagram (**7**; also **6**, **9**, **10**, **11**), with each of whose p disjoint constituents can be associated a unique residue class modulo p. What was not realized is that the converse theorem is also valid. The first part of Chung's conjecture concerning ordinary representations follows immediately from this theorem, while the proof of the second part concerning modular representations requires a detailed study of the independence of Chung's identities which are satisfied by the rows of the decomposition matrix $D = (d_{ij})$.

Note added April 1, 1952. Since the writing of the present paper the problem of the modular representations of the symmetric group has been approached from a fresh point of view by D. E. Littlewood (Proc. Royal Soc. (London), (A), vol. 209 (1951), 333-353) and the present author (Proc. Nat. Acad. Sci., vol. 38 (1952), 129-133; 424-426). Chung's characterization of an indecomposable has been made explicit, while new light has been thrown on the modular irreducible representations themselves. So far as the contents of this paper is concerned, the new approach associates a residue (mod p) with *each* node of a disjoint constituent of the star diagram rather than with the upper left-hand corner node only, but the proof of the Main Theorem is largely unaffected.

2. The Main Theorem. The notion of a p-hook was introduced by Nakayama (**4**) and has proved most fruitful in studying the modular representation theory of the symmetric group S_n. If as many p-hooks as possible are removed from a diagram $[\alpha]$ containing n nodes, then the residue $[\alpha_0]$ is called the *p-core* of $[\alpha]$ and we have the basic theorem of the modular theory (**4**, p. 423; **1**; **6**):

2.1. *Two irreducible representations* $[\alpha]$ *and* $[\beta]$ *of* S_n *belong to the same p-block if and only if they have the same p-core.*

To proceed further it is necessary to know something of the p-hook structure of $[\alpha]$, i.e., of the star diagram $[\alpha]^*$ of $[\alpha]$. If b p-hooks are removable from $[\alpha]$

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we shall say (**9**) that $[\alpha]$ is of *weight b*. We shall write

2.2 $$[\alpha]^*_p = [\lambda_0]\cdot[\lambda_1]\cdot \ldots \cdot[\lambda_{p-1}],$$

where the $[\lambda_r]$ are the disjoint right constituents of $[\alpha]^*$ which are determined explicitly in terms of the p-chains (**7**; **10**) of $[\alpha]$. We assume that $[\lambda_r]$ contains b_r nodes, where

2.3 $$b = b_0 + b_1 + \ldots + b_{p-1},$$

and r is the leg length of the p-hook represented by its upper left-hand corner node. We shall call r the *class* of the constituent $[\lambda_r]$. We prove the following

2.4. MAIN THEOREM. *A diagram* $[\alpha]$ *of weight b exists and is uniquely determined by assigning*:

(i) *its p-core* $[\alpha_0]$;
(ii) *its star diagram* $[\alpha]^*$;
(iii) *the class of each disjoint constituent* $[\lambda_r]$.

As a preliminary to the proof it is necessary to remind the reader of a fundamental result of Young which can be stated in the form (**7**, p. 283)

2.5 $$[\alpha] = \sum [\alpha'],$$

where $[\alpha']$ is obtained by removing a node from $[\alpha]$, and the summation indicates that this is to be done in all possible ways. From a group-theoretic point of view, the equation 2.5 gives the reduction of $[\alpha]$ when S_n is restricted to the subgroup S_{n-1}. It is convenient to describe this process of removing a node as *differentiating* $[\alpha]$; conversely, we may speak of $[\alpha]$ as the *integral* of the set of $[\alpha']$'s (**2**, p. 337; **3**).

Proof. (a) *Suppose first that* $[\alpha]^*$ *has just one constituent* $[\lambda]$ *and it is of class r*. Certainly the theorem is true for $b = 1$, as was shown by Nakayama (**4**) and Chung (**2**); we shall assume it to be true for all diagrams of weight less than $b > 1$ and prove it for diagrams $[\alpha]$ of weight b. Let us differentiate $[\lambda]$, obtaining a derived set $[\lambda']_i$, where $i = 1, 2, \ldots, k$.

Assume that $k > 1$. Clearly the class of each $[\lambda']_i$ is also r, since this depends on the last p-hook removable, i.e., on the upper left-hand corner node of $[\lambda]$. By our inductive assumption, a unique diagram $[\alpha']_i$ is associated with each such $[\lambda']_i$ of class r. Since each node removable from $[\lambda]$ belongs to the same constituent, these nodes represent (**4**, §§4, 5; **7**, p. 288) non-overlapping p-hooks of $[\alpha]$. Thus each $[\alpha']_i$ can be obtained by adding $k - 1$ non-overlapping p-hooks to a certain diagram $[\bar{\alpha}]$ which is the residue left when all k p-hooks are removed from $[\alpha]$. Considering the set of k diagrams $[\alpha']_i$ together, it follows that the k p-hooks can be added simultaneously to $[\bar{\alpha}]$ to yield $[\alpha]$. The uniqueness of $[\alpha]$ is thus proved for $k \geqslant 2$.

Assume that $k = 1$. In this case we know that $[\lambda]$ must be of the form $[x^y]$. Again by our inductive hypothesis, we know that a unique diagram exists having as star diagram $[\bar{\lambda}] = [(x-1)^{y-1}]$. Such a diagram is an $(x+y-1)p$-core,[1] since $[(x-1)^{y-1}]$ is an $(x+y-1)$-core, and so there can be added to this core an $(x+y-1)p$-hook chosen so as to have the desired representation $[\lambda] - [\bar{\lambda}]$, of class r. This completes the proof of the theorem in case (a).

(b) *If* $[\alpha]^*$ *has more than one constituent* $[\lambda]$, we may again consider the set of k derived diagrams which we can divide into at most p subsets with k_r ($r = 0, 1, \ldots, p-1$) diagrams in the rth set which differ only in the constituent of class r.

Assume that $k_r > 1$ *and* $k_s > 1$. The argument of case (a) is applicable to the rth set and we obtain a unique diagram $[\alpha^r]$. Similarly, we obtain a unique diagram $[\alpha^s]$ from the sth set. In order to prove that $[\alpha^r] = [\alpha^s]$ we note that the two diagrams can differ only in the positions of at most two p-hooks, in view of our inductive assumption; and each of these can be added separately to a common part $[\alpha_1]$ in a unique manner. If these added p-hooks do not overlap then certainly $[\alpha^r] = [\alpha^s]$. If they do overlap, they can be added in succession, the resulting diagram being obtained (**4**, §3) by moving the common part down and to the right one place, leaving the non-overlapping parts in their original positions. Thus again, $[\alpha^r] = [\alpha^s]$.

The extension of the argument to the case where one or more of the k_r's is unity presents no added difficulty, and after at most p steps the theorem is proved in case (b) also.

It is to be noted that the conditions (i), (ii), (iii) in the above theorem are independent. The significance of (i) has already been seen in 2.1. With regard to (ii) and (iii), it is clear that each partition 2.3 of b leads to a class of star diagrams and the number of diagrams in each class depends again on the number of partitions of each b_r. It is also evident that each b_r is to be associated with a residue class r modulo p, and the number of representations belonging to a given block will depend on the number of ways this association can be made. To sum up, we have the following corollary of the Main Theorem:

2.6. *The number* ρ *of ordinary irreducible representations in any p-block of weight b is independent of the p-core and is determined by the conditions* (ii) *and* (iii), *i.e., by the number of possible star diagrams and the number of different ways the p distinct residue classes can be associated with the disjoint constituents.*

3. Chung's identities between the rows of D**.** In the general modular theory the splitting of an ordinary irreducible representation of a finite group G into its modular components is given by the decomposition matrix $D = (d_{ij})$. If we suppose the representations arranged in blocks B_k, the matrix D takes the form

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[1]Nakayama's original theorem (**5**, p. 414) was phrased so as to apply to hooks of prime length p, but it and the hook structure theorem (**7**, p. 287) defining the star diagram, as well as the converse being proved here, are all valid for hooks of composite length. The argument at this point in the proof requires just this generality.

3.1 $$\begin{pmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & \cdots & \\ 0 & & & D_\omega \end{pmatrix},$$

where the reduction of the representations belonging to B_k is given by D_k. For the symmetric group this arrangement is easy to achieve in view of 2.1.

Since the indecomposable representations of the regular representation of G, or alternatively, the modular representations of G belonging to B_k, are associated with the columns of D_k, it follows that there will be a set of identities holding between the rows of D_k. The number of these identities for D as a whole is $\rho - \sigma$, where ρ is the number of classes of G and σ is the number of these which are p-regular. *Thus $\rho - \sigma$ is equal to the number of p-singular classes of G.* If we denote by ρ_k the number of ordinary irreducible representations in the block B_k, and by σ_k the number of indecomposables of B_k, then we have

3.2 $$\rho - \sigma = \sum_{k=1}^{\omega} (\rho_k - \sigma_k),$$

where the number of identities satisfied by the rows of D_k is $\rho_k - \sigma_k$, in view of 3.1. We can construct the following table, which, though it does not go very far, yields a convenient summary of our knowledge at the present time (**4**; **2**, p. 235).[2]

3.3

Weight	ρ_k	σ_k	$\rho_k - \sigma_k$
0	1	1	0
1	p	$p-1$	1
2	$\frac{1}{2}p(p+3)$	$\frac{1}{2}(p-1)(p+2)$	$p+1$

In what follows we shall be concerned, for the most part, with representations belonging to a fixed block of weight b, so we may drop the subscript k.

The problem of actually constructing the set of identities associated with B was solved by Chung (**2**, §3). He began with the fundamental character relation

3.4 $$\sum_{\alpha} \chi_\alpha(R)\chi_\alpha(S) = 0, \qquad \text{all } [\alpha],$$

which is to remain valid for any p-regular element of S of S_n. Thus R must be p-singular and we may set it equal to $P_k V$, where P_k is a cycle of length kp. Applying the Murnaghan-Nakayama recursion formula and the orthogonality relations for the characters of $S_n - kp$, we obtain

3.5 $$\sum_{\alpha} a_{\alpha\beta_k}\chi_\alpha(S) = 0 \qquad (k = 1, 2, \ldots, b),\ [\alpha] \subset \mathrm{B},$$

[2]Chung's determination of the *number* of ordinary and modular irreducible representations belonging to a p-block of weight 2 with zero core is applicable to the case of arbitrary p-core, in consequence of the theorems proved in this paper.

where $[\beta_k]$ is the residual diagram left after removing a kp-hook of parity $a_{\alpha\beta k}$ from $[\alpha]$. Of course all $[\beta_k]$ have p-core $[\alpha_0]$, so *the summation in* 3.5 *may be limited to those* $[\alpha]$*'s belonging to* B. Writing

$$\chi_\alpha(S) = \sum_\lambda d_{\alpha\lambda}\phi_\lambda(S),$$

where $\phi_\lambda(S)$ is the character of S in the irreducible modular representation λ, Chung obtained the identities

3.6 $$\sum_\alpha a_{\alpha\beta_k} d_{\alpha\lambda} = 0 \qquad (k = 1, 2, \ldots, b), [\alpha] \subset \text{B},$$

for all λ in B.

Conversely, let us assume the existence of a relation

3.7 $$\sum_\alpha a_\alpha d_{\alpha\lambda} = 0, \qquad [\alpha] \subset \text{B}.$$

Retracing our steps, we conclude the existence of a relation

3.8 $$\sum_\alpha a_\alpha \chi_\alpha(S) = 0, \qquad [\alpha] \subset \text{B},$$

for any p-regular element S of S_n. If we consider the columns of the character table as a mutually orthogonal set of vectors spannnig the space, then it follows that *any equation* 3.8 *must be linearly dependent on the set of equations* 3.4. But we have seen how these can be broken up into sets of equations 3.6 applicable to the separate blocks. Thus we have proved that

3.9. *Chung's system of identities satisfied by the rows of D is complete.*

4. Star ordering of Young diagrams. Let us denote the set of Young diagrams $[\alpha]$ belonging to B, or alternatively, the set of their representative star diagrams according to 2.4, by the symbol (b, α_0). If we remove a p-hook from each $[\alpha]$, i.e., a node from each of the representative star diagrams of (b, α_0), it is not difficult to see that the residual diagrams $[\beta_1]$ are just those of the set $(b - 1, \alpha_0)$. Similarly, removing a $2p$-hook, where possible, from the $[\alpha]$'s yields the diagrams $[\beta_2]$ of the set $(b - 2, \alpha_0)$. That we can identify the $[\beta_k]$ in this simple way follows from our Main Theorem. Proceeding in this manner we can write Chung's relation matrix in the form

4.1 $$A = (a_{\alpha\beta_k})\begin{cases}[\alpha] \text{ ranging over the set } (b, \alpha_0),\\ [\beta] \text{ ranging over the set } (b - k, \alpha_0) \text{ with } k = 1, 2, \ldots, b.\end{cases}$$

The coefficients $a_{\alpha\beta k}$ are $\pm$ 1 or zero according as the kp-hook removed is of even or odd leg length or does not exist, to yield the residual diagram $[\beta_k]$ of the set $(b - k, \alpha_0)$ at the head of the column in question. We shall denote the matrix consisting of rows (b, α_0) and columns $(b - k, \alpha_0)$ by $M(b - k, \alpha_0)$. The equations 3.6 may be summarized in the equation

4.2 $$A'D = 0,$$

where A' is the transpose of A.

Chung arranged the $[\alpha]$'s of (b, α_0) according to their *natural* or dictionary ordering. Largely for its intrinsic interest we define now a *star ordering* which will depend on the ordering of the associated star diagrams. For a diagram $[\beta]$ we may write

4.3 $$[\beta]_p^* = [\mu_0]\cdot[\mu_1]\cdot \ldots \cdot[\mu_{p-1}]$$

as in 2.2, where $[\mu_r]$ contains c_r nodes and

4.4 $$b = c_0 + c_1 + \ldots + c_{p-1}.$$

We shall say that $[\alpha]$ *precedes* $[\beta]$ *in the star order if:*

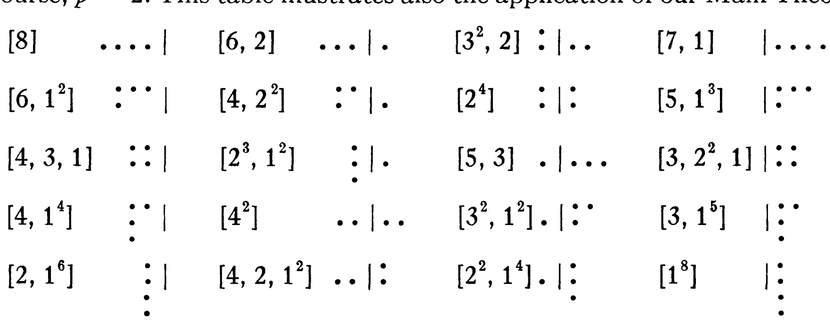
(i) $b_0 = c_0,\ b_1 = c_1, \ldots, b_r > c_r$;

or if

(ii) $b_r = c_r\ (r = 0, 1, \ldots, p-1)$ *and*

$[\lambda_0] = [\mu_0],\ [\lambda_1] = [\mu_1], \ldots, [\lambda_s]$ *precedes* $[\mu_s]$ *in the natural order.*

As an illustration we shall rearrange the rows of Chung's table (**2**, p. 317) according to this ordering. The constituent of the star diagram preceding the vertical stroke is assumed to be of class 0, and that following it of class 1; here, of course, $p = 2$. This table illustrates also the application of our Main Theorem.



If, in addition to the rows, the columns of each set $(b - k, \alpha_0)$ are rearranged in star order, then we shall denote the resulting matrix by A^*. Similarly we may star order the rows of D to obtain D^*. Of course these changes could have been made by using a transforming matrix, but it is clear that the relation $A^{*\prime} D^* = 0$ still holds.

5. The independence of Chung's identities. The proof of the following lemma follows immediately from the independence of the representations of the set $(b - k, \alpha_0)$.

5.1. *The columns belonging to any set* $(b - k, \alpha_0)$ *of* A, *that is, of* $M(b - k, \alpha_0)$, *are linearly independent.*

While the columns of any given set are linearly independent, yet the columns belonging to different sets may very well be dependent, as Chung showed. The question arises: do Chung's relations between the columns of different sets exhaust the possibilities?

As referred to already, the character of any element R in $[\alpha]$ can be obtained by considering the removal of successive hooks from $[\alpha]$ which correspond to the cycles of R (**7**, p. 290). The $\pm$ 1's which appear in the row $[\alpha]$ of A and lie in $M(b - k, \alpha_0)$ pick out those representations of S_{n-kp} which contribute to $\chi_\alpha(R)$. Thus for columns of $M(b - k_1, \alpha_0)$ to be dependent on columns of $M(b - k_2, \alpha_0)$, with $k_1 \neq k_2$, implies a relation of the form

$$5.2 \qquad \sum_i a_i \operatorname{col}_i M(b - k_1, \alpha_0) = \sum_j b_j \operatorname{col}_j M(b - k_2, \alpha_0).$$

To say that 5.2 is valid for all $[\alpha]$ of B means that it is valid for those elements of the form $P_{k_1} P_{k_2} V$ which contribute to both sides of the equation. For such elements we can represent both sides of 5.2 in the form

$$5.3 \qquad \sum_x c_x \operatorname{col}_x M(b - k_1 - k_2, \alpha_0)^\circ$$

where

$$5.4 \qquad M(b - k_1 - k_2, \alpha_0)^\circ = M(b - k_1, \alpha_0) M(\overline{b - k_1} - k_2, \alpha_0)$$
$$= M(b - k_2, \alpha_0) M(\overline{b - k_2} - k_1, \alpha_0).$$

The expression 5.3 implies the restriction of the k's given by Chung (**2**, p. 220). Clearly the number of independent expressions 5.3 is just the number of irreducible representations of $S_{n-k_1p-k_2p}$ with p-core $[\alpha_0]$, i.e., the number of members of the set $(b - k_1 - k_2, \alpha_0)$. Taking all possible k_1 and k_2 we obtain precisely Chung's relations between the identities.

5.5. *All linear relations between the columns of A are expressible in terms of the independent relations obtained by Chung.*

In §3 we denoted the number of representations belonging to a block B_i by ρ_i, and the number of indecomposables or modular irreducible representations belonging to B_i by σ_i. Thus the number of identities satisfied by the rows of D_i will be $\rho_i - \sigma_i$, and it follows from 5.1 and 5.5 that this number is equal to the number of Chung's identities which are linearly independent. According to Chung's procedure, this number $\rho_i - \sigma_i$ is a function of the following two determinations:

(a) The number of distinct sets $k_1, k_2, \ldots, k_r$ with $k_i \neq k_j$, $i \neq j$, and $k_i > 0$ for all i and j, and such that $\sum k_i \leqslant b$;

(b) The number of members of the sets $(b - k, \alpha_0)$.

Now it is clear that the number of solutions of (a) depends only on n and b and not on $[\alpha_0]$, and the number of members of $(b - k, \alpha_0)$ in (b) is also independent of $[\alpha_0]$ by our Main Theorem. Thus we have the following result:

5.6. *The number of indecomposables or modularly irreducible representations belonging to any p-block is independent of the p-core.*

The two results 2.6 and 5.6 provide a proof of Chung's conjecture.

REFERENCES

1. R. Brauer and G. de B. Robinson, *On a conjecture by Nakayama*, Trans. Royal Soc. Canada, Sec. III, vol. 40 (1947), 11-25.
2. J. H. Chung, *On the modular representations of the symmetric group*, Can. J. Math., vol. 3 (1951), 309-327.
3. F. D. Murnaghan, *The characters of the symmetric group*, Proc. Nat. Acad. Sci., vol. 37 (1951), 55-58.
4. T. Nakayama, *On some modular properties of irreducible representations of a symmetric group* I, Jap. J. Math., vol. 17 (1941), 89-108.
5. ———, II, *ibid*, 411-423.
6. T. Nakayama and M. Osima, *Note on blocks of symmetric groups*, Nagoya Math. J., vol. 2 (1951), 111-117.
7. G. de B. Robinson, *On the representations of the symmetric group* III, Amer. J. Math., vol. 70 (1948), 277-294.
8. ———, *Induced representations and invariants*, Can. J. Math., vol. 2 (1950), 334-343.
9. ———, *On the modular representations of the symmetric group*, Proc. Nat. Acad. Sci., vol. 37 (1951), 694-696.
10. R. A. Staal, *Star diagrams and the symmetric group*, Can. J. Math., vol. 2 (1950), 79-92.
11. R. M. Thrall and G. de B. Robinson, *Supplement to a paper by G. de B. Robinson*, Amer. J. Math., vol. 73 (1951), 721-724 (cf. **7** above).

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