A NOTE ON REGULAR MEASURES

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As is well known, every Borel measure in a metric space S is regular,[1] provided that S is the union of a sequence of open sets of finite measure.[2] It seems, however, not yet to have been noticed that this theorem can be easily extended to all spaces with Urysohn's "F-property", i.e., spaces in which every closed set is a countable intersection of open sets (we call such spaces "F-spaces"). Indeed, various theorems are unnecessarily restricted to metric spaces, while weaker assertions are made about F-spaces.[3] This seems to justify the publication of the following simple proof which extends the theorem stated above to F-spaces.

Let m be a (non-negative) measure defined on a σ-field[4] M of subsets of a topological space S. A set $E \in M$ is said

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1) i.e., such that the measure mE of each measurable set E is the g.l.b. of the measures of all open sets which contain E. A Borel measure is a measure defined on the Borel field in a topological space S, i.e. on the least set family containing all open sets in S and closed under countable unions and complementation.

2) Cf. [3], p. 660.

3) Thus, e.g., Schaerf [4] limits his Theorem 8 to metric spaces, while a weaker Theorem 3 is proved for F-spaces. Saks [2], p. 153, likewise unnecessarily weakens his assertions about measures in F-spaces.

4) Called σ-algebra in [1].

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to be regular if, for every real $\varepsilon > 0$, there are two sets $F, G \in M$ such that $F \subseteq E \subseteq G$, $m(E-F) < \varepsilon$, $m(G-E) < \varepsilon$, with F closed, and G open in S. If all members of M are regular, the measure m is referred to as strongly regular. As is readily seen, each such measure is regular, but the converse is not true. The triple (S, M, m) is called a measure space. $(S \in M)$.

LEMMA. *In every measure space (S, M, m), the family R of all regular sets is a field of sets. If, further, $m(S) < \infty$, then R is a σ-field.*

Proof. R is obviously closed under complementation, as follows by replacing the sets E, F, G in the definition of a regular set by $S-E$, $S-G$ and $S-F$, respectively. To prove closure under unions, let $A, B \in R$. Then there are sets $F, F', G, G' \in M$ such that $F \subseteq A \subseteq G$, $F' \subseteq B \subseteq G'$, $m(A-F) < \frac{1}{2}\varepsilon$, $m(B-F') < \frac{1}{2}\varepsilon$, $m(G-A) < \frac{1}{2}\varepsilon$, $m(G'-B) < \frac{1}{2}\varepsilon$, with G, G' open and F, F' closed in S. As is readily seen, the open set $G \cup G'$ and the closed set $F \cup F'$ satisfy the conditions of regularity with respect to $A \cup B$. Thus $A \cup B \in R$, and R is a set field.

Now suppose that m is finite. It suffices to show that R is closed under countable disjoint unions. Let then $A = \bigcup_{n=1}^{\infty} A_n$ be a disjoint union, with all A_n regular. Then, given $\varepsilon > 0$, one can find, for each n, a closed set $F_n \in M$ and an open set $G_n \in M$ such that $F_n \subseteq A_n \subseteq G_n$, $m(G_n - A_n) < \varepsilon/2^n$, $m(A_n - F_n) < \varepsilon/2^{n+1}$, $n = 1, 2, \ldots$. Put $E = \bigcup_{n=1}^{\infty} F_n$, so that $E \in M$, $E \subseteq \bigcup_{n=1}^{\infty} A_n = A$ and $m(E) = \sum_{n=1}^{\infty} mF_n$. As m is finite, $m(E - \bigcup_{n=1}^{p} F_n) < \frac{1}{2}\varepsilon$ for some positive integer p. Also, $m(A-E) = m(\bigcup_{n=1}^{\infty} A_n - \bigcup_{n=1}^{\infty} F_n) \leq m\left(\bigcup_{n=1}^{\infty} (A_n - F_n)\right) < \frac{1}{2}\varepsilon$. Let $F = \bigcup_{n=1}^{p} F_n \subseteq A$ and $G = \bigcup_{n=1}^{\infty} G_n \supseteq A$. Then, as is readily seen, $m(A-F) < \varepsilon$ and $m(G-A) < \varepsilon$. Thus $A \in R$, q.e.d.

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THEOREM. Let m be a Borel measure in an F-space S. If S is the union of a sequence of open sets of finite measure, then m is strongly regular, and so is its least complete extension $\bar{m}$.

Proof. Suppose first that $m(S) < \infty$. It suffices to show that all open sets in S are regular.[5] Let then E be an open set. By the F-property (taking the complements) it follows that $E = \bigcup_{n=1}^{\infty} F_n$ for some closed sets F_n. Hence $\lim_{n\to\infty} m(\bigcup_{k=1}^{n} F_k) = mE < \infty$. Therefore, given $\varepsilon > 0$, there is a positive integer p such that $m(E - \bigcup_{k=1}^{p} F_k) < \varepsilon$. The regularity of E now follows by setting $F = \bigcup_{k=1}^{p} F_k$ and $G = E$ in the definition of a regular set. Thus all open sets (and, hence, all Borel sets) are regular. The theorem then holds for finite Borel measures.

Next let $S = \bigcup_{n=1}^{\infty} G_n$, with all G_n open sets of finite measure, and let B be any Borel set. Define $B_n = B \cap G_n$, $n = 1, 2, \ldots$. Clearly, each B_n is a Borel set of finite measure, and $B = \bigcup_{n=1}^{\infty} B_n$. Fixing any one of the open sets G_n $(m(G_n) < \infty)$, treat it as a subspace and let B' be the (relative) Borel field of that subspace. Then B' consists of all sets of the form $Y \cap G_n$, with Y a Borel set in S.[6]

5) Here we are using the second part of the Lemma and the definition of the Borel field as the least σ-field containing all open sets.

6) Cf. [1], p. 25.

In particular, $B_n = B \cap G_n \in B'$. As G_n is open, all members of B' are also Borel sets in S; similarly for open sets in G_n. Since $m(G_n) < \infty$, the restriction of m to B' is a finite Borel measure in G_n. Thus, by the first part of the proof, B_n is regular in G_n. Therefore (unfixing n), one can find, for each n, an open set $G'_n \supseteq B_n$ such that $m(G'_n - B_n) < \varepsilon / 2^n$, $n = 1, 2, \ldots$. Let $G = \bigcup_{n=1}^{\infty} G'_n$. Then $G \supseteq \bigcup_{n=1}^{\infty} B_n = B$, and

$m(G-B) \le m\left(\bigcup_{n=1}^{\infty} (G'_n - B_n) \right) < \varepsilon$. By applying the same process to the complement of B in S, one obtains a closed set $F \subseteq B$ with $m(B-F) < \varepsilon$. Thus, indeed, each Borel set B in S is regular. It follows that m is strongly regular, as asserted. The same applies to its completion $\overline{m}$ since all $\overline{m}$-measurable sets, by definition, are Borel sets modified by subsets of sets of measure zero, and this, as is easily seen, does not affect their regularity. This completes the proof of the theorem.

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