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ON DIRECT PRODUCTS OF ABELIAN GROUPS

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In this paper we investigate the properties of the product (or complete direct sum) of torsion Abelian groups. The chief results concern products of Abelian primary groups (p-groups). Given a set of p-groups, $[G_\lambda]$, over an index set Λ, the product of these groups is written $\prod_{\lambda\in\Lambda}G_\lambda$, the torsion subgroup of the product of these p-groups is written $T[\prod G_\lambda]$, and the discrete direct sum of the p-groups is written $\sum G_\lambda$.

Definition. $\sum G_\lambda$ is said to be an essentially bounded decomposition if and only if there exists an integer $M > 0$ such that $MG_\lambda = 0$ for all but a finite number of G_λs; otherwise the decomposition is essentially unbounded.

Notation, for the most part, will be that of Fuchs [**1**].

The main results of this paper are the following.

(1) The cardinal number of $\prod G_\lambda$ equals the cardinal number of $T[\prod G_\lambda]$.

(2) $T[\prod G_\lambda]$ is torsion-complete if and only if each G_λ is torsion-complete.

(3) If the set of p-groups $[G_\lambda]$ is reduced, then the following are equivalent:

(a) $\sum G_\lambda$ is an essentially bounded decomposition,

(b) $\prod G$ equals $T[\prod G_\lambda]$,

(c) $T[\prod G_\lambda]$ is a direct summand of $\prod G_\lambda$,

(d) The quotient group $\prod G_\lambda/T[\prod G_\lambda]$ is reduced.

(4) For reduced p-groups, $[G_\lambda]$, the quotient group $T[\prod G_\lambda]/\sum G_\lambda$ is divisible if and only if a basic subgroup of $\sum G_\lambda$ is also basic in $T[\prod G_\lambda]$.

(5) For (reduced) p-groups $[G_\lambda]$, the following are equivalent:

(a) $T[\prod G_\lambda]/\sum G_\lambda$ is reduced,

(b) $\sum G_\lambda$ is an essentially bounded decomposition,

(c) $T(\prod G_\lambda)/\sum G_\lambda$ is bounded.

(6) If $T(\prod_1^\infty G_n)$ is a reduced p-group, it has an essentially unbounded decomposition if and only if some G_n has an essentially unbounded decomposition.

(7) If $T(\prod_1^\infty G_n)$ equals an infinite direct sum of isomorphic groups where all G_ns are countable reduced p-groups, then $\sum_1^\infty G_n$ is essentially bounded.

(8) A countably infinite direct product of isomorphic p-groups can be decomposed into an infinite direct sum of isomorphic groups if and only if the product is the direct sum of a divisible group and a bounded group.

Lemmas which are proved in this paper and which are important in their own right are the following.

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(1) *If $G = A \oplus B = C \oplus D$ are two direct sum decompositions of an Abelian group G, where C is an unbounded direct sum of cyclic groups of infinite rank, then A or B contains a direct summand which is a direct sum of cyclic groups of infinite rank.*

(2) *If $G = A \oplus \bar{B} = C \oplus D$ is an Abelian group, where $\bar{B}$ is an unbounded torsion-complete p-group, then C or D contains a summand which is an unbounded torsion-complete p-group.*

(3) *If $G = A \oplus B$ is an Abelian p-group without elements of infinite height, and G contains an unbounded torsion-complete group, then A or B contains an unbounded torsion-complete group.*

(4) *Every unbounded pure subgroup of a direct sum of cyclic p-groups contains an unbounded summand of the group.*

1. Preliminary propositions. The following propositions are interesting in their own right or will be used in subsequent parts of the paper. Proofs will be omitted whenever they are obvious.

1.1. PROPOSITION. *If $[G_\lambda]$ is a set of torsion groups, where $G_\lambda = D_\lambda \oplus R_\lambda$, D_λ divisible, R_λ reduced, then $\prod G_\lambda = \prod D_\lambda \oplus \prod R_\lambda$, where the first summand is divisible, and the second is reduced.*

1.2. PROPOSITION. *Let $G_\lambda = \sum_{i=1}^{\infty} G_{\lambda p_i}$ be a decomposition of a torsion group G_λ into a direct sum of its primary components for each λ in an index set Λ and where $p_1 < p_2 < \ldots$ is a set of prime numbers. Then*

$$T(\prod G_\lambda) = \sum_{i=1}^{\infty} T(\prod_\Lambda G_{\lambda p_i}).$$

Proof. The torsion subgroup of a product is certainly the direct sum of its primary components. Now, for given p_i, the p_i-component of $T(\prod G_\lambda)$ in our primary sum decomposition is clearly $T(\prod_\Lambda G_{\lambda p_i})$.

It is due to these first two propositions that our study deals with the complete direct sum of groups which are usually reduced and always p-groups, unless otherwise noted.

1.3. PROPOSITION. *If $[G_\lambda]$ is a set of p-groups and D_λ is the divisible hull of G_λ for each λ in Λ, then $T(\prod D_\lambda)$ is the divisible hull of $T(\prod G_\lambda)$.*

Proof. This is clear, once we observe that $T(\prod D_\lambda)[p] = \prod(D_\lambda[p]) = \prod G_\lambda[p] = T(\prod G_\lambda)[p]$, and that $T(\prod D_\lambda)$ is divisible.

Remark. Notice that $\prod D_\lambda$ need not be the divisible hull of $\prod G_\lambda$. To see this, consider the case where each G_λ is cyclic of order p. Then if Λ has infinite cardinality, $\prod G_\lambda$ is bounded while $\prod D_\lambda$ is mixed.

1.4a. PROPOSITION. *Given p-groups* $[G_\lambda]_\Lambda$, *let* $G_\lambda = S_{\lambda n} + G_{\lambda n}$, *where* $S_{\lambda n}$ *is a maximal* p^n*-bounded direct summand of* G_λ *for every* $\lambda \in \Lambda$. *Then* $\prod_\Lambda S_{\lambda n}$ *is a maximal* p^n*-bounded direct summand of* $T = T(\prod G_\lambda)$.

Proof. That $T = T(\prod_\Lambda S_{\lambda n}) \oplus T(\prod_\Lambda G_{\lambda n}) = \prod_\Lambda S_{\lambda n} \oplus T(\prod_\Lambda G_{\lambda n})$ is obvious. Now suppose that $\langle x\rangle$ is a direct summand of $T(\prod_\Lambda G_{\lambda n})$ and $o(x) = p^k \leqq p^n$. Let $x = (g_1, g_2, \ldots, g_\lambda, \ldots)$, $g_\lambda \in G_{\lambda n}$. Some g_λ in this expansion, say g_i, generates a pure cycle $\langle g_i\rangle$ of order p^k in G_{in}. Hence $\langle g_i\rangle$ is a direct summand of G_{in} and $S_{in} \oplus \langle g_i\rangle$ is a larger p^n-bounded direct summand of G_i than S_{in}.

1.4b. PROPOSITION. *Given p-groups* $[G_\lambda]_\Lambda$, *let* $B_\lambda = \sum_{n=1}^\infty B_{\lambda n}$ *be a basic subgroup of* G_λ, *where* $B_{\lambda n}$ *is a direct sum of cyclic groups of order* p^n, *for each* λ *in* Λ. *Then* $\hat{B} = \sum_{n=1}^\infty \prod_\Lambda B_{\lambda n}$ *is basic in* $T = T(\prod G_\lambda)$.

Proof. $\hat{B}$ is clearly pure in T and a direct sum of cyclic groups. We must show that $T/\hat{B}$ is divisible. Let x in T be mapped to $\bar{x}$ in $T/\hat{B}$. Let $o(x) = p^k$, $x = (g_1, g_2, \ldots, g_\lambda, \ldots)$, $g_\lambda \in G_\lambda$. By the Baer Decomposition Theorem [**1**, p. 98, Theorem 29.3], each g_λ may be written $b_\lambda + b_\lambda^* + p^k g_\lambda'$, where $b_\lambda \in B_{\lambda 1} + \ldots + B_{\lambda k}$, $b_\lambda^* + p^k g' \in G_{\lambda k} \in \{B_{\lambda k}^*, p^k G_\lambda\}$. Since $0 = p^k x = p^k g_\lambda = p^k b_\lambda^* + p^{k+k} g_\lambda'$, p divides b^*. Thus each $g_\lambda = b_\lambda + p\hat{g}_\lambda$ for some $b_\lambda \in \sum_{n=1}^k B_{\lambda n}$, $\hat{g}_\lambda \in G_\lambda$. Hence $x = (b_1, b_2, \ldots, b_\lambda, \ldots) + p(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_\lambda, \ldots)$, where $(b_1, b_2, \ldots, b_\lambda, \ldots)$ is in

$$\prod_\Lambda \sum_{n=1}^k B_{\lambda n} = \sum_{n=1}^k \prod_\Lambda B_{\lambda n} \subset \hat{B}.$$

In $T/\hat{B}$, $\bar{x} = p\hat{g}$, $\hat{g}$ being the image of $(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_\lambda, \ldots)$. Thus $T/\hat{B}$ is divisible.

1.5a. PROPOSITION. $|\prod G_\lambda| = |T(\prod G_\lambda)|$, *if all* G_λ*s are p-groups and* Λ *is any index set.*

Proof. (i) If the index set Λ is finite, the groups are identical and have the same cardinal number.

(ii) If the index set Λ is infinite, then

$$|\prod G_\lambda| = \prod|G_\lambda| \leqq \prod (\aleph_0|G_\lambda[p]|) = \aleph_0\prod|G_\lambda[p]| = \prod |G_\lambda[p]|$$
$$= |\prod G_\lambda[p]| = |T(\prod G_\lambda[p])| \leqq |T(\prod G_\lambda)|,$$

and the proposition is again true.

1.5b. PROPOSITION. *For prime numbers:* $p_1 < p_2 < \ldots$, $|\prod_1^\infty C(p_i)|$ *is greater than* $|\sum_1^\infty C(p_i)|$ *which equals* $|T(\prod_1^\infty C(p_i))|$, *where* $C(p_i)$ *is a cyclic group of order* p_i, *for each* i.

1.6. PROPOSITION. *Let* $[G_i]_1^\infty$ *be a set of unbounded reduced p-groups. Then* $T(\prod_1^\infty G_i)$ *has an unbounded torsion-complete direct summand.*

Proof. Let $G_i = \langle g_i\rangle \oplus G_i'$, where $o(g_i) < o(g_{i+1})$, $i = 1, 2, 3, \ldots$. Then $T(\prod G_i) = T(\prod \langle g_i\rangle) \oplus T(\prod G_i')$, where $T(\prod \langle g_i\rangle)$ is torsion-complete and unbounded.

Remark. Here, the subgroup $T(\prod \langle g_i \rangle)$ is not a direct summand of $\prod G_i$, since $T(\prod \langle g_i \rangle) \subset \prod \langle g_i \rangle \subset \prod G_i$ and $\prod \langle g_i \rangle / T(\prod \langle g_i \rangle)$ is not reduced.

1.7. PROPOSITION. *If $B_\lambda \subset G_\lambda \subset \bar{B}_\lambda$, where B_λ is basic in the torsion-complete p-group $\bar{B}_\lambda$, and G_λ is pure in $\bar{B}_\lambda$* [**1**, p. 112], *then $T(\prod G_\lambda)$ is pure in $T(\prod \bar{B}_\lambda)$.*

1.8a. PROPOSITION. *Given a set of p-groups $[G_\lambda]$, if the elements of infinite height in G_λ, $\prod G_\lambda$, and $T(\prod G_\lambda)$ are designated by G_λ^1, $\prod^1$, and T^1, respectively, then $\prod^1 = \prod G_\lambda^1$ and $T^1 = T(\prod G_\lambda^1)$.*

1.8b. PROPOSITION. *If the elements of infinite height in the p-group G_λ are designated by G_λ^1, then $T(\prod G_\lambda)/T(\prod G_\lambda^1)$ is isomorphic to a pure subgroup of $T(\prod G_\lambda/G_\lambda^1)$.*

Proof. Map $P = \prod G_\lambda$ to $P' = \prod G_\lambda/G_\lambda^1$. Now $T = T(\prod G_\lambda)$ is mapped to T', a subgroup of $T(\prod G_\lambda/G_\lambda^1)$. We let $K = \prod G_\lambda^1$, the kernel of the map. Then T maps to $\{T, K\}/K \cong T/(T \cap K)$. But $T \cap K = T(\prod G_\lambda^1)$. Thus $T(\prod G_\lambda)/T(\prod G_\lambda^1) \cong T'$. We now show that T' is pure in P'. Let $p^n g' = t'$, $g' \in P'$, $t' \in T'$. If g' is the image of g in P and t' of t in T, then there exists $k \in K$ such that $p^n g = t + k$. Since k has infinite height and T is pure, there exists $x \in T$ such that $p^n x = t$. Thus $p^n x' = t'$, where x maps to x' in T' and T' is pure.

1.9. PROPOSITION. *If $\Lambda = \Lambda_\alpha + \Lambda_\beta + \ldots$ is a partitioning of the index set Λ into subsets indexed by $N = [\alpha, \beta, \ldots]$, then*

$$T(\prod G_\lambda) \cong T\left(\prod_{\alpha \in N}\left[T\left(\prod_{\lambda \in \Lambda_\alpha} G_\lambda\right)\right]\right) \quad \textit{and} \quad \prod G_\lambda \cong \prod_{\alpha \in N}\left[\prod_{\lambda \in \Lambda_\alpha} G_\lambda\right]$$

for any set of p-groups $[G_\lambda]_\Lambda$.

1.10. PROPOSITION. *For p-groups $[G_\lambda]$, if $\prod G_\lambda \neq T(\prod G_\lambda)$, then*

$$|\prod G_\lambda / T(\prod G_\lambda)| \geqq 2^{\aleph_0}.$$

Proof. Each G_λ may be considered as a module over the p-adic integers. By defining multiplication by scalar component-wise, $\prod G_\lambda$ may be considered as a module over the p-adic integers with $T(\prod G_\lambda)$ as its submodule. Thus $\prod G_\lambda/T(\prod G_\lambda)$ is also a module over the p-adics. This quotient, if not zero, is torsion-free and contains a copy of the p-adics which is uncountable.

2. $T(\prod G_\lambda)$ and torsion completion. A direct sum of cyclic groups $\sum_1^\infty B_n$ completely determines its torsion completion $T(\prod B_n)$ (see [**1**, p. 115, Corollary 34.2]). We might think that the same relationship exists between $\sum G_\lambda$ and $T(\prod G_\lambda)$ in general. That this is not so is made clear by the following example.

2.1. *Example*. Let $I = [1, 2, 3, \ldots]$. Let $G_1 = C_1(p^1)$; $G_i = C_i(p^1) \oplus C_i(p^i)$ for $i = 2, 3, 4, \ldots$, where $C_j(p^i)$ is a cyclic group of order p^i for every $j \in I$.

Likewise, let $H_1 = \sum_1^\infty C_i(p)$ and $H_i = C_i(p^i)$, $i = 2, 3, 4, \ldots$. Then $\sum G_i = \sum H_i$, yet $T(\prod G_i)$ is not isomorphic to $T(\prod H_i)$, though both have the same cardinality and both are torsion-complete.

However, if in the example above, the number of cyclic summands of every power had been finite, then for every decomposition $\sum G_i = \sum H_i$, if $T(\prod G_i)$ and $T(\prod H_i)$ are torsion-complete, they are isomorphic. This would be true since $\sum G_i$ and $\sum H_i$ would then be basic in $T(\prod G_i)$ and $T(\prod H_i)$, respectively, which in turn are the torsion completions of these subgroups. In fact, if $\sum G_i = \sum H_i$ is a direct sum of cyclic groups where cycles of power p^k for given k appear in only a finite number of G_is and H_is, then $T(\prod G_i)$ and $T(\prod H_i)$ are isomorphic if both are torsion-complete.

Although $\sum G_i = \sum H_i$ is a direct sum of cyclic p-groups and the number of cyclic summands of each power is finite, $T(\prod G_i)$ may still not be isomorphic to $T(\prod H_i)$, if the latter groups are not both torsion-complete. Let us illustrate.

2.2. *Example.* Let $G = \sum G_i$, where $G_1 = \sum_1^\infty C(p^i)$, $G_i = 0$, for $i = 2, 3, 4, \ldots$, and let $H = \sum_1^\infty H_i$, where $H_i = C(p^i)$, $i = 1, 2, 3, \ldots$. Then $\hat{G} = T(\prod G_i) = G_1$, and $\hat{H} = T(\prod H_i) = T(\prod C(p^i))$. Here, G equals H, but $\hat{G}$ is not isomorphic to $\hat{H}$.

On the other hand, $T(\prod G_i)$ may equal $T(\prod H_i)$, yet $\sum G_i$ may not be isomorphic to $\sum H_i$. Again, we give an example.

2.3. *Example.* Let $G_1 = T(\prod C(p^i))$, and $G_i = 0$, for $i = 2, 3, 4, \ldots$; let $H_i = C(p^i)$, $i = 1, 2, 3, \ldots$. Then $T(\prod G_i) = T(\prod C(p^i)) = T(\prod H_i)$. But $\sum G_i = G_1 = T(\prod C(p^i))$ and $\sum H_i = \sum C(p^i)$, and these two groups are not isomorphic.

Along these lines, however, we do have the following positive theorem.

2.4. THEOREM. *For p-groups $[G_\lambda]$, $T = T(\prod G_\lambda)$ is torsion-complete if and only if each G_λ is torsion-complete.*

Since each G_λ is a direct summand of $T(\prod G_\lambda)$, it is clear that, if $T(\prod G_\lambda)$ is torsion-complete, then so is each G_λ. We will prove the converse three times: first directly, then more quickly employing propositions of § 1, and finally by homological methods.

Proof 1. Let each G_λ be torsion-complete, and let $[g_n]$ be a bounded Cauchy sequence in $T = T(\prod G_\lambda)$. Let $g_n = (g_1^n, g_2^n, \ldots, g_\lambda^n, \ldots)$, $g_\lambda^n \in G_\lambda$, for every n. Then for each λ, $[g_\lambda^n]_n$ approaches limit g^λ in G. Now, the element $g = (g^1, g^2, \ldots, g^\lambda, \ldots)$, being bounded, is in $T(\prod G_\lambda)$. Each G_λ, being torsion-complete, is without elements of infinite height and by Proposition 1.8a, $T = T(\prod G_\lambda)$ is also without elements of infinite height. Now,

$$g^n - g = (g_1^n - g^1, g_2^n - g^2, \ldots, g_\lambda^n - g^\lambda, \ldots)$$

is in $T(\prod_\Lambda p^n G_\lambda) \subset p^n T(\prod G_\lambda)$ for every n. Therefore $[g_n]$ converges to g, and T is torsion-complete.

Proof 2. More directly, we might arrive at the same conclusion by first letting $G_\lambda = T(\prod_n B_{\lambda n})$ where $\sum_{n=1}^{\infty} B_{\lambda n}$ is basic in G_λ, as in Proposition 1.4b, for each λ. Then $T = T(\prod G_\lambda) = T(\prod_\Lambda[T(\prod_n B_{\lambda n})]) = T(\prod_\Lambda \prod_n B_{\lambda n})$. But, by Proposition 1.9, $T(\prod_\Lambda \prod_n B_{\lambda n})$ is isomorphic to $T(\prod_n \prod_\Lambda B_{\lambda n})$ which is torsion-complete.

Proof 3. Let $T = T(\prod G_\lambda)$, and $\Pi = \prod G_\lambda$, and $\Pi/T = \prod G_\lambda/T(\prod G_\lambda)$. Consider the exact sequence: $0 \to T \to \Pi \to \Pi/T \to 0$ and

$$0 \to \operatorname{Hom}(Z(p^\infty), T) \to \operatorname{Hom}(Z(p^\infty), \Pi) \to \operatorname{Hom}(Z(p^\infty), \Pi/T)$$
$$\to \operatorname{Pext}(Z(p^\infty), T) \to \operatorname{Pext}(Z(p^\infty), \Pi) \to \operatorname{Pext}(Z(p^\infty), \Pi/T) \to 0$$

(see [**2**]). It is well known that a reduced p-group G is torsion-complete if and only if $\operatorname{Pext}(Z(p^\infty), G) = 0$. Now, $\operatorname{Hom}(Z(p^\infty), \Pi/T)$ is zero, since $Z(p^\infty)$ is torsion and Π/T is torsion-free. $\operatorname{Pext}(Z(p^\infty), \prod G_\lambda) = \prod \operatorname{Pext}(Z(p^\infty), G_\lambda)$ which equals zero, since each G_λ is torsion-complete. Hence,

$$\operatorname{Pext}(Z(p^\infty), T) = 0.$$

T must then be torsion-complete, as claimed.

2.5. COROLLARY. *If $G_\lambda = T(\prod_n B_{\lambda n})$ for every λ in Λ, as in Proof 2 of Theorem 2.4, then $T(\prod G_\lambda)$ is the torsion completion of its basic subgroup $\sum_n \prod_\Lambda B_{\lambda n}$, and $T(\prod G_\lambda) \cong T(\prod_n \prod_\Lambda B_{\lambda n})$.*

Remark. If $G = \sum G_\lambda$ and each G_λ is a torsion-complete p-group, then $T(\prod G_\lambda)$ is torsion-complete, but not necessarily the smallest torsion-complete group containing G. Using the notation of Corollary 2.5, we can express $T(\prod G_\lambda)$ as $T(\prod_n \prod_\Lambda B_{\lambda n})$. Now $T(\prod_n \sum_\Lambda B_{\lambda n})$ is torsion-complete, contains G, and $\sum_\Lambda B_{\lambda n}$ need not equal $\prod_\Lambda B_{\lambda n}$ for every n.

2.6. COROLLARY. *If $\sum G_\lambda \subset H \subset T(\prod G_\lambda)$ and H is a torsion-complete p-group, then so is $T(\prod G_\lambda)$.*

Proof. Each G_λ is a direct summand of $T(\prod G_\lambda)$ and hence of H. Since H is torsion-complete, so is each G_λ and by Theorem 2.4, $T(\prod G_\lambda)$ is then torsion-complete.

3. Essentially bounded decompositions.

Definition. $G = \sum G_\lambda$ will be called an essentially bounded decomposition of G if there exists $M > 0$ such that $MG_\lambda = 0$ for almost all λ (for all but a finite number of λ). Otherwise, the decomposition will be called essentially unbounded.

3.1. THEOREM. *For a set of reduced p-groups $[G_\lambda]$, the following statements are equivalent:*

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(a) $G = \sum G_\lambda$ *is an essentially bounded decomposition of* G,
(b) $\prod G_\lambda = T(\prod G_\lambda)$,
(c) $T(\prod G_\lambda)$ *is a direct summand of* $\prod G_\lambda$,
(d) $(\prod G_\lambda)/T(\prod G_\lambda)$ *is reduced.*

Proof. We shall establish this theorem by showing that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). This is clear by the definition above.

(b) $\Rightarrow$ (c). This is clear.

(c) $\Rightarrow$ (d). If $\prod G_\lambda = T(\prod G_\lambda) \oplus K$, then K is reduced, and $\prod G_\lambda/T(\prod G_\lambda)$, isomorphic to K, is also reduced.

(d) $\Rightarrow$ (a). Suppose that $G = \sum G_\lambda$ is not essentially bounded. We could then find a set of elements $[g_{\lambda_i}]_{i=1}^{\infty}$ from a countable subset $[G_{\lambda_i}]$ of $[G_\lambda]$ such that $g_{\lambda_i} \subset G_{\lambda_i}$ and $o(g_{\lambda_i}) < o(g_{\lambda_{i+1}})$ for every i. Writing the indices $[\lambda_1, \lambda_2, \ldots]$ consecutively in $\prod G_\lambda$, we consider the summand $\prod_{i=1}^{\infty} G_{\lambda_i}$. Now $g = (g_{\lambda_1}, 0, pg_{\lambda_3}, \ldots, p^{i-1}g_{\lambda_{2i+1}}, \ldots)$ is in $\prod G_{\lambda_i} \backslash T(\prod G_{\lambda_i}) \subset \prod G_\lambda \backslash T(\prod G_\lambda)$. The image of g in $\prod G_\lambda/T(\prod G_\lambda)$ has infinite height therein. This quotient group, then, contains a divisible subgroup, since it is torsion-free. Thus $\prod G_\lambda/T(\prod G_\lambda)$ would not be reduced.

3.2. COROLLARY. *If* $\sum G_\lambda$ *is essentially bounded, and each* G_λ *is a direct sum of cyclic groups, then* $T(\prod G_\lambda)$ *is a direct sum of cyclic groups.*

3.3. COROLLARY. *For p-groups* $[G_\lambda]$, $T(\prod G_\lambda)$ *is a direct summand of* $\prod G_\lambda$ *if and only if* $\sum R_\lambda$ *is an essentially bounded decomposition, where* $G_\lambda = D_\lambda \oplus R_\lambda$, D_λ *divisible,* R_λ *reduced.*

3.4. THEOREM. *For reduced p-groups* $[G_\lambda]_\Lambda$, *the following statements are equivalent:*

(a) $T(\prod G_\lambda)/\sum G_\lambda$ *is divisible;*

(b) *For any given order* p^k, *only a finite number of* G_λ*s have cyclic summands of this order;*

(c) *If* B_λ *is basic in* G_λ *for every* λ, *then* $\sum B_\lambda$ *is basic in* $T(\prod G_\lambda)$.

Proof. We shall prove this theorem in the following manner: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Suppose that we could split off cyclic summands of the same order p^k from an infinite subset of $[G_\lambda]$, say from G_λ, $\lambda \in A$, where $\Lambda = A + B$, $|A| \geqq \aleph_0$. We could then write $G_\lambda = \langle e_\lambda \rangle \oplus G_\lambda'$, where $o(e_\lambda) = p^k$ for every $\lambda \in A$. Then

$$\frac{T\left(\prod G_\lambda\right)}{\sum G_\lambda} = \frac{T\left(\prod_A G_\lambda\right)}{\sum_A G_\lambda} \oplus \frac{T\left(\prod_B G_\lambda\right)}{\sum_B G_\lambda},$$

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and

$$\frac{T\left(\prod_A G_\lambda\right)}{\sum_A G_\lambda} = \frac{T\left(\prod_A (\langle e_\lambda\rangle \oplus G_\lambda')\right)}{\sum_A (\langle e_\lambda\rangle \oplus G_\lambda')}$$

$$\cong \frac{\prod_A \langle e_\lambda\rangle}{\sum_A \langle e_\lambda\rangle} \oplus \frac{T\left(\prod_A G_\lambda'\right)}{\sum_A G_\lambda'}.$$

Since A is infinite, $\prod_A \langle e_\lambda\rangle/\sum_A \langle e_\lambda\rangle$ is a non-zero sum of cyclic groups. Thus $T(\prod G_\lambda)/\sum G_\lambda$ is not divisible.

(b) $\Rightarrow$ (c). By Proposition 1.4b, $\sum_n \prod_\Lambda B_{\lambda n}$ is basic in $T(\prod G_\lambda)$, where $B_\lambda = \sum_n B_{\lambda n}$ is basic in G_λ. By condition (b), then, $\sum_n B_{\lambda n} = \prod_n B_{\lambda n}$ for every n. Hence, the basic subgroup of $T(\prod G_\lambda)$ is

$$\sum_n \prod_\Lambda B_{\lambda n} = \sum_n \sum_\Lambda B_{\lambda n} = \sum_\Lambda \sum_n B_{\lambda n} = \sum_\Lambda B_\lambda.$$

(c) $\Rightarrow$ (a). If $\sum B_\lambda$ is basic in $T(\prod G_\lambda)$, $T(\prod G_\lambda)/\sum B_\lambda$ is divisible and its homomorphic image $(T(\prod G_\lambda)/\sum B_\lambda)/(\sum G_\lambda/\sum B_\lambda) \cong T(\prod G_\lambda)/\sum G_\lambda$ is divisible.

3.5. COROLLARY. *Theorem* 3.4 *remains true for p-groups in general.*

Proof. Using the statement and notation of Proposition 1.1,

$$T(\prod G_\lambda)/\sum G_\lambda \cong (T(\prod D_\lambda)/\sum D_\lambda) \oplus (T(\prod R_\lambda)/\sum R_\lambda),$$

where $G_\lambda = D_\lambda \oplus R_\lambda$, D_λ divisible, R_λ reduced. The left summand is divisible. Since cyclic summands and basic subgroups appear in the reduced part of groups, Theorem 3.4 applies to the right summand.

3.6. THEOREM. *For reduced p-groups* $[G_\lambda]$, *the following statements are equivalent:*

(a) $T(\prod G_\lambda)/\sum G_\lambda$ *is reduced,*

(b) $\sum G_\lambda$ *is an essentially bounded decomposition,*

(c) $T(\prod G_\lambda)/\sum G_\lambda$ *is bounded.*

Proof. We shall establish the following implications: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). If $\sum G_\lambda$ is not essentially bounded, we can find a subset $[G_{\lambda_i}]_{i=1}^\infty$ in $[G_\lambda]$ and $e_{\lambda_i} \in G_{\lambda_i}$ such that $o(e_{\lambda_i}) < o(e_{\lambda_{i+1}})$ and $G_{\lambda_i} = \langle e_{\lambda_i}\rangle \oplus G_{\lambda_i}'$ for $i = 1, 2, 3, \ldots$. We now have a summand of $T(\prod G_\lambda)/\sum G_\lambda$ which is isomorphic to $T(\prod_i \langle e_{\lambda_i}\rangle)/\sum_i \langle e_{\lambda_i}\rangle$ and this summand is divisible. Thus $T(\prod G_\lambda)/\sum G_\lambda$ is not reduced.

(b) $\Rightarrow$ (c). This is clear.

(c) $\Rightarrow$ (a). This is clear.

3.7. COROLLARY. *Theorem* 3.6 *is true for p-groups in general.*

Proof. As in Corollary 3.5,

$$T(\prod G_\lambda)/\sum G_\lambda \cong (T(\prod D_\lambda)/\sum D_\lambda) \oplus (T(\prod R_\lambda)/\sum R_\lambda).$$

If the left summand is non-zero, none of the conditions of Theorem 3.6 are satisfied. If the left summand is zero, Theorem 3.6 applies.

3.8. COROLLARY. *For a set of p-groups* $[G_\lambda]$, *the reduced part of* $T(\prod G_\lambda)/\sum G_\lambda$ *is bounded if and only if, using the notation of Proposition* 1.4b, $\sum_\Lambda B_{\lambda n}$ *differs from* $\prod_\Lambda B_{\lambda n}$ *for at most a finite number of ns.*

Proof. (a) Suppose that $\sum_\Lambda B_{\lambda n} \neq \prod_\Lambda B_{\lambda n}$ implies $n \leqq N$. Then, as in Proposition 1.4a, $T(\prod G_\lambda) = T(\prod_\Lambda S_{\lambda N}) \oplus T(\prod_\Lambda G_{\lambda N})$ and $T(\prod G_\lambda)/\sum G_\lambda \cong ((\prod S_{\lambda N})/\sum S_{\lambda N}) \oplus (T(\prod G_{\lambda N})/\sum G_{\lambda N})$. The left summand is bounded. Now in $T(\prod G_{\lambda N})$, $\sum_\Lambda B_{\lambda n}$ equals $\prod_\Lambda B_{\lambda n}$, since in $T(\prod G_{\lambda N})$ we have n greater than N and since $\sum_\Lambda B_{\lambda n} \neq \prod_\Lambda B_{\lambda n}$ implies $n \leqq N$. Thus, $T(\prod G_{\lambda N})/\sum G_{\lambda N}$ is divisible (by Theorem 3.4). The quotient group is then the direct sum of a divisible group and a bounded group.

(b) If $\sum_\Lambda B_{\lambda n} \neq \prod_\Lambda B_{\lambda n}$ for an infinite number of ns, then we can find a bounded direct summand of the quotient group isomorphic to $\prod S_{\lambda n}/\sum S_{\lambda n}$ for arbitrarily large n. The reduced part of $T(\prod G_\lambda)/\sum G_\lambda$ would not then be bounded.

3.9. THEOREM. *For reduced p-groups* $[G_\lambda]$, $T(\prod G_\lambda)/\sum G_\lambda$ *is a direct summand of* $\prod G_\lambda/\sum G_\lambda$ *if and only if* $\sum_\Lambda B_{\lambda n} \neq \prod_\Lambda B_{\lambda n}$ *for at most a finite number of ns, where* $\sum_n B_{\lambda n}$ *is basic in* G_λ *for every* λ.

Proof. If $\sum_\Lambda B_{\lambda n} \neq \prod_\Lambda B_{\lambda n}$ for at most a finite number of ns, then by Corollary 3.8, the reduced part of $T(\prod G_\lambda)/\sum G_\lambda$ is bounded and $T(\prod G_\lambda)/\sum G_\lambda$ is divisible plus bounded. Since $T(\prod G_\lambda)/\sum G_\lambda$ is the torsion subgroup of $\prod G_\lambda/\sum G_\lambda$, it is a direct summand of $\prod G_\lambda/\sum G_\lambda$ by [**6**, Theorem 8].

On the other hand, suppose that $\sum_\Lambda B_{\lambda n} \neq \prod_\Lambda B_{\lambda n}$ for an infinite number of ns. Without loss of generality, we may suppose that this is true for all n. Then we can find an infinite number of infinite and mutually disjoint subsets Λ_i, $1 \leqq i < \infty$, of Λ, and pure cycles $\langle e_\lambda{}^i \rangle$ of order p^i in G_λ for $\lambda \in \Lambda_i$. If λ is not in Λ_i for any $i \geqq 1$, we will say that λ is in Λ_0, and let

$$\Lambda = \Lambda_0 + \Lambda_1 + \ldots + \Lambda_i + \ldots .$$

Let x_i be the element $(e_1{}^i, e_2{}^i, \ldots, e_\lambda{}^i, \ldots)$ in $\prod_{\Lambda_i} G_\lambda$ for each $i \geqq 1$. Let x be the tuple in $\prod_{i=0}^\infty (\prod_{\Lambda_i} G_\lambda) = \prod G_\lambda$, where

$$x = (0, x_1, 0, px_3, 0, p^2 x_5, \ldots, p^i x_{2i+1}, \ldots).$$

If $T(\prod G_\lambda)/\sum G_\lambda$ is a summand of $\prod G_\lambda$, we may write

$$\prod G_\lambda/\sum G_\lambda = (T(\prod G_\lambda)/\sum G_\lambda) \oplus (K/\sum G_\lambda).$$

The order of x is infinite. If x is mapped to $\hat{x}$ in $\prod G_\lambda/\sum G_\lambda$, $p^k \hat{x}$ is in $K/\sum G_\lambda$, for some k. If $h(p^k x) = p^j$, then $h(p^k \hat{x}) = p^j$, since no x_i is in $\sum G_\lambda$. However, there exists a in $T(\prod G_\lambda)$ such that $h(p^k x - a) > p^j$. Let $\hat{a}$ be the image of a

in $T(\prod G_\lambda)/\sum G_\lambda$. Now $h(p^k\hat{x} - \hat{t}) > p^j$ in $\prod G_\lambda/\sum G_\lambda$. Since $p^k\hat{x}$ is in $K/\sum G_\lambda$ and $\hat{t}$ is in $T(\prod G_\lambda)/\sum G_\lambda$, the height of $p^k\hat{x}$ would be greater than p^j, a contradiction. We conclude that $T(\prod G_\lambda)/\sum G_\lambda$ is not a summand of $\prod G_\lambda/\sum G_\lambda$, if $\sum_\Lambda B_{\lambda n} \neq \prod_\Lambda B_{\lambda n}$ for an infinite number of ns.

4. Decomposition theorems. Since we are more familiar with direct sums than with direct products of p-groups, it is worthwhile to know when a direct product of p-groups can be decomposed into a direct sum of groups. It is natural to ask the same question for the torsion subgroup of a product of p-groups.

Two cases with easy answers come immediately to mind. If the p-groups have a common bound, then their direct product is also bounded and a direct sum of cyclic groups. If all the p-groups are divisible, the product, and the torsion subgroup of the product, of these p-groups can each be written as a direct sum of copies of $Z(p^\infty)$ and the rationals. Hence we will be more concerned with collections of p-groups which are reduced, and where the direct sum of a collection is essentially unbounded (as defined in § 2).

4.1. THEOREM. *Neither the direct product, nor its torsion subgroup, of a collection of reduced p-groups whose direct sum is essentially unbounded, is a direct sum of countable groups.*

We first must prove a lemma.

4.2. LEMMA. *If $G = H \oplus K$ is a reduced group, and A and T are subgroups of G such that $A \subset H$ and T/A is divisible, then $T \subset H$.*

Proof. First, $T/H \cap T$ is reduced since $K \cong G/H \supset \{T, H\}/H \cong T/H \cap T$, and K is reduced. Now $A \subset H \cap T \subset T$, and T/A divisible, implies $T/H \cap T$, a homomorphic image of T/A, is divisible. Since $T/H \cap T$ is both divisible and reduced, it equals zero. Hence $T = H \cap T$, or T is contained in H.

Proof. of Theorem 4.1. Given the set $[G_\lambda]$ whose direct sum is essentially unbounded, we find a subset $[G_{\lambda_i}]_{i=1}^\infty$ and $g_{\lambda_i} \in G_{\lambda_i}$ such that $o(g_{\lambda_i}) < o(g_{\lambda_{i+1}})$ for every i. If $\prod G_\lambda$ or $T(\prod G_\lambda)$ equals $\sum_{\mu \in M} H_\mu$, where each H_μ is countable, then the set $[g_{\lambda_i}]_{i=1}^\infty$ is contained in a countable subset of $[H_\mu]_M$. $T(\prod_{i=1}^\infty \langle g_{\lambda_i}\rangle)$ is in both $\prod G_\lambda$ and $T(\prod G_\lambda)$. Since $T(\prod_i \langle g_{\lambda_i}\rangle)/\sum_i \langle g_{\lambda_i}\rangle$ is divisible, $T(\prod_i \langle g_{\lambda_i}\rangle)$ would be contained in the direct sum of the same countable subset of $[H_\mu]$ as $[g_{\lambda_i}]$ by Lemma 4.2. But this is impossible, since $T(\prod_i \langle g_{\lambda_i}\rangle)$ is uncountable.

Remark. It is of interest to know when a group is a direct sum of reduced countable groups, for such a group is fully starred as noted by Irwin and Richman [**4**, p. 446].

4.3. THEOREM. *If $G = A \oplus B = C \oplus D$, C is an unbounded direct sum of* *cyclic p-groups, then A or B has a direct summand which is an unbounded direct sum of cyclic p-groups.*

Before proceeding to the proof we first establish some lemmas.

4.4. LEMMA. *If p-group G contains* $H = \sum_1^\infty \langle x_i \rangle$, *where* $o(x_i) < o(x_{i+1})$ *for every i, then H is pure if and only if* $\langle x_i \rangle$ *is pure for each i.*

Proof. If H is pure, then $\langle x_i \rangle$ as a summand is pure. The converse follows easily from a consideration of the socle elements and [**6**, p. 20, Lemma 7].

4.5. LEMMA. *If* $H = \{x_i\}_1^\infty$, *is a p-group, where* $o(x_i) < o(x_{i+1})$ *and* $\langle x_i \rangle$ *is pure for each i, then H is a direct sum, i.e.* $H = \sum_1^\infty \langle x_i \rangle$.

Proof. Consider the set $[p^{k_i - 1} x_i]_{i=1}^\infty$, where $o(x_i) = p^{k_i}$ for each i. Let $\sum_{i=1}^N a_i(p^{k_i-1}x_i) = 0$. Suppose that $a_j(p^{k_j-1}x_j)$ is the first non-zero term on the left. Then

$$a_j(p^{k_j-1}x_j) = -\sum_{i=j+1}^{N} a_i(p^{k_i-1}x_i)$$

is non-zero. Now $\langle x_j \rangle$ is pure and $h(a_j p^{k_j-1} x_j) = k_j - 1$. But since $o(x_i) < o(x_{i+1})$, each term on the right has height greater than $k_j - 1$, a contradiction. Thus $[p^{k_i-1}x_i]_{i=1}^\infty$ are linearly independent and as a result the x_is are linearly independent or $H = \sum \langle x_i \rangle$ is direct.

4.6. LEMMA. *If p-group G is a direct sum of cyclic groups and* $H = \sum_1^\infty \langle h_j \rangle$ *is a pure unbounded subgroup of G, then there exists* $H_1 = \sum_{n=1}^\infty \langle h_{j_n} \rangle$ *which is an unbounded summand of G.*

Proof. We may restrict ourselves to the case where $o(h_j) < o(h_{j+1})$ and where $G = \sum \langle x_i \rangle$, $o(x_i) \leqq o(x_{i+1})$. Let $\langle u_j \rangle = \langle h_j \rangle[p]$. Now

$$u_{j_1} = u_1 \in \sum_1^{N_1} \langle x_i \rangle$$

and is purifiable in this summand, i.e., there exists y_1 such that

$$\langle y_1 \rangle \perp \sum_1^{N_1} \langle x_i \rangle \text{ and } \langle y_1 \rangle[p] = \langle u_1 \rangle.$$

There exists u_{j_2} such that $h(u_{j_2}) > N_1$ and $u_{j_2} \in \sum_{i>N_1}^{N_2} \langle x_i \rangle$ and $\langle u_{j_2} \rangle = \langle y_2 \rangle[p]$, where $\langle y_2 \rangle$ is a summand of $\sum_{i>N_1}^{N_2} \langle x_i \rangle$. By induction we may find a

$$u_{j_n} \in \sum_{i>N_n}^{N_{n+1}} \langle x_i \rangle \quad \text{and} \quad \langle y_n \rangle \perp \sum_{i>N_n}^{N_{n+1}} \langle x_i \rangle,$$

where $\langle y_n \rangle[p] = \langle u_{j_n} \rangle$. Clearly, $\sum_1^\infty \langle y_n \rangle \perp \sum \langle x_i \rangle$. Now,

$$\sum \langle y_n \rangle[p] = \sum \langle u_{j_n} \rangle[p] = \sum_{n=1}^{\infty} \langle h_{j_n} \rangle[p].$$

By a theorem of Irwin and Walker [**5**, p. 1373, Theorem 16], if two pure subgroups have the same socle and one subgroup is a summand, then so is the other. Thus $\sum_{n=1}^\infty \langle h_{j_n} \rangle$ is a summand of $\sum \langle x_i \rangle$, as desired.

4.7. LEMMA. *If* $G = A \oplus B = C \oplus D$, *where* $C = \sum \langle c_i \rangle$ *is a* p-*group, and if* $c_j = a_j + b_j$, $a_j \in A$, $b_j \in B$, *and* $a_j = c_a{}^j + d_j$, $b_j = c_b{}^j - d_j$, $c_a{}^j \in C$, $c_b{}^j \in C$, $d_j \in D$, *then either* $o(a_j) = o(c_a{}^j) = o(c_j)$ *and* $\langle c_a{}^j \rangle$ *is pure or* $\langle c_b{}^j \rangle$ *is pure with* $o(c_b{}^j) = o(c_j) = o(b_j)$.

Proof. Let $c_a{}^j = \sum x_i c_i$, $c_b{}^j = \sum y_i c_i$. Then

$$p^{k-1}c_j = p^{k-1}(c_a{}^j + c_b{}^j) = p^{k-1}(x_j + y_j)c_j,$$

where $o(c_j) = p^k$. Since $\langle c_j \rangle$ is pure, $h(p^{k-1}c_j) = k - 1$ and $(x_j + y_j, p) = 1$. Thus $(x_j, p) = 1$ or $(y_j, p) = 1$. Suppose that $(x_j, p) = 1$. Then

$$o(c_j) = p^k \leqq o(c_a{}^j) \leqq o(a_j) \leqq o(c_j), \quad \text{and} \quad o(c_j) = o(c_a{}^j) = o(a_j).$$

Also $k - 1 \leqq h(p^{k-1}c_a{}^j) \leqq h(p^{k-1}x_j c_j) = k - 1$. Thus $\langle c_a{}^j \rangle$ is pure.

Proof of Theorem 4.3. Let $G = A \oplus B = C \oplus D$, where $C = \sum_1^\infty \langle c_i \rangle$ is an unbounded p-group. Let $c_i = a_i + b_i$, $a_i \in A$, $b_i \in B$, for every i, where $a_i = c_a{}^i + d_i$, $b_i = c_b{}^i - d_i$, $c_a{}^i \in C$, $c_b{}^i \in C$, $d_i \in D$. By Lemma 4.7, for each i, either $o(a_i) = o(c_i) = o(c_a{}^i)$ and $\langle c_a{}^i \rangle$ is pure or $o(b_i) = o(c_i) = o(c_b{}^i)$ and $\langle c_b{}^i \rangle$ is pure. Let us suppose the former case to be true for an infinite number of c_is of properly increasing orders. Thus, for notational purposes, let us restrict ourselves to $C = \sum_1^\infty \langle c_i \rangle$, where $c_i = a_i + b_i$, $a_i = c_a{}^i + d_i$, $b_i = c_b{}^i - d_i$, and $o(c_i) = o(a_i) = o(c_a{}^i)$, $\langle c_a{}^i \rangle$ is pure, $o(c_i) < o(c_{i+1})$ for each i. Since $o(c_a{}^i) < o(c_a{}^{i+1})$ and $\langle c_a{}^i \rangle$ is pure for each i, by Lemma 4.5, $H = \sum \langle c_a{}^i \rangle$ is a direct sum, and is pure by Lemma 4.4. By Lemma 4.6, H contains an unbounded subgroup $H_1 = \sum_{k=1}^\infty \langle c_a{}^{i_k} \rangle$ which is a summand of C. Now $G = A \oplus B = H_1 \oplus H_2 \oplus D$, where $C = H_1 \oplus H_2$. Consider

$$G/(H_2 \oplus D) \cong \sum_{k=1}^{\infty} \langle c_a{}^{i_k} \rangle.$$

Since, for each k, $a_{i_k} = c_a{}^{i_k} + d_{i_k}$, $o(a_{i_k}) = o(c_a{}^{i_k})$, and a_{i_k} is mapped to $c_a{}^{i_k}$ in the natural map $G \to G/(H_2 \oplus D)$, then $G = \sum_{k=1}^\infty \langle a_a{}^{i_k} \rangle \oplus H_2 \oplus D$, by the proof of [**6**, Theorem 5]. Here $\sum_{k=1}^\infty \langle a_a{}^{i_k} \rangle$ is unbounded and direct. Since $G \supset A \supset \sum \langle a_a{}^{i_k} \rangle$, $\sum \langle a_a{}^{i_k} \rangle$ is a direct summand of A, and the proof is complete.

4.8. COROLLARY. *If reduced* p-*groups,* H *and* K, *are essentially finitely indecomposable groups* (*i.e., have no essentially unbounded decompositions*), *then* $H \oplus K$ *is essentially finitely indecomposable.*

4.9. LEMMA. *If* $G = A \oplus B = C \oplus D$, *where* C *is a direct sum of an infinite number of cyclic groups, the orders of the cycles being powers of different prime numbers, then* A *or* B *has a direct summand which is a direct sum of an infinite number of cyclic groups, the orders of the cycles being powers of different prime numbers.*

Proof. (a) Suppose that $C = \sum_1^\infty \langle c_i \rangle$, $o(c_i) = p_i{}^{k_i}$, $p_1 < p_2 < \ldots$, and $c_i = a_i + b_i$, $a_i \in A$, $b_i \in B$. Since the orders of the c_is are relatively prime, and $p_i{}^{k_i}c_i = 0 = p_i{}^{k_i}a_i = p_i{}^{k_i}b_i$, it follows that

$$a_i = x_i{}^i c_i + d_i', \qquad b_i = y_i{}^i c_i + d_i \qquad \text{in } C \oplus D.$$

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Now $(x_i{}^i, p_i) = 1$ for an infinite number of is or $(y_i{}^i, p_i) = 1$ for an infinite number of is. Let us suppose that $(y_i{}^i, p_i) = 1$ for an infinite number of is. In fact, without compromising our proof, let us suppose this to be true for all $y_i{}^i$, $i = 1, 2, 3, \ldots$.

(b) We now claim that $G = \sum_1^\infty \langle b_i \rangle \oplus D$. We first show that

$$\langle b_1 \rangle + \langle b_2 \rangle + \ldots + D$$

generates G. Since $(y_i{}^i, p_i) = 1$, and $y_i{}^i c_i = b_i - d_i$, each c_i is in

$$\langle b_1 \rangle + \langle b_2 \rangle + \ldots + D$$

and thus $G = C + D = \langle b_1 \rangle + \langle b_2 \rangle + \ldots + D$. Secondly, we show that $\sum \langle b_i \rangle + D$ is direct. If $\sum m_i b_i + d = 0$, then $\sum m_i b_i - \sum m_i d_i = \sum m_i y_i{}^i c_i = -d - \sum m_i d_i = 0$, since $C \cap D = 0$. Thus $m_i y_i{}^i c_i = 0$ and $m_i c_i = 0$, for each i. Since $c_i = a_i + b_i$ in $A + B$, $m_i b_i = 0$, and the sum is direct.

(c) Since $G \supset B \supset \sum \langle b_i \rangle$ and $G = \sum \langle b_i \rangle \oplus D$, we conclude that $\sum \langle b_i \rangle$, which is unbounded, is a direct summand of B.

Remark. If $G = A \oplus B = C \oplus D$ and C is an unbounded direct sum of cyclic groups of infinite rank, then A or B has an unbounded direct sum of cyclic groups of infinite rank as a direct summand. Here we generalize Lemmas 4.7 and 4.9. If C is free, the statement is still true.

4.10. LEMMA. *If $B = \sum \langle b_i \rangle$ is a direct sum of cyclic p-groups, where $b_i \in \prod_{n \geqq i}^\infty G_n$ in $\prod_1^\infty G_n$, then there exists a subgroup H such that*

$$\prod G_n \supset H \supset \sum \langle b_i \rangle$$

and H is a torsion-complete group.

Proof. If one writes out the b_is as tuples, the components appearing in each G_n is finite, and the lemma follows immediately.

4.11. LEMMA. *If the p-group $T = T(\prod_1^\infty G_n)$ has as a direct summand an unbounded direct sum of cyclic groups $\sum_1^\infty \langle c_i \rangle$, where $c_i \in \sum G_n$, for every i, then some G_n has an unbounded direct sum of cyclic groups as a direct summand.*

Proof. (a) Let $c_i = g_1{}^i + g_2{}^i + \ldots + g_{N_i}{}^i$, $o(c_i) < o(c_{i+1})$ for every i, in $\sum G_n$. If $T = \sum \langle c_i \rangle + D$, let $g_j{}^i = g_{j1}{}^i c_1 + \ldots + y_{jN_ij}{}^i c_{N_ij} + d_j{}^i = c_j{}^i + d_j{}^i$, where $g_{j1}{}^i c_1 + \ldots + y_{jN_ij}{}^i c_{N_ij} = c_j{}^i$ in $\sum \langle c_i \rangle$ and $d_j{}^i \in D$. Now

$$(y_{1i}{}^i + y_{2i}{}^i + \ldots + y_{N_ii}{}^i, p) = 1 \quad \text{for each } i.$$

Hence $(y_{ji}{}^i, p) = 1$, for some $j < N_i$.

(b) Let us take one $g_j{}^i$ for each c_i such that $(y_{ji}{}^i, p) = 1$ and hence $o(c_i) = o(g_{ji}{}^i) = o(c_j{}^i)$. If $o(c_i) = p^k$, then

$$k - 1 \leqq h(p^{k-1} c_j{}^i) \leqq h(p^{k-1} y_{ji}{}^i c_i) = k - 1.$$

Thus $\langle c_j{}^i \rangle$ is pure in $\sum \langle c_i \rangle$, and $\sum \langle c_j{}^i \rangle$ is direct and pure by Lemmas 4.5 and 4.4. By Lemma 4.6, we can find a subset $[c_{j_k}{}^{i_k}]_{k=1}^\infty$ such that $\sum \langle c_{j_k}{}^{i_k} \rangle$ is a summand of $\sum \langle c_i \rangle$. As in the proof of Theorem 4.3, the corresponding subset $\sum \langle g_{j_k}{}^{i_k} \rangle$ is a summand of T. If $[j_1, j_2, \ldots, j_k, \ldots]$ contains a properly increasing subset, the corresponding elements in $[g_{j_k}{}^{i_k}]_{k=1}^{\infty}$ satisfy the condition of Lemma 4.9, and yet generate an unbounded direct sum of cyclic groups, say K, which is a summand of $T(\prod G_n)$. We then have, by Lemma 4.9, a torsion-complete group H such that K is in H and K is a summand H. This contradicts the torsion completeness of H. Thus $[j_1, j_2, \ldots, j_k, \ldots]$ is bounded, and we can find an infinite subset $[g_{j_k}{}^{i_k}]$ and finite N such that $\sum \langle g_{j_k}{}^{i_k} \rangle$ is an unbounded direct summand of $G_1 + \ldots + G_N$. A finite application of Theorem 4.3 completes the proof.

4.12. Lemma. *If the p-group $T = T(\prod_1^\infty G_n)$ has an unbounded direct sum of cyclic groups as a direct summand, then some G_n has an unbounded direct sum of cyclic groups as a direct summand.*

Proof. (a) Let $T = \sum_1^\infty \langle c_i \rangle \oplus D$, $o(c_i) < o(c_{i+1})$ for every i. Let $c_i = a_i + b_i$, $a_i \in G_1 + \ldots + G_i$, $b_i \in T(\prod_{i+1}^\infty G_n)$, for every i.

(b) Exactly as in Theorem 4.3, we can prove that either $[a_i]_1^\infty$ or $[b_i]_1^\infty$ contains a subset which generates an unbounded direct sum of cyclic groups, say K, which is a summand of $T(\prod G_n)$. Now $[b_i]_1^\infty$ cannot contain such a subset, for, by Lemma 4.10, there is a torsion-complete group H such that $\{b_i\} \subset H \subset T(\prod G_n)$. Then K, which is not torsion-complete would be a summand of H.

(c) Thus, $[a_i]_1^\infty$ contains a subset which generates an unbounded summand K of $T(\prod G_n)$, where K is a direct sum of cyclic groups. Since

$$K \subset \{a_i\} \subset \sum G_n,$$

we may use Lemma 4.11 to complete our proof.

4.13. Theorem. *The reduced p-group $T(\prod_1^\infty G_n)$ has an essentially unbounded decomposition if and only if some G_n has the same property.*

Proof. If $G_i = \sum_1^\infty H_n$ is an essentially unbounded decomposition of G_i for some i, then $T(\prod G_n) = \sum_1^\infty H_i \oplus T(\prod_{n \neq i} G_n)$ is an essentially unbounded decomposition. If $T(\prod_1^\infty G_n) = \sum_1^\infty H_n$ is an essentially unbounded decomposition, then we can split off an unbounded direct sum of cyclic groups from the right side as a direct summand. Lemma 4.12 completes the proof.

We now turn our attention to the problem of when $\prod G_\lambda$ or $T(\prod G_\lambda)$ is a direct sum of isomorphic groups. If all G_λ in $[G_\lambda]$ have a common bound or are all divisible, and of suitable rank, such a decomposition is possible. Again, given $[H_n]_1^\infty$, where each $H_n \cong T(\prod_2^\infty G_n)$ and $G_1 = \sum_2^\infty H_n$, then $T(\prod_1^\infty G_n) \cong \sum_1^\infty H_n$, and all H_ns are isomorphic. For unbounded reduced G_λs, things are more complicated. Before proceeding, we first establish some preliminary facts.

4.14. Lemma. *If p-group $G = A \oplus B$, and $\sum \langle c_i \rangle$ is a pure direct sum of cyclic groups of properly ascending orders, and $c_i = a_i + b_i$, $a_i \in A$, $b_i \in B$,* *then there exists a pure direct sum of cyclic groups* $\sum_1^\infty \langle x_i \rangle$, *where* $x_i = a_i$ *or* $x_i = b_i$, *and* $o(x_i) = o(c_i)$ *for each* i.

Proof. For each i, either $\langle a_i \rangle$ or $\langle b_i \rangle$ is a pure cycle of the same order as $\langle c_i \rangle$. Let $\langle x_i \rangle$ be this pure cycle for each i. First, $\sum \langle x_i \rangle$ is a direct sum, by Lemma 4.5. Then $\sum \langle x_i \rangle$ is pure by Lemma 4.4.

4.15. LEMMA. *If p-group* $G = \bar{B} \oplus K = A \oplus C$, *where* $\bar{B}$ *is an unbounded torsion-complete group, and* $B = \sum \langle a_i + c_i \rangle$ *is a direct sum of cyclic groups of properly ascending orders which is basic in* $\bar{B}$ *and where* $\sum \langle c_i \rangle$ *is a pure direct sum in* C *and* $o(a_i + c_i) = o(c_i)$ *for every* i, *then* C *contains a copy of* $\bar{B}$ *as a summand.*

Proof. Let $\pi: \bar{B} \to C$ be the natural projection in $G = A \oplus C$ of $\bar{B}$. Now the image $\pi(\bar{B})$ is isomorphic to $\bar{B}$ and pure in C. We first show that the kernel of the map is zero. Let a be in $\bar{B} \cap A$. Since $\bar{B}/B$ is divisible, for any n, we can find $a' + c'$ in $\bar{B}$, $a' \in A$, $c' \in C$, and b in B such that $a = p^n(a' + c') + b$. If $b = \sum x_i(a_i + c_i)$, then $p^n c' = -\sum x_i c_i$. Since $\sum \langle c_i \rangle$ is pure direct, p^n divides x_i, where $x_i c_i \neq 0$. This, in turn, implies that a is p^n-divisible. Since n is arbitrary, $h(a) = \infty$ and $a = 0$ for $\bar{B}^1 = 0$. Thus $\pi(\bar{B}) \cong \bar{B}$. We now show that $\pi(\bar{B})$ is pure in C. Suppose that $x \in \pi(\bar{B})$ and $x = p^k c$, $c \in C$. Then x is the image of some $a + x$ in $\bar{B}$, $a \in A$. Since $\bar{B}/B$ is divisible, $a + x = p^k(a' + c') + a'' + c''$ for some $a' + c' \in B$, $a'' + c'' \in B$. Since $p^k c = x = p^k c' + c''$, and $c'' \in \sum \langle c_i \rangle$ which is pure, then, $c'' = p^k c'''$ for some $c''' \in \sum \langle c_i \rangle \subset \pi(\bar{B})$. Therefore, $x = p^k(c' + c''')$, $c' + c''' \in \pi(\bar{B})$. The Kulikov-Papp Theorem [**1**, p. 117, Theorem 34.6] completes the proof.

4.16. THEOREM. *If* $G = \bar{B} \oplus K = A \oplus C$, *where* $\bar{B}$ *is an unbounded torsion-complete p-group, then* A *or* C *contains an unbounded torsion-complete p-group as a summand.*

Proof. Let $B = \sum \langle b_i \rangle$ be basic in $\bar{B}$. And we may suppose that $o(b_i) < o(b_{i+1})$. Since, if $b_i = a_i + c_i$, $a_i \in A$, $c_i \in C$, $o(b_i) = o(a_i)$ and $\langle a_i \rangle$ is pure or $o(b_i) = o(c_i)$ and $\langle c_i \rangle$ is pure, we may, by splitting B, suppose the former or latter case to be true for all b_is. Therefore, let us suppose that $o(b_i) = o(c_i)$ and $\langle c_i \rangle$ to be pure for all b_is. By the preceding lemma, then, C contains a copy of $\bar{B}$ as a summand.

4.17. THEOREM. *If* $G = \bar{B} \oplus K = \sum H_\lambda$, *where* $\bar{B}$ *is an unbounded torsion-complete p-group, then some* H_λ *has an unbounded torsion-complete p-group as a summand.*

Proof. Suppose that $B = \sum \langle b_i \rangle$ is basic in $\bar{B}$ and that $o(b_i) < o(b_{i+1})$ for each i. Let $b_i = h_1{}^i + \ldots + h_j{}^i + \ldots + h_N{}^i$, $h_j{}^i \in H_j$. Then there exists j_i such that $o(b_i) = o(h_{j_i}{}^i)$ and $\langle h_{j_i} \rangle$ is pure in H_{j_i}. Consider the set $[j_i]_{i=1}^\infty$. If we have an infinite number of distinct numbers in the set, we may suppose all to be distinct and split off $H' = \sum \langle h_{j_i}{}^i \rangle$ as a summand, letting

$G = H' \oplus M$. Now, if $b_i = h_i' + m_i$, $h_i' \in H'$, $m_i \in M$, then $o(b_i) = o(h_i')$ and $\langle h_i' \rangle$ is pure. By Lemma 4.15, H' contains a copy of $\bar{B}$, which is false. Therefore, $[j_i]_{i=1}$ is bounded, say by N. Then if $G = \sum_1^N H_i + \sum_{i>N} H_i$, by the previous argument, $\sum_1^N H_i$ contains an unbounded torsion-complete summand. A finite application of Theorem 4.16 completes the proof.

4.18. COROLLARY. *If $G = \sum_1^\infty G_n$ is an essentially unbounded decomposition of a reduced p-group, and if $T(\prod_1^\infty G_n) = \sum H_\lambda$, then some H_λ has an unbounded torsion-complete p-group as a summand.*

Proof. This is an immediate consequence of Proposition 1.6 and Theorem 4.17.

In the above discussion, we note that Λ may be any index set.

4.19. THEOREM. *If $T(\prod_1^\infty G_n) = \sum_1^\infty H_n$, where all G_ns are countable and reduced p-groups, and $H_m \cong H_n$ for all m and n, then $\sum G_n$ is essentially bounded.*

Case 1. *The group $T(\prod G_n)$ has no non-zero elements of infinite height.*

Proof of Case 1. If $\sum G_n$ is not essentially bounded, $|p^n T(\prod G_n)| > \aleph_0$ for all n. Since $\sum H_n$ is a countable direct sum of isomorphic groups, $|p^n H_n| > \aleph_0$ for all n. Since H_n is reduced, $|(p^n H_n)[p]| > \aleph_0$ for all n. Consider $pH_1[p]$. It is uncountable. Since G_1 is countable, some distinct elements, x and y in $pH_1[p]$ have the same G_1-component when expressed as an $\aleph_0$-tuple in $T(\prod G_n)$. Now $h_1 = x - y \neq 0$ is in $pH_1[p] \cap T(\prod_2 G_n)$. Similarly, we can find $h_i \neq 0$ in $p^i H_i[p] \cap T(\prod_{i+1} G_n)$. By the purity of $T(\prod_{i+1} G_n)$, then, $h_i = p^i(0, \ldots, 0, g_{i+1}^i, \ldots, g_n^i, \ldots)$ for elements $g_n^i \in G_n$ and all i. Form

$$x = (0, pg_2^1, p^1 g_3^1 + p^2 g_3^2, \ldots, pg_n^1 + p^2 g_n^2 + \ldots + p^{n-1} g_n^{n-1}, \ldots)$$

in $T(\prod G_n)[p]$. Let $g_n = h_1 + \ldots + h_{n-1}$, for every $n \geqq 2$. Then $g_n - x \in p^n T(\prod G_n)$ for every $n \geqq 2$. If $x = x_1 + \ldots + x_N$ in $\sum_1^N H_n$, then $g_n - x = (h_1 - x_1, \ldots, h_N - x_N, h_{N+1}, h_{N+2}, \ldots, h_{n-1}, \ldots)$. Since our group has no elements of infinite height, $g_n - x$ has bounded height as n approaches infinity, a contradiction. Case 1 is proved.

4.20. LEMMA. *For p-group G_λ, if K_λ is high in G_λ for each λ, then $T(\prod K_\lambda)$ is high in $T(\prod G_\lambda)$.*

Proof. A subgroup is called high in a group, we recall, if it is maximal with respect to disjointness from the subgroup of elements of infinite height in that group. Let G_λ^1 be this latter subgroup in G_λ and K_λ maximal with respect to $K_\lambda \cap G^1 = 0$ for each λ. We must show that $T(\prod K_\lambda)$ is maximal with respect to $T(\prod K_\lambda) \cap T(\prod G_\lambda^1) = 0$. Suppose that $x \neq 0$ is in $T(\prod G_\lambda)$ such that $\{x, T(\prod K_\lambda)\} \cap T(\prod G_\lambda^1) = 0$. We may suppose that $px = 0$, $x \notin T(\prod K_\lambda)$. Let $x = (g_1, \ldots, g_\lambda, \ldots)$, where $g_\lambda \in G_\lambda$. If $g_\lambda \notin K_\lambda$, then there is a k_λ in K_λ such that $o(k_\lambda) = p$ and $k_\lambda + g_\lambda$ has infinite height. If g_λ is in K_λ, let $k_\lambda = -g_\lambda$, then $\{x, T(\prod K_\lambda)\}$ has as a non-zero element

$y = (g_1 + k_1, \ldots, g_\lambda + k_\lambda, \ldots)$ where each component has infinite height. Then y has infinite height, which contradicts $\{x, T(\prod K_\lambda)\} \cap T(\prod G_\lambda^1) = 0$.

Remark. By a similar argument, $\prod K_\lambda$ can be shown to be high in $\prod G_\lambda$.

Case 2 (of Theorem 4.19). *The group* $T(\prod G_\lambda)$ *has non-zero elements of infinite height.*

Proof of Case 2. Let $T(\prod_1^\infty G_n) = \sum_1^\infty H_n$. For each n, let K_n be high in G_n and H_n' high in H_n. By Lemma 4.20, $T(\prod K_n)$ is high in $T(\prod G_n)$. Also $\sum H_n'$ is high by a similar argument. Since $\sum G_n$ is an essentially unbounded decomposition, $|p^i T(\prod G_n)| > \aleph_0$ for all i, and since $\sum K_n$ is essentially unbounded, $|p^i T(\prod K_n)| > \aleph_0$ for all i. By [**3**, p. 1380, Theorem 5], $p^i T(\prod K_n)$ and $p^i(\sum H_n')$ are high in $p^i T(\prod G_n)$ and hence have the same cardinality. Therefore $|p^i \sum H_n'| > \aleph_0$ and $|p^i H_n'| > \aleph_0$ for all i. We may now revert to Case 1, being careful to take each h_i from H_i'. Such a subgroup is without elements of infinite height and the contradiction of Case 1 will repeat itself now in Case 2.

4.21. THEOREM. *Let G be a p-group without elements of infinite height. If $G = H \oplus K$ contains an unbounded torsion-complete group, then either H or K contains an unbounded torsion-complete group.*

The following two lemmas yield a straightforward proof of the theorem.

4.22. LEMMA. *Let N be a subgroup of a torsion-complete group $\bar{B}$ such that $\bar{B}/N$ is reduced. Then N is torsion-complete.*

Proof. From the exact sequence: $0 \to N \to \bar{B} \to \bar{B}/N \to 0$, we obtain the exact sequence:

$$(1) \quad 0 = \mathrm{Hom}(Z(p^\infty), \bar{B}/N) \to \mathrm{Ext}(Z(p^\infty), N) \\ \to \mathrm{Ext}(Z(p^\infty), \bar{B}) \to \mathrm{Ext}(Z(p^\infty), \bar{B}/N) \to 0.$$

Now, as is well-known, a reduced p-group G is torsion-complete if and only if $\mathrm{Pext}(Z(p^\infty), G) = 0$. Moreover, $\mathrm{Pext}(Z(p^\infty), G)$ is the subgroup of elements of infinite height in $\mathrm{Ext}(Z(p^\infty), G)$. Since $\mathrm{Pext}(Z(p^\infty), \bar{B}) = 0$, $\mathrm{Ext}(Z(p^\infty), \bar{B})$ has no elements of infinite height, whence, by (1), $\mathrm{Ext}(Z(p^\infty), N)$ has no elements of infinite height, or $\mathrm{Pext}(Z(p^\infty), N) = 0$. Thus N is torsion-complete, as stated.

4.23. LEMMA. *Let N be a p^n-bounded subgroup of the torsion-complete p-group $\bar{B}$. Then if $(\bar{B}/N)^1 = 0$, $\bar{B}/N$ is torsion-complete.*

Proof. To see this, we show that $\mathrm{Pext}(Z(p^\infty), \bar{B}/N) = 0$. From the exact sequence (1), we see that

$$\mathrm{Ext}\left(Z(p^\infty), \frac{\bar{B}}{N}\right) \cong \frac{\mathrm{Ext}(Z(p^\infty), \bar{B})}{\mathrm{Ext}(Z(p^\infty), N)} \cong \frac{\mathrm{Ext}(Z(p^\infty), \bar{B})}{N},$$

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where, since N is cotorsion, $N \cong \operatorname{Ext}(Z(p^\infty), N)$. Now, since $(\bar{B}/N)^1 = 0$, $\operatorname{Pext}(Z(p^\infty), \bar{B}/N)$ is torsion-free.

Suppose that $g + N = p^r g_r + N$ is an element of infinite height and infinite order in $\operatorname{Ext}(Z(p^\infty), \bar{B})/N$. Then $p^n g = p^{r+n} g_r$ for all positive integers r, so that $0 \neq p^n g$ is an element of infinite height in $\operatorname{Ext}(Z(p^\infty), \bar{B})$, a contradiction. Thus $\operatorname{Pext}(Z(p^\infty), \bar{B}/N) = 0$, and $\bar{B}/N$ is torsion-complete, as stated.

Proof of the Theorem 4.21. Consider the projection $\pi_H: \bar{B} \to H$ in the decomposition $G = H \oplus K$. The kernel of this map is $\bar{B} \cap K$. Now if $\bar{B} \cap K$ is unbounded, it is the sought-after torsion-complete group, in K, by Lemma 4.22. On the other hand, if $\bar{B} \cap K$ is bounded, then $\bar{B}/\bar{B} \cap K \cong \pi_H(\bar{B}) \subset H$ is unbounded and torsion-complete by Lemma 4.23. This completes the proof.

4.24. Corollary. *Let $G = \sum_1^N H_i$ be a p-group without elements of infinite height. If G contains an unbounded torsion-complete group, then so does some H_i.*

4.25. Theorem. *If $G = \sum H_\lambda$ is a p-group without elements of infinite height, which contains an unbounded torsion-complete group, then some H_λ contains an unbounded torsion-complete group.*

Proof. (a) Let $\bar{B}$ be an unbounded torsion-complete group in G. Let $B = B_1 \oplus B_2$ be basic in $\bar{B}$, where $|B_1| = \aleph_0$. $\bar{B} = \bar{B}_1 \oplus \bar{B}_2$, by [**1**, p. 115, Theorem 34.3]. Thus, we may for our purposes, assume that $\bar{B}$ is the torsion completion of a countable basic subgroup B. Then B is contained in a countable subsum of the H_λs. By Lemma 4.2, $\bar{B}$ is in the countable subsum of the H_λs also. We suppose, then, that $\bar{B} \subset \sum_1^\infty H_n$.

(b) Consider $S_i = p^i\bar{B} \cap p^i(\sum_{n=i}^\infty H_n)$, $i = 1, 2, 3, \ldots$. We shall show that $S_i = 0$ for some i. If $S_i \neq 0$ for each i, we can find a set of positive integers: $N_1 < N_2 < \ldots$ and a set $[p^i b_i]_1^\infty$ with each b_i in $\bar{B}$,

$$p^i b_i \in p^i\left(\sum_{N_{i-1}+1}^{N_i} H_n\right).$$

We can also assume that $o(p^i b_i) = p$, for every i. Consider $g_n = \sum_{i=1}^{n-1} p^i b_i$. Then $g_{n+1} - g_n \in p^n\bar{B}$, for every n and the Cauchy sequence $[g_n]_1^\infty$ has a limit g in $\bar{B}$. Suppose that $g = (h_1 + \ldots + h_N)$ in $\sum_1^N H_n$. Then,

$$g_n - g = \sum_1^{n-1} p^i b_i - (h_1 + \ldots + h_N).$$

Since the $p^i b_i$s are from mutually disjoint subsums of $\sum_1^\infty H_n$, the height of $g_n - g$ is bounded as n approaches infinity and $g_n - g \notin p^n\bar{B} \subset p^n G$, for every n, a contradiction. Thus, for $N > 0$,

$$\sum_1^{N-1} p^N H_n = G \Big/ \left(\sum_N^\infty p^N H_n\right) \supset \left\{p^N\bar{B}, \sum_N^\infty p^N H_n\right\} \Big/ \sum_N^\infty p^N H_n \cong p^N\bar{B}/\{0\} \cong p^N\bar{B}.$$

Since $p^N\bar{B}$ is unbounded and torsion-complete, $\sum_1^{N-1} H_n$ contains an unbounded torsion-complete group. Corollary 4.24 completes the proof.

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4.26. THEOREM. *If $T(\prod_1^\infty G_n) = \sum_1^\infty H_n$ is a p-group without elements of infinite height where no G_n contains an unbounded torsion-complete group and $H_n \cong H_m$, for every n and m, then $\sum G_n$ is essentially bounded.*

Proof. Suppose that $\sum G_n$ is essentially unbounded. By Proposition 1.6, $T(\prod G_n)$ has an unbounded torsion-complete summand. By Theorem 4.25, each H_n has an unbounded torsion-complete subgroup, say $\bar{B}_n$. We write $\sum H_n = \sum_1^\infty \bar{B}_n \oplus K$. Consider the natural projection $\pi_1: \bar{B}_1 \to G_1$ in $G_1 \oplus T(\prod_2^\infty G_n)$. Since G_1 contains no unbounded torsion-complete group, π_1 is not one-to-one. We can find then $b_1 \in \bar{B}_1 \cap T(\prod_2^\infty G_n)$ with $o(b_1) = p$. Similarly, for each i, consider

$$p^{i-1}(T(\prod G_n)) = T(\prod p^{i-1}G_n) = \sum p^{i-1}\bar{B}_n \oplus p^{i-1}K,$$

and the projection $\pi_i: p^{i-1}\bar{B}_i \to p^{i-1}G_1 + \ldots + p^{i-1}G_i$. By Corollary 4.24, $p^{i-1}G_1 + \ldots + p^{i-1}G_i$ contains no unbounded torsion-complete group, and the projection π_i is not one-to-one, for any i. We then find, for each $i = 1, 2, 3, \ldots, p^{i-1}b_i \in p^{i-1}B_i \cap T(\prod_{i+1}G_n)$ with $o(p^{i-1}b_i) = p$. Letting $g_n = \sum_1^n p^{i-1}b_i$, $[g_n]_1^\infty$ is a bounded Cauchy sequence which converges in $T(\prod G_n)$. If $g_n \to g = b + k$, $b \in \sum_1^N \bar{B}_n$, $k \in K$, then, for large n,

$$\begin{aligned} g - g_n &= b + k - \sum_1^n p^{i-1}b_i \\ &= b - \sum_1^N p^{i-1}b_i - p^N b_{N+1} - p^{N+1}b_{N+2} - \ldots - p^{n-1}b_n + k. \end{aligned}$$

As n approaches infinity, $H(g - g_n)$ is bounded, which contradicts convergence. The theorem is thus proved.

4.27. COROLLARY. *Neither the product, nor its torsion subgroup, of a countably infinite collection of unbounded direct sums of cyclic p-groups equals an infinite direct sum of isomorphic groups.*

4.28. THEOREM. *A countable direct product of isomorphic p-groups can be decomposed into an infinite direct sum of isomorphic groups if and only if the product is the direct sum of a divisible group and a bounded group.*

Proof. (a) Let $\prod_1^\infty G_n$ be a countable direct product of isomorphic p-groups. If $\prod G_n = D \oplus B$, where D is divisible and B is bounded, the ranks of D and/or B are infinite. We then express D, B, and consequently $\prod G_n$, as a direct sum of isomorphic groups.

(b) Suppose that $\prod_1^\infty G_n = \sum_1^\infty H_n$, where $[G_n]_1^\infty$ is a set of isomorphic p-groups, $[H_n]_1^\infty$ is a set of isomorphic groups. Suppose that $\prod G_n$ is not the direct sum of a divisible and a bounded group. Then the reduced part of each G_n is unbounded. We can find $g_n \in G_n$ for every n such that $\langle g_n \rangle$ is pure and $o(g_n) < o(g_{n+1})$. Now $x = (g_1, g_2, \ldots, g_n, \ldots)$ is in $\prod G_n$ and $\langle x \rangle$ is a p-pure cycle of infinite order. Thus $\sum H_n$, and in fact each H_i contains a pure (p-pure) cycle of infinite order, say $\langle h_i \rangle$. Now there exists k_i such that $p^{k_i}h_i = y_i$ is in $\prod_{n>i} G_n$ for each i. Letting $a_n = y_1 + py_2 + \ldots + p^{n-1}y_n$, there exists a non-zero a in $\prod G_n$ such that $a_n - a \in p^n \prod G_n$ for every n. However, as n increases, $a_n - a$ has bounded p-height, since a is in a finite sum of H_ns and $a_{n+1} - a_n$ is in H_{n+1} and of finite p-height. This contradiction completes the proof.

Note. We can replace countable by infinite in the preceding theorem, since a countable product will split off and the proof remain intact.

5. Open questions. Many questions about direct products of Abelian p-groups remain to be answered. Especially relevant to the work in this paper are the following.

(1) Under what conditions does a product (torsion subgroup of a product) of p-groups decompose into an infinite direct sum of isomorphic groups?

(2) If the torsion subgroup of a product of p-groups equals $A \oplus B$, does the product equal $A' \oplus B'$, where $A' \supset A$ and $B' \supset B$?

(3) For p-groups G_λ, when do epimorphisms exist of the following type: (a) $\prod G_\lambda \to T(\prod G_\lambda)$, (b) $\prod G_\lambda \to \sum G_\lambda$, (c) $T(\prod G_\lambda) \to \sum G_\lambda$? We note that, for bounded G_λs and unbounded product, the epimorphisms (a) and (b) do not exist since $\prod G_\lambda$ would be cotorsion as well as any homomorphic image of it; (c) would exist since $\sum G_\lambda$ would be a direct summand of a basic subgroup of $T(\prod G_\lambda)$.

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