J. Aust. Math. Soc. **107** (2019), 110–132
doi:10.1017/S1446788718000241

TAME DISCRETE SUBSETS IN STEIN MANIFOLDS

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(Received 6 October 2017; accepted 21 June 2018; first published online 26 December 2018)

Communicated by F. Larusson

Abstract

Rosay and Rudin introduced the notion of 'tameness' for discrete subsets of $\mathbf{C}^n$. We generalize the notion of tameness for discrete sets to arbitrary Stein manifolds, with special emphasis on complex Lie groups.

2010 *Mathematics subject classification*: primary 32M10; secondary 32M17.

Keywords and phrases: tame discrete set, semisimple complex Lie group.

1. Introduction

For discrete subsets in $\mathbf{C}^n$ the notion of being 'tame' was defined in the important paper of Rosay and Rudin [3]. A discrete subset $D \subset \mathbf{C}^n$ is called tame if and only if there exists an automorphism ϕ of $\mathbf{C}^n$ such that $\phi(D) = \mathbf{N} \times \{0\}^{n-1}$. (In this paper a subset D of a topological space X is called a 'discrete subset' if every point p in X admits an open neighborhood W such that $W \cap D$ is finite.)

We want to introduce and study a similar notion for complex manifolds other than $\mathbf{C}^n$.

Therefore, we propose a new definition, show that it is equivalent to that of Rosay and Rudin if the ambient manifold is $\mathbf{C}^n$ and deduce some standard properties.

To obtain good results, we need some knowledge of the automorphism group of the respective complex manifold. For this reason we get our best results in the case where the manifold is biholomorphic to a complex Lie group. We concentrate on semisimple complex Lie groups, since every simply connected complex Lie group is biholomorphic to a direct product of $\mathbf{C}^n$ and a semisimple complex Lie group.

Definition 1.1. Let X be a complex manifold. An infinite discrete subset D is called (weakly) *tame* if for every exhaustion function $\rho : X \to \mathbf{R}^+$ and every map $\zeta : D \to \mathbf{R}^+$ there exists an automorphism ϕ of X such that $\rho(\phi(x)) \geq \zeta(x)$ for all $x \in D$.

Andrist and Ugolini [1] have proposed a different notion, namely the following definition.

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DEFINITION 1.2. Let X be a complex manifold. An infinite discrete subset D is called (strongly) tame if for every injective map $f : D \to D$ there exists an automorphism ϕ of X such that $\phi(x) = f(x)$ for all $x \in D$.

It is easily verified that 'strongly tame' implies 'weakly tame'. For $X \simeq \mathbf{C}^n$ and $X \simeq \mathrm{SL}_n(\mathbf{C})$ both tameness notions coincide. Furthermore, for $X = \mathbf{C}^n$ both notions agree with tameness as defined by Rosay and Rudin.

However, for arbitrary manifolds 'strongly tame' and 'weakly tame' are not equivalent.

In this article, unless explicitly stated otherwise, tame always means weakly tame, that is, tame in the sense of Definition 1.1.

2. Comparison between $\mathbf{C}^n$ and semisimple complex Lie groups

For tame discrete sets in $\mathbf{C}^n$ in the sense of Rosay and Rudin, the following facts are well known.

(1) Any two tame sets are equivalent.
(2) Every discrete subgroup of $(\mathbf{C}^n, +)$ is tame as a discrete set.
(3) Every discrete subset of $\mathbf{C}^n$ is the union of two tame ones.
(4) There exist nontame subsets in $\mathbf{C}^n$.
(5) Every injective self-map of a tame discrete subset of $\mathbf{C}^n$ extends to a biholomorphic self-map of $\mathbf{C}^n$.
(6) If v_k is a sequence in $\mathbf{C}^n$ with $\sum_{k=1}^{\infty}(1/\|v_k\|^{2n-1}) < \infty$, then $\{v_k : k \in \mathbf{N}\}$ is a tame discrete subset.

(See [3] for (1), (3), (4), (5), [5] for (6). For $n = 2$, (2) is implied by Proposition 4.1 of [2]. The proof given there generalizes easily to arbitrary dimension n.)

For discrete subsets in semisimple complex Lie groups we are able to prove the following properties.

(1) Any two tame discrete subsets in $\mathrm{SL}_n(\mathbf{C})$ are equivalent (Proposition 10.6).
(2) Certain discrete subgroups may be verified to be tame discrete subsets. In particular, $\mathrm{SL}_2(\mathbf{Z}[i])$ is a tame discrete subset (Corollary 11.3) and also every discrete subgroup of a one-dimensional Lie subgroup of $\mathrm{SL}_n(\mathbf{C})$ (Proposition 12.2) and every discrete subgroup of a maximal torus (Corollary 10.11).
(3) Every discrete subset of $\mathrm{SL}_n(\mathbf{C})$ is the union of n tame discrete subsets (Corollary 10.10).
(4) Every semisimple complex Lie group admits a nontame discrete subset (Proposition 9.9).
(5) Every injective self-map of a tame discrete subset of $\mathrm{SL}_n(\mathbf{C})$ extends to a biholomorphic self-map of $\mathrm{SL}_n(\mathbf{C})$ (Proposition 10.6).
(6) For every semisimple complex Lie group S, there exists a 'threshold sequence', that is, there exist a sequence of numbers $R_k > 0$ and an exhaustion function

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τ such that every sequence g_k with $\tau(g_k) > R_k$ defines a tame discrete subset (Proposition 9.7).

3. Results for other manifolds

While tame discrete sets in semisimple complex Lie groups behave in a way very similar to those in $\mathbf{C}^n$, for arbitrary complex manifolds the situation is quite different.

(1) On $\mathbf{C}^n \setminus \{(0, \ldots, 0)\}$ ($n \geq 2$) there exist discrete subsets which may not be realized as a finite union of tame discrete subsets (Corollary 6.4).

(2) On $\Delta \times \mathbf{C}$ there are (weakly) tame discrete sets which are not strongly tame. There are permutations of tame discrete sets which do not extend to biholomorphic self-maps of the ambient manifold (Propositions 5.1 and 5.3).

(3) On $\Delta \times \mathbf{C}$ there exist inequivalent tame discrete subsets (Corollary 5.2).

(4) On $\mathbf{C}^n \setminus \{(0, \ldots, 0)\}$ there is no 'threshold sequence' (Corollary 6.3).

4. Preparations

PROPOSITION 4.1. *Let X be a complex manifold and let D be an infinite discrete subset. Then D is tame if and only if there exists* one *exhaustion function ρ such that the following property holds. For every map $\zeta : D \to \mathbf{R}^+$, there exists an automorphism ϕ of X such that $\rho(\phi(x)) \geq \zeta(x)$ for all $x \in D$.*

(In the Definition 1.1 for being tame it is required that this property holds for *every* exhaustion function.)

PROOF. Assume that the property holds with respect to a given exhaustion function ρ. We have to show that D is tame, that is, that the property holds with respect to every exhaustion function. Let τ be an arbitrary exhaustion function and let $\zeta : D \to \mathbf{R}$ be a map. We choose a map $\zeta_0 : D \to \mathbf{R}$ in such a way that

$$\zeta_0(x) > \sup\{\rho(p) : \tau(p) < \zeta(x)\} \quad \forall x \in D.$$

By assumption, there is an automorphism ϕ of X such that $\rho(\phi(x)) \geq \zeta_0(x)$ for all $x \in D$. By the construction of ζ_0,

$$\phi(x) \notin \{p \in X : \tau(p) < \zeta(x)\} \quad \forall x \in D.$$

Hence, $\tau(\phi(x)) \geq \zeta(x)$ for all $x \in D$, as desired. □

DEFINITION 4.2. Let X be a complex manifold. Two sequences $A(k)$, $B(k)$ in X are called *equivalent* if there exists a holomorphic automorphism ϕ of the complex manifold X such that $\phi(A(k)) = B(k)$ for all $k \in \mathbf{N}$.

A sequence $A(k)$ in X is called *tame* if the set $\{A(k) : k \in \mathbf{N}\}$ is a tame discrete subset of X.

PROPOSITION 4.3. *Let X be a complex manifold. Assume that the automorphism group* $\mathrm{Aut}_{\mathcal{O}}(X)$ *is a finite-dimensional Lie group with countably many connected components. Then X does not admit any tame discrete subset.*

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PROOF. Let $D = \{p_n : n \in \mathbf{N}\}$ be an infinite discrete subset of X. Fix an exhaustion function $\rho : X \to \mathbf{R}^+$. Let K_n be an increasing sequence of compact subsets of $G = \mathrm{Aut}_{\mathcal{O}}(X)$ which exhausts G, that is, $\bigcup_n K_n = G$. Define $c_n = \max\{\rho(x) : x \in K_n(p_n)\}$. Choose $\zeta_n > c_n$. By construction, if $\rho(\phi(p_n)) > \zeta_n$ for some $n \in \mathbf{N}$, then $\phi \in G \setminus K_n$. Since $\bigcap_n (G \setminus K_n) = \{\}$, it follows that there is no $\phi \in \mathrm{Aut}_{\mathcal{O}}(X)$ with $\rho(\phi(p_n)) > \zeta_n$ for all $n \in \mathbf{N}$. Thus, D is not tame. □

COROLLARY 4.4. *Complex manifolds which are hyperbolic in the sense of Kobayashi (for example, bounded domains in Stein manifolds) do not admit tame subsets.*

COROLLARY 4.5. *Let $\bar{X}$ be a compact complex manifold, $\dim_{\mathbf{C}}(X) \geq 2$, and let S be a finite subset. Then $X = \bar{X} \setminus S$ contains no tame discrete subset.*

PROOF. Every automorphism of X extends to an automorphism of $\bar{X}$ and the automorphism group of $\bar{X}$ is a finite-dimensional Lie group by the theorem of Bochner and Montgomery. □

COROLLARY 4.6. *There are no tame discrete subsets in Riemann surfaces.*

PROPOSITION 4.7. *A discrete subset D of $\mathbf{C}^n$ is tame in the sense of Definition 1.1 if and only if it is tame in the sense of Rosay and Rudin, that is, if and only if there exists a holomorphic automorphism ϕ of $\mathbf{C}^n$ such that $\phi(D) = \mathbf{Z} \times \{0\}^{n-1}$.*

PROOF. For every map $\xi : \mathbf{Z} \to \mathbf{R}$, there exists a holomorphic function f on $\mathbf{C}$ with $f(n) = \xi(n)$ for all $n \in \mathbf{Z}$. The automorphism $z \mapsto (z_1, z_2 + f(z_1), z_3, \ldots, z_n)$ maps $\mathbf{Z} \times \{0\}^{n-1}$ to $\{(n, \xi(n), 0, \ldots, 0) : n \in \mathbf{Z}\}$. Using this fact, it is clear that $\mathbf{Z} \times \{0\}^{n-1}$ is tame in the sense of Definition 1.1. Therefore, being tame in the sense of Rosay and Rudin implies being tame in the sense of Definition 1.1.

Conversely, if a discrete set $D = \{a_k : k \in \mathbf{N}\}$ is tame in the sense of Definition 1.1, then there exists a biholomorphic map ϕ of $\mathbf{C}^n$ such that $\|\phi(a_k)\|^{2n-1} > k^2$ for all k. Then the proposition below implies that D is tame in the sense of Rosay and Rudin, since $\sum_k k^{-2} < \infty$. □

PROPOSITION 4.8. *Let v_k be a sequence in $\mathbf{C}^n$. If*

$$\sum_{k=1}^{\infty} \frac{1}{\|v_k\|^{2n-1}} < \infty,$$

then $D = \{v_k : k \in \mathbf{N}\}$ is a tame (in the sense of Rosay and Rudin) discrete subset of $\mathbf{C}^n$.

PROOF. See [5]. □

DEFINITION 4.9. Let X be a complex manifold with an exhaustion function ρ. A sequence of positive real numbers R_n is called a 'threshold sequence' for (X, ρ) if every discrete subset D with

$$\#\{x \in D : \rho(x) \leq R_n\} < n \quad \forall n \in \mathbf{N}$$

is tame.

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In other words: if R_n is a threshold sequence, then every sequence x_k in X with $\rho(x_k) \geq R_k$ defines a tame discrete subset of X.

It follows from [5] that every sequence (R_k) with $\sum_k (R_k)^{-(2n-1)} < \infty$ is a threshold sequence for $\mathbf{C}^n$ (with respect to the exhaustion function $\rho(x) = \|x\|$).

If a complex manifold X with exhaustion function ρ admits a threshold sequence R_k and $\tilde{\rho}$ is a different exhaustion function, we may define a threshold sequence $\tilde{R}_k$ for $(X, \tilde{\rho})$ as follows. We need to ensure that $\rho(x) > R_k$ implies that $\tilde{\rho}(x) > \tilde{R}_k$. Hence, we may define

$$\tilde{R}_k = \max\{\tilde{\rho}(x) : \rho(x) \leq R_k\}.$$

Thus, if there exists a threshold sequence for one exhaustion function, then there also exists a threshold sequence for any other exhaustion function on the same complex manifold, that is, whether or not there exists a threshold sequence depends only on the complex manifold, not on the exhaustion function.

We will see that there exist threshold sequences for every semisimple complex Lie group (Proposition 9.7).

In contrast, there is no threshold sequence for $\mathbf{C}^n \setminus \{(0, \ldots, 0)\}$ (Corollary 6.3).

PROPOSITION 4.10. *Let X be a complex manifold for which there exists a threshold sequence. Let $A \subset X$ be an unbounded (that is, not relatively compact) subset. Then A contains a subset which is a tame discrete subset of X.*

PROOF. This is obvious. □

5. The case $X = \Delta \times \mathbf{C}$

We start by deducing a description of the automorphisms. Let $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. On $X = \Delta \times \mathbf{C}$ there is a natural equivalence relation: two points (x, y), (z, w) can be separated by a bounded holomorphic function if and only if $x \neq z$. The projection onto the first factor is therefore equivariant for every automorphism of X. Using this fact, one easily verifies that every automorphism of X can be written in the form

$$(z, w) \mapsto (\phi(z), f(z)w + g(z))$$

with $\phi \in \mathrm{Aut}_{\mathcal{O}}(\Delta)$, $f \in \mathcal{O}^*(\Delta)$ and $g \in \mathcal{O}(\Delta)$.

PROPOSITION 5.1. *Let $X = \Delta \times \mathbf{C}$ with $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. Let $\pi : \Delta \times \mathbf{C} \to \Delta$ denote the projection onto the first factor.*

A discrete subset $D \subset X$ is tame if and only if

$$\pi(D) = \{z \in \Delta : \exists (z, w) \in D\}$$

is discrete in Δ and

$$\pi^{-1}(p) = \{(z, w) \in D : z = p\}$$

is finite for all $p \in \Delta$.

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PROOF. The discrete sets fulfilling these conditions are tame due to Proposition 8.4.

Suppose conversely that D is a discrete subset which does not fulfill all of these conditions. Then there exists a divergent sequence (p_n, q_n) in D with $\lim p_n = p \in \Delta$. We fix an exhaustion function τ on X such that $\tau(z, w) = |w| + (1 - |z|)^{-1}$. We choose R_n such that $R_n > 2^n|q_n|$. By the tameness assumption, there exists an automorphism of X given as

$$(z, w) \mapsto (\phi(z), f(z)w + g(z))$$

such that

$$\tau(\phi(p_n), f(p_n)q_n + g(p_n)) > R_n \quad \forall n.$$

Since $\lim p_n = p$,

$$\lim(1 - \phi(p_n))^{-1} = (1 - \phi(p))^{-1}.$$

Thus, there exists a constant K such that $(1 - \phi(p_n))^{-1} < K$ for all sufficiently large n. Then

$$|f(p_n)q_n + g(p_n)| > R_n - K > 2^n|q_n| - K.$$

On the other hand, we obtain (using $\lim p_n = p$)

$$\lim \frac{1}{|q_n|}|f(p_n)q_n + g(p_n)| = |f(p)|.$$

This yields a contradiction, since

$$\frac{1}{|q_n|}|f(p_n)q_n + g(p_n)| > 2^n - \frac{K}{|q_n|} \quad \forall n,$$

but $\lim(2^n - K/|q_n|) = +\infty$. □

COROLLARY 5.2. *The manifold $X = \Delta \times \mathbf{C}$ admits inequivalent tame discrete subsets.*

PROOF. Every automorphism of the unit disc preserves the Poincaré metric. Hence, it is clear that there are many inequivalent discrete subsets in the unit disc. By the proposition, for each discrete subset $D \subset \Delta$ we obtain a tame discrete subset D' in X via $D' = D \times 0\}$. □

Thus, $X = \Delta \times \mathbf{C}$ admits many (weakly) tame discrete subsets. In contrast, there are no strongly tame discrete subsets.

PROPOSITION 5.3. *There are no strongly tame discrete subsets in $X = \Delta \times \mathbf{C}$.*

PROOF. Suppose that D is strongly tame. Then it is tame. Due to Proposition 5.1, $\pi(D) \subset \Delta$ is infinite. Thus, for any two points $v, w \in D$, we can find an injective map $F : D \to D$ with $\pi(F(v)) \neq \pi(F(w))$. It follows that $\pi|_D$ is injective. However, now any injective self-map of $\pi(D)$ is induced by an injective self-map of D. Thus, the assumption of D being strongly tame in X implies that $\pi(D)$ is strongly tame (and therefore tame) in Δ, which is impossible (cf. Corollary 4.4). □

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6. The case $\mathbf{C}^n \setminus \{(0,\ldots,0\}$

We start with a preparation.

Lemma 6.1. *Let ϕ be a C^1-diffeomorphism of $\mathbf{C}^n$ fixing the origin. Then there exist an open neighborhood W of the origin $(0,\ldots,0)$ and constants $C_1, C_2 > 0$ such that*

$$C_1\|v\| \le \|\phi(v)\| \le C_2\|v\| \quad \forall v \in W.$$

Proof. Let U be a convex relatively compact open neighborhood of $(0,\ldots,0)$. Define $C = \sup_{v\in U} \max\{\|D\phi\|_v, \|D(\phi^{-1})\|_v\}$. Then $\|\phi(v)\| \le C\|v\|$ and $\|\phi^{-1}(v)\| \le C\|v\|$ for all $v \in U$. This implies the assertion with $C_2 = C$, $C_1 = 1/C$ and $W = U \cap \phi^{-1}(U)$. □

Proposition 6.2. *A discrete subset D in $X = \mathbf{C}^n \setminus \{(0,\ldots,0\}$ is tame if and only if it is discrete and tame considered as a subset of $\mathbf{C}^n$.*

Proof. We recall that holomorphic automorphisms of X extend to holomorphic automorphisms of $\mathbf{C}^n$.

We fix the exhaustion function $\tau : X \to \mathbf{R}^+$ given by $\tau(x) = \max\{\|x\|, \|x\|^{-1}\}$.

Assume that D is tame and discrete in $\mathbf{C}^n$. Then $D_1 = D \cup \{0\}$ is likewise tame. Enumerate the elements of D such that $D = \{a_k : k \in \mathbf{N}\}$. Let a strictly increasing sequence R_k be given. Choose elements $v_k \in \mathbf{C}^n$ such that $R_{k+1} \ge \|v_k\| > R_k$ for all $k \in \mathbf{N}$. Since D_1 is tame, there exists an automorphism ϕ of $\mathbf{C}^n$ such that $\phi(0) = 0$ and $\phi(a_k) = v_k$ for all $k \in \mathbf{N}$. This shows: for any such sequence R_k, there exists an automorphism of X such that $\|\phi(a_k)\| > R_k$ for all k, that is, D is tame in X.

Now assume that D is a discrete subset in X which is not discrete in $\mathbf{C}^n$. Then there exists a sequence $\gamma_k \in D$ with $\lim \gamma_k = (0,\ldots,0)$. We want to show that D is not tame. We choose a map $\zeta : D \to \mathbf{R}$ such that $\zeta(\gamma_k) > k/\|\gamma_k\|$ for all k. We claim that there exists no holomorphic automorphism ϕ of X such that $\tau(\phi(\gamma_k)) \ge \zeta(\gamma_k)$ for all $k \in \mathbf{N}$. Indeed, each holomorphic automorphism of X extends to an automorphism of $\mathbf{C}^n$ fixing the origin $(0,\ldots,0)$. It follows that there exist a neighborhood W of $(0,\ldots,0)$ in $\mathbf{C}^n$ and constants $C_2 > C_1 > 0$ such that

$$C_2\|v\| \ge \|\phi(v)\| \ge C_1\|v\| \quad \forall v \in W.$$

Assume that W is contained in the unit ball. Then $\tau(v) = 1/\|v\|$ for all $v \in W$ and therefore

$$\frac{1}{C_2\|v\|} \le \tau(\phi(v)) \le \frac{1}{C_1\|v\|} \quad \forall v \in W \setminus \{(0,\ldots,0)\}.$$

This implies that $\tau(\gamma_k) < \zeta(\gamma_k)$ for all k with $\gamma_k \in W$ and $k > 1/C_1$. Hence, D cannot be tame. □

Corollary 6.3. *There is no threshold sequence for $X = \mathbf{C}^n \setminus \{(0,\ldots,0)\}$.*

Proof. Let v_n be any sequence in $\mathbf{C}^n \setminus \{(0,\ldots,0)\}$ which converges to the origin in $\mathbf{C}^n$. If there existed a threshold sequence, (v_n) would contain a tame subsequence, in contradiction to the above proposition. □

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COROLLARY 6.4. *There exist discrete subsets of X which cannot be realized as the union of finitely many tame discrete subsets.*

PROOF. Just take any discrete subset of X which in $\mathbf{C}^n$ has an accumulation point in $(0, \ldots, 0)$. □

7. Some preparation

LEMMA 7.1. *Let G be a connected complex Lie group. Then there exists a surjective holomorphic map from some complex vector space $\mathbf{C}^n$ onto G.*

COROLLARY 7.2. *Let G be a connected complex Lie group, X a Stein complex manifold and $D \subset X$ a discrete subset. Then every map $f : D \to G$ extends to a holomorphic map $F : X \to G$.*

PROOF. We fix a surjective holomorphic map $\Phi : \mathbf{C}^n \to G$. Then, given a map $f : D \to G$, there is a 'lift' $g : D \to \mathbf{C}^n$, that is, a map $g : D \to \mathbf{C}^n$ such that $f = \Phi \circ g$. The existence of the desired map F now follows from the classical fact in complex analysis that the value of a holomorphic function can be described on a fixed discrete set. □

8. π-tame sets

DEFINITION 8.1. Let X, Y be complex manifolds and $\pi : X \to Y$ a holomorphic map. An infinite discrete subset $D \subset X$ is called *π-tame* if there exists an automorphism ϕ of X such that the restriction of the map $\pi \circ \phi$ to D is proper.

REMARK 8.2. A map from a discrete space D into a locally compact space Y is *proper* if and only if it has discrete image and finite fibers.

REMARK 8.3. If D is π-tame and D' is discrete infinite with $D' \setminus D$ being finite, then D' is π-tame.

PROPOSITION 8.4. *Let H be a noncompact connected complex Lie group and let $\pi : X \to Y$ be a H-principal bundle. If Y is Stein, then every π-tame discrete subset $D \subset X$ is tame.*

PROOF. Let ρ be an exhaustion function on X. Assume that $\pi|_D$ is proper. Let $q \in \pi(D)$ and $X_q = \pi^{-1}(q)$. Observe that $X_q \cap D$ is finite and that $\rho|_{X_q}$ is unbounded. Hence, there exists an element $h_q \in H$ (acting on X_q by the principal H-action of the principal bundle) with

$$\rho(ph_q) \geq \zeta(p) \quad \forall p \in X_q \cap D.$$

Next we choose a holomorphic map $F : Y \to H$ with $F(q) = h_q$ for all $q \in \pi(D)$. (Such a map F exists due to Corollary 7.2.) Now we can define the desired automorphism ϕ as the principal action of $F(q)$ on X_q for each $q \in Y$. (In other words: $\phi : x \mapsto x \cdot F(\pi(x))$.) □

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Proposition 8.5. *Let $\pi : X \to Y$ be a principal bundle with noncompact structure group G. Assume that there exists a nonconstant holomorphic function on Y. Then there exists a tame discrete subset in X.*

Proof. Let f be a nonconstant holomorphic function on Y and $W = f(Y)$. Note that W is an open domain in $\mathbf{C}$ and in particular Stein. Choose a sequence x_n in X such that $f(\pi(x_n))$ is without accumulation points in W. Assume that ρ is an exhaustion function on X and let $\zeta : D \to \mathbf{R}^+$ be a given map where $D = \{x_n : n \in \mathbf{N}\}$. Since G is noncompact and acting with closed orbits, we can find elements $g_n \in G$ such that $\rho(x_n g_n) > \zeta(x_n)$. Recall that for every connected complex Lie group G, there exists a surjective holomorphic map $h : \mathbf{C}^N \to G$. Using this and the fact that $f(\pi(D))$ is discrete in W, it is clear that there exists a holomorphic map $F : W \to G$ with $F(f(\tau(x_n))) = g_n$ for all $n \in \mathbf{N}$. Now $\phi : x \mapsto x \cdot F(f(\pi(x)))$ defines an automorphism of X such that $\rho(\phi(x)) > \zeta(x)$ for all $x \in D$. Since $\zeta : D \to \mathbf{R}^+$ was arbitrary, we may conclude that D is tame. □

9. Generic projections

9.1. Generalities.

Proposition 9.1. *Let V, W be finite-dimensional (real) Hilbert spaces, $V \neq \{0\}$, and let $\pi : V \to W$ be a linear map. Let K be a connected compact Lie group which acts linearly and orthogonally on V. We assume that $\ker \pi$ contains no nontrivial K-invariant vector subspace. Let μ denote the Haar measure on K (normalized, that is, $\mu(K) = 1$). Then, for every $r, \delta > 0$, there exists a number $R > 0$ such that*

$$\mu\{k \in K : \|\pi(k(v))\| < r\} < \delta$$

for all $v \in V$ with $\|v\| > R$.

Proof. There is no loss in generality in assuming that $\|\pi(v)\| \leq \|v\|$ for all $v \in V$. Let $S = \{v \in V : \|v\| = 1\}$. Note that S is compact. We define an auxiliary function $f : K \times S \to \mathbf{R}$ as

$$f(k, v) = \|\pi(k \cdot v)\|.$$

Since K does not stabilize any nontrivial vector subspace of $\ker \pi$, it is clear that

$$\forall v \in S : \exists k \in K : f(k, v) > 0.$$

Thus, $\{(k, v) : f(k, v) = 0\}$ is a real-analytic subset of $K \times S$ not containing any of the fibers of the projection $pr_2 : K \times S \to S$. Consequently,

$$\mu(\{k \in K : f(k, v) = 0\}) = 0 \tag{9.1}$$

for every $v \in S$. (Here we use the fact that nowhere-dense real-analytic subsets of K are of Haar measure zero.)

Next we define

$$\Omega(v, r) = \{k \in K : f(k, v) < r\}$$

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and

$$h(v,r) = \mu(\Omega(v,r))$$

for $(v,r) \in S \times (0,1]$. Evidently, $r \mapsto h(v,r)$ is a monotonically increasing function for every fixed v.

On the other hand, $\|\pi(v)\| \leq \|v\|$ for all $v \in V$ implies that

$$\|\pi(kv) - \pi(kw)\| \leq \|v - w\| \quad \forall k \in K, v, w \in V.$$

Thus, $|f(k,v) - f(k,w)| \leq \|v - w\|$ for all $k \in K$, $v, w \in S$. This in turn implies that

$$\Omega(v,r) \subset \Omega(w, r + \epsilon)$$

for $v, w \in S$ with $\|v - w\| < \epsilon$. From this,

$$h(v,r) \leq h(w, r + \|v - w\|) \quad \forall v, w, r. \tag{9.2}$$

Next we define another auxiliary function $g\colon (0,1] \to [0,1]$:

$$r \mapsto g(r) = \sup_{v \in S} h(v,r).$$

We claim that $\lim_{r\to 0} g(r) = 0$.

First we note that the limit $\lim_{r\to 0} g(r)$ exists, because g is monotonically increasing and bounded from below: $g \geq 0$.

Define $c = \lim_{r\to 0} g(r)$. Then there exists a sequence r_n in $(0,1]$ with $\lim_{n\to\infty} r_n = 0$ and $g(r_n) \geq c$ for all n.

Now assume that $c > 0$. Then the definition of g implies the existence of a sequence v_n in S with $h(v_n, r_n) > c/2$. Since S is compact, we may assume that v_n is convergent: $\lim_{n\to\infty} v_n = v$.

Due to inequality (9.2),

$$h(v, r_n + \|v_n - v\|) \geq h(v_n, r_n) > c/2.$$

Since $\lim_{n\to\infty}(r_n + \|v_n - v\|) = 0$, σ-additivity of the Haar measure now implies that

$$\mu\{k \in K : f(k,v) = 0\} = \mu\Big(\bigcap_n \{k : f(k,v) < r_n + \|v_n - v\|\}\Big) \geq c/2 > 0,$$

contradicting (9.1).

Therefore, the claim must hold, that is, $\lim_{r\to 0} g(r) = 0$.

Now we can prove the statement of the proposition. Given $r, \delta > 0$, we choose $\rho \in (0,1]$ such that $g(\rho) < \delta$. Then we define $R = r/\rho$.

Let $v \in V$ with $\|v\| > R$ and define $v_0 = (v/\|v\|) \in S$. We observe that $\|\pi(k(v))\| < r$ is equivalent to $\|\pi(k(v_0))\| < (r/\|v\|)$.

Now $g(\rho) < \delta$ implies that $h(w,\rho) < \delta$ for all $w \in S$. Thus, $h(v_0, \rho) < \delta$, that is,

$$\mu\{k \in K : \|\pi(k(v_0))\| < \rho\} < \delta,$$

which is equivalent to

$$\mu\{k \in K : \|\pi(k(v))\| < \rho\|v\|\} < \delta.$$

This completes the proof, since $\rho\|v\| > \rho R = r$. □

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Remark 9.2. Instead of a compact real Lie group K with its Haar measure μ we may take an arbitrary connected real Lie group with an arbitrary probability measure of Lebesgue measure class.

Proposition 9.3. *Let K be a Lie group acting on a manifold S and let $\pi : S \to Y$ be a continuous map. Let $\rho : S \to \mathbf{R}^+$ be an exhaustion function and let $\tau : Y \to \mathbf{R}^+$ be an arbitrary continuous function.*

Assume that K is endowed with a probability measure μ such that the following property holds.

($*$) *For every $r, \delta > 0$, there exists a number $R > 0$ such that*

$$\mu\{k \in K : \tau(\pi(kx)) < r\} < \delta$$

for all $x \in S$ with $\rho(x) > R$.

Then there exists a sequence $R_n > 0$ such that for every sequence x_n in S with $\rho(x_n) > R_n$ for all n we can find an element $k \in K$ such that $\tau(\pi(kx_n)) \geq n$ for all $n \in \mathbf{N}$.

Proof. Using ($*$), we may (for every $n \in \mathbf{N}$) choose $R_n > 0$ such that

$$\mu\{k \in K : \tau(\pi(kx_n)) < n\} < 2^{-(n+1)}$$

for all x_n with $\rho(x_n) > R_n$.

Since $\sum_{n\in\mathbf{N}} 2^{-(n+1)} = \frac{1}{2} < 1$, the set

$$\{k \in K : \tau(\pi(kx)) \geq n \; \forall n \in \mathbf{N}\}$$

has positive measure (and is therefore nonempty) for every sequence x_n in S with $\rho(x_n) > R_n$. □

Corollary 9.4. *Choose R_n as above and let $D = \{x_n : n \in \mathbf{N}\}$ be a discrete subset of S for which $\rho(x_n) > R_n$ for all $n \in \mathbf{N}$. Let Ω denote the set of all $k \in K$ for which π restricts to a proper map on $k(D)$, that is, for which $\pi|_{k(D)}$ has finite fibers and maps to a discrete subset of Y. Then $K \setminus \Omega$ has measure zero.*

Proof. By construction,

$$\mu\{k \in K : \tau(\pi(kx_n)) < n\} < 2^{-(n+1)}.$$

Let Ω_N (for a given $N \in \mathbf{N}$) denote the set

$$\{k \in K : \tau(\pi(kx_n)) \geq n \; \forall n \geq N\}.$$

We observe that π restricted to $k(D)$ is proper for any k for which there exists a number N such that $k \in \Omega_N$. By construction,

$$\mu(K \setminus \Omega_N) \leq 2^{-(N+1)}.$$

This implies that

$$K \setminus \Omega \subset \bigcap_N (K \setminus \Omega_N)$$

is a set of measure zero, because $\mu(K \setminus \Omega) \leq 2^{-(N+1)}$ for all $N \in \mathbf{N}$. □

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9.2. Generic projections on semisimple complex Lie groups.

Proposition 9.5. *Let S be a semisimple complex Lie group, $T \simeq (\mathbf{C}^*)^d \subset S$, and let K be a maximal compact subgroup of S. Let π denote the projection $\pi : S \to S/T$. Then there are finitely many regular functions f_i on S/T such that:*

(1) *the map $(f_1, \dots, f_s) : S/T \to \mathbf{C}^s = W$ is a proper embedding;*
(2) *all functions of the form $x \mapsto f_i(\pi(xk))$ $(k \in K, i \in I = \{1, \dots, s\})$ generate a finite-dimensional vector subspace V of $\mathbf{C}[S]$;*
(3) *the natural map from S to V^* which associates to each point $p \in S$ the evaluation homomorphism $f \mapsto f(p)$ defines a proper finite morphism from S to V^*;*
(4) *the kernel of the natural map $V^* \to W^*$ contains no nontrivial K-invariant vector subspace.*

Proof. First we note that $Y = S/T$ is an affine variety, because both S and T are reductive. This yields property (1).

Now we define V as in (2). Finite dimensionality follows from standard results on transformation groups.

We obtain a natural map from S to V^* which associates to each point $p \in S$ its evaluation map $f \mapsto f(p)$.

We have to show (3). As a preparation, we consider $Z = \bigcap_{k \in K} kTk^{-1}$. This is an algebraic subgroup of S, obviously normalized by K and therefore normal in S. Since S is semisimple, $T \simeq (\mathbf{C}^*)^d$ cannot be normal in S. Hence, $\dim(Z) = 0$ and consequently Z is finite.

Let $x, y \in S$. If $f(\pi(xk)) = f(\pi(yk))$ for all $f \in W$, $k \in K$, then necessarily $\pi(xk) = \pi(yk)$ for all $k \in K$. This is equivalent to $y^{-1}x \in kTk^{-1}$. It follows that $F(x) = F(y)$ for all $F \in V$ if and only if $y^{-1}x \in Z$. Thus, $S \to V^*$ is induced by an injective map from S/Z to V^*.

We still have to check properness.

We recall that the base S/T is properly embedded into W^*. Choose a point $\lambda \in W^*$ which is in the image of S/T, that is, is the image of some sT and consider its preimage in V^*.

Taking note that all maps between S, S/T, V^* and W^* are S-equivariant, we have to show that T-orbits in the fiber in V^* over $\lambda \in W^*$ are closed.

Using theory of toric varieties, we know that if a T-orbit in V^* is not closed, there is a subgroup $\mathbf{C}^* \simeq T_0 \subset T$ with a nonclosed T_0-orbit.

However, each such T_0 is contained in some $\mathrm{SL}_2(\mathbf{C}) \simeq H \subset S$. Hence, there exists an element $k \in K$ such that conjugation by k induces the automorphism $z \mapsto 1/z$ of $\mathbf{C}^* \simeq T_0$. It follows that the closure of a nontrivial T_0-orbit in V^* is either the orbit itself or isomorphic to $\mathbf{P}^1$. Since $\mathbf{P}^1(\mathbf{C})$ as a compact Riemann surface cannot be embedded in a complex vector space like V^*, it follows that T_0-orbits are closed. Therefore, T-orbits are closed and consequently the image of S in V^* is closed. This verifies (3).

Finally, we have to show (4). Assume that C is such a K-invariant vector subspace of the kernel of the map $V^* \to W^*$.

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The linear form on V which is defined by an element $c \in C$ must vanish on W, because C is contained in the kernel. But K-invariance of C implies that c must in fact vanish on the smallest vector subspace of V which contains W and is K-invariant. However, this is V itself and hence $C = \{0\}$. □

Lemma 9.6. *Let S be a semisimple complex Lie group and Z its center. Let $\tau : S \to S/Z$ denote the natural projection. Let $D \subset S$ be a tame discrete subset. Let R_n be a sequence of positive real numbers. Let $\rho : S/Z \to \mathbf{R}^+$ be an exhaustion function.*

Then there exists a biholomorphic automorphism ϕ of S such that:

(1) *τ is injective on $\phi(D)$;*
(2) *the condition $\#\{x \in D : \rho(\phi(x)) \le R_n\} \le n$ holds for all $n \in \mathbf{N}$.*

Proof. First note that $\tau(g) = \tau(h)$ if and only if $g^{-1}h \in Z$.

We choose a one-dimensional unipotent subgroup U of S and fix an isomorphism of complex Lie groups $\zeta : (\mathbf{C}, +) \to U$. Let $\pi : S \to S/U$=Y be the natural projection. The quotient S/U is a quasi-affine variety, because U is unipotent. In particular, $Y = S/U$ admits an injective holomorphic map into some $\mathbf{C}^m$. As a consequence, there is a finite-dimensional complex vector space V of holomorphic functions on Y separating the points. Let B be the unit ball in V for some (arbitrary) norm on V. Note that B is relatively compact and open.

To each $F \in V$ we may associate a holomorphic map from S to U via $g \mapsto \zeta(F(\pi(g)))$. Using the action on U by right multiplication on S, we obtain a way to associate to each element $F \in V$ a biholomorphic self-map $\Phi_F(g) = g \cdot \zeta(F(\pi(g)))$. (The automorphisms constructed in this way are in some sense analogous to 'shears' on $\mathbf{C}^n$ or 'replicas' on flexible varieties.)

Observe that $\pi(g \cdot \zeta(F(\pi(g)))) = \pi(g)$. Hence, $\phi_{-F} \circ \phi_F = \mathrm{id}_S$. This confirms that ϕ_F is indeed an automorphism.

Since ρ is an exhaustion function, $K_r = \{x \in S : \rho(x) \le r\}$ is compact for all $R > 0$. As a consequence, we may define

$$\tilde{R}_n = \max_{F \in \bar{B}, \rho(x) \le R_n} \rho(\phi_{-F}(x)).$$

Observe that $\rho(p) > \tilde{R}_n$ implies that $\rho(\phi_F(p)) > R_n$ for all $F \in B$.

Recall that D is assumed to be tame. By definition of tameness, there exists a biholomorphic automorphism α of S such that

$$\#\{p \in D : \rho(\alpha(p)) \le \tilde{R}_n\} \le n \quad \forall n \in \mathbf{N}.$$

Hence, we may (replacing D by $\alpha(D)$) assume that

$$\#\{p \in D : \rho(p) \le \tilde{R}_n\} \le n \quad \forall n \in \mathbf{N}.$$

For every $g, h \in D$, we define a subset $\Omega_{g,h} \subset V$ as follows:

$$\Omega_{g,h} = \{F \in V : (\phi_F(g))^{-1}\phi_F(h) \notin Z\}.$$

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The condition

$$(\phi_F(g))^{-1}\phi_F(h) \notin Z$$

is equivalent to

$$g^{-1}h \notin F(g)ZF(h).$$

Since Z is finite and the functions F separate the points on S/U, it is clear that $\Omega_{g,h}$ is a dense open subset of V. As the Fréchet space V has the Baire property, it follows that

$$W = B \cap \left(\bigcap_{g \in D} \bigcap_{h \in D} \Omega_{g,h} \right)$$

is again dense and in particular not empty. Choose an element $F \in W$ and define $D' = \phi_F(D)$. By construction, $\rho(p) > \tilde{R}_n$ (for any $p \in S$, $n \in \mathbf{N}$) implies that $\rho(\Phi_F(p)) > R_n$. Hence, $\rho(\Phi_F(p)) \leq R_n$ implies that $\rho(p) \leq \tilde{R}_n$ for any $n \in \mathbf{N}$, $p \in X$, $F \in B$ and consequently

$$\#\{\{p \in D' : \rho(p) \leq R_n\} \leq n\} \leq \#\{\{p \in D : \rho(p) \leq \tilde{R}_n\} \leq n\} \leq n \quad \forall n \in \mathbf{N}$$

for every $n \in \mathbf{N}$. This implies the statement. □

Proposition 9.7. *Let S be a connected semisimple complex Lie group with center Z and exhaustion function $\rho : S \to \mathbf{R}^+$. Let K be a maximal compact subgroup with Haar measure μ. Let T be a 'maximal torus' of S in the sense of algebraic groups, that is, a maximal connected reductive commutative complex Lie subgroup of S.*

For every discrete subset $D \subset S$, let Ω_D denote the set of all $k \in K$ for which the following holds:

(1) *the natural projection map π_k from S to S/kTk^{-1} restricted to D is proper, that is, has discrete image and finite fibers;*
(2) *for every $x, y \in D$, we have $\pi_k(x) \neq \pi_k(y)$ unless $x^{-1}y \in Z$.*

Then for every tame discrete set D there is a biholomorphic self-map ϕ of S such that $\mu(K \setminus \Omega_{\phi(D)}) = 0$.

Moreover, there exists a sequence R_n such that every discrete subset $D \subset S$ is tame if $\#\{p \in D : \rho(p) \leq R_n\} \leq n$, that is, R_n is a 'threshold sequence' as defined in Definition 4.9.

Proof. We use Proposition 9.5 to obtain an embedding $\alpha : S/T \hookrightarrow W^*$, a proper map $\beta : S \to V^*$ and a linear map $L : V^* \to W^*$. Conjugation by K on S induces a natural K-action on V^*, because the dual space V of V^* consists of functions on S.

Due to property (4) of Proposition 9.5, we may invoke Proposition 9.1 to obtain the following statement.

For every $\delta, r > 0$, there exists a number $R > 0$ such that

$$\|v\| > R \quad \Longrightarrow \quad \mu(\{k \in K : \|L(k(v))\| < r\}) < \delta \quad \forall v \in V^*.$$

Next we want to apply Proposition 9.3 with $Y = W^*$.

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Since β is proper, $\rho'(p) = \|\beta(p)\|$ defines an exhaustion function on S. We observe that for any $R > 0$, there exists a number $R' > 0$ such that $\rho(p) > R'$ implies that $\rho'(p) > R$.

Therefore, Proposition 9.3 combined with its corollary implies that there is a sequence r_n of positive numbers such that we have a conull set Ω'_D for which (1) holds provided $\#\{p \in D : \rho(p) \leq r_n\} \leq n$ for all n. Since D is required to be tame, there is no loss in generality in assuming the condition $\#\{p \in D : \rho(p) \leq r_n\} \leq n$ for all n.

We still have to discuss property (2) of Ω_D. Let $p, q \in D$ and consider

$$C_{p,q} = \{k \in K : \pi_k(p) = \pi_k(q)\}.$$

If $C_{p,q} = K$, then

$$pq^{-1} = \bigcap_k kTk^{-1} = Z.$$

Thus, $C_{p,q}$ is a nowhere-dense real-analytic subset of K if $pq^{-1} \notin Z$. This proves the statement on Ω_D, because nowhere-dense real-analytic subsets are of Haar measure zero and there are only countably many choices for $(p, q) \in D \times D$.

Finally, the statement on the threshold sequence is a direct consequence. □

Proposition 9.8. *Let S be a semisimple complex Lie group with maximal torus T and a complex-analytic subset $E \subset S$. Let $D \subset S$ be a tame discrete subset. Then there exists an automorphism ϕ of S such that for both quotients of S by T, the quotient by the right action as well as the quotient by the left action, the projection map restricts to an injective map from S to a discrete subset of the respective quotient manifold. In addition, ϕ may be chosen such that $\phi(D) \cap E = \{\}$.*

Proof. Let R_n be a threshold sequence and let Z denote the center of S. Due to Lemma 9.6, we may without loss of generality assume that the natural projection $\tau : S \to S/Z$ is injective on D. Now let Ω_D be defined as in Proposition 9.7.

Define $\alpha : S \to S$ as $\alpha(g) = g^{-1}$ and $D' = \alpha(D)$.

Due to Proposition 9.7, we know that $K \setminus \Omega_D$ and $K \setminus \Omega_{D'}$ are sets of Haar measure zero.

Now define

$$U = (K \times K) \setminus \bigcup_{g \in D} \{(k, h) \in K \times K : kgh \in E\}.$$

For each $dg \in D$, the set $\{(k, h) \in K \times K : kgh \in E\}$ is a nowhere-dense real-analytic subset and therefore a set of Haar measure zero. Since D is countable, it follows that $(K \times K) \setminus U$ is a set of Haar measure zero. We observe that

$$M = (\Omega_D \times \Omega_{D'}) \cap U$$

is a conull set and therefore nonempty. Now we choose $(k, h) \in M$ and define $\phi : g \mapsto kgh^{-1}$. □

Proposition 9.9. *Every semisimple complex Lie group S admits a nontame discrete subset.*

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PROOF. The group S is Stein as a complex manifold. Due to [4], there exists a discrete subset $D_0 \subset S$ such that $\mathrm{Aut}_{\mathcal{O}}(S \setminus D_0) = \{\mathrm{id}\}$. On the other hand, if D is a tame discrete subset in S, then there exists a biholomorphic self-map ϕ of S such that $\pi(\phi(D))$ is discrete in S/T, where T is a maximal torus and $\pi : S \to S/T$ denotes the natural projection. But this implies that S admits many holomorphic automorphisms fixing each element of D: we may just take any holomorphic map $f : S/T \to T$ such that $f(q) = e_T$ for all $q \in \pi(\phi(D))$ and define

$$x \mapsto \phi^{-1}(\phi(x) \cdot f(\pi(\phi(x)))).$$

Thus, $\mathrm{Aut}_{\mathcal{O}}(S \setminus D_0) = \{\mathrm{id}\}$ prevents D_0 from being tame. □

10. The case $\mathrm{SL}_n(\mathbf{C})$

DEFINITION 10.1. A sequence $A(k)$ in $S = \mathrm{SL}_n(\mathbf{C})$ is called *well-placed* if the matrix coefficients $A_{i,j}(k)$ of the elements $A(k)$ fulfill the following two conditions:

(1) $A_{i,j}(k) \neq 0$ for all $k \in \mathbf{N}$, $1 \leq i, j \leq n$;
(2) for every $2 \leq j \leq n$ and every $1 \leq h \leq n$, the sequences $\alpha_k = |A_{1,h}(k)/A_{j,h}(k)|$ and $\beta_k = |A_{h,1}(k)/A_{h,j}(k)|$ are unbounded and strictly increasing.

PROPOSITION 10.2. *If a sequence $A(k)$ is well-placed, then $D = \{A(k) : k \in \mathbf{N}\}$ is a tame discrete subset of S and both natural projections $\pi : S \to S/T$ and $\pi' : S \to T\backslash S$ map D injectively onto a discrete subset of the respective quotient manifold.*

PROOF. The torus T is the subgroup of diagonal matrices. Its action by left multiplication on S may be identified with $(\mathbf{C}^*)^{n-1}$ acting on the coefficients $A_{i,j}$ of elements A of $\mathrm{SL}_n(\mathbf{C})$ via $A_{j,k} \mapsto A_{j,k}\lambda_{k-1}$ for $k \geq 2$ and $A_{j,1} \mapsto A_{j,1}\Pi_k\lambda_k^{-1}$. We may identify S/T with $(\mathbf{P}^{n-1}(\mathbf{C}))^n \setminus Z$, where the projection map $\pi : S \to S/T$ is realized by projecting the columns of the matrix A to their respective equivalence classes in $\mathbf{P}^n(\mathbf{C})$ and where the 'bad locus' Z consists of those elements $([v_1], \ldots, [v_n]) \in (\mathbf{P}^{n-1}(\mathbf{C}))^n$ for which $v_1, \ldots, v_n$ are not linearly independent. The assumption of $A(k)$ being well-placed implies that

$$\lim_{k\to\infty} \pi(A(k)) = ([e], \ldots, [e])$$

with $e = (1, 0 \ldots, 0)$. Therefore, $\lim_{k\to\infty} \pi(A(k)) \in Z$, that is, $\{\pi(A(k)) : k \in \mathbf{N}\}$ is discrete in S/T. Injectivity follows from the requirement that any sequence $k \mapsto |A_{j,1}(k)/A_{j,h}(k)|$ is strictly increasing. Tameness is due to Proposition 8.4, taking into account that S/T is Stein due to the theorem of Matsushima. The statement for the left quotient is derived in the same way. □

LEMMA 10.3. *Let $A(k)$ be a well-placed sequence $A(k)$. Assume that $\lambda_i(k) \in \mathbf{C}^*$ are given for $1 \leq i \leq n$, $k \in \mathbf{N}$ such that:*

(1) $|\lambda_1(k)| \geq |\lambda_j(k)|$ *for all* $j \in \{2, \ldots, n\}$, $k \in \mathbf{N}$*;*
(2) $|\lambda_1(k+1)\lambda_j(k)| \geq |\lambda_1(k)\lambda_j(k+1)|$ *for all* $j \in \{2, \ldots, n\}$, $k \in \mathbf{N}$*;*
(3) $\Pi_{i=1}^n \lambda_i(k) = 1$ *for all* $k \in \mathbf{N}$.

Define $B_{i,j}(k) = \lambda_i A_{i,j}(k)$. Then $B(k)$ is well-placed and equivalent to $A(k)$.

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Proof. The conditions on the $\lambda_i(k)$ ensure that $B(k)$ is well-placed. Proposition 10.2 implies that the sequence $A(k)$ is mapped injectively onto a discrete subset of S/T by the natural projection $\pi : S \to S/T$. Now the $\lambda_i(k)$ define a map g from $D = \{A(k) : k \in \mathbf{N}\}$ to T such that $B(k) = A(k) \cdot g(A(k))$ for all $k \in \mathbf{N}$. Hence, $B(k)$ is equivalent to $A(k)$. □

Proposition 10.4. *Let $D(k)$ be a tame sequence in $S = \mathrm{SL}_n(\mathbf{C})$. Then there is a sequence $A(k)$ in S such that:*

(1) *$D(k)$ and $A(k)$ are equivalent;*
(2) *$A(k)$ is well-placed.*

Proof. Due to Proposition 9.8, we may assume that the projection map $\pi : S \to S/T$ restricts to an injective map from D onto a discrete subset of S/T. Moreover, we may assume that all the matrix coefficients $A_{i,j}(k)$ of every element $A(k)$ are nonzero, again by Proposition 9.8. (Noting that the set E of all $A \in \mathrm{SL}_n(\mathbf{C})$ with at least one matrix coefficient being zero is a complex analytic subset of S.) The torus T is noncompact and S/T is Stein. Thus, for every sequence $\zeta(k) \in T$, we can find an automorphism ϕ of the complex manifold S such that $\phi(A(k)) = A(k) \cdot \zeta(k)$. From this we deduce the assertion. □

Proposition 10.5. *Let $A(k)$, $B(k)$ be well-placed sequences in $S = \mathrm{SL}_n(\mathbf{C})$. Let $\tau : \mathrm{SL}_n(\mathbf{C}) \to \mathbf{C}^n \setminus \{(0, \ldots, 0)\}$ be the map which associates to each matrix its first column vector. Then there exist well-placed sequences $C(k)$, $D(k)$ in S such that:*

(1) *A and C are equivalent;*
(2) *B and D are equivalent;*
(3) *$\tau(C(k)) = \tau(D(k))$ for all $k \in \mathbf{N}$;*
(4) *$\{\tau(C(k)) : k \in \mathbf{N}\}$ is discrete in $\mathbf{C}^n$.*

Proof. We want to use Lemma 10.3. For this purpose we define recursively sequences $\lambda_j(k)$ and $\mu_j(k)$ in $(\mathbf{C}^*)^n$ such that:

(1) $|\lambda_j(k)| \leq 1$ and $|\mu_j(k)| \leq 1$ for all $2 \leq j \leq n$;
(2) $|\lambda_j(k)| \leq |\lambda_j(k-1)/\lambda_1(k-1)|$ for $2 \leq j \leq n$;
(3) $|\mu_j(k)| \leq |\mu_j(k-1)/\mu_1(k-1)|$ for $2 \leq j \leq n$;
(4) $\lambda_j(k) A_{j,1}(k) = \mu_j(k) B_{j,1}$ for $2 \leq j \leq n$;
(5) $\Pi_{j=1}^n \lambda_j(k) = 1 = \Pi_{j=1}^n \mu_j(k)$.

Next we choose recursively sequences $\tilde{\lambda}_j(k)$ and $\tilde{\mu}_j(k)$ in $(\mathbf{C}^*)^n$ such that:

(1) $|\tilde{\lambda}_j(k)| \leq 1$ and $|\tilde{\mu}_j(k)| \leq 1$ for all $2 \leq j \leq n$;
(2) $|\tilde{\lambda}_j(k)| \leq |\tilde{\lambda}_j(k-1)/\tilde{\lambda}_1(k-1)|$ for $2 \leq j \leq n$;
(3) $|\tilde{\mu}_j(k)| \leq |\tilde{\mu}_j(k-1)/\tilde{\mu}_1(k-1)|$ for $2 \leq j \leq n$;
(4) $\tilde{\lambda}_1(k)\lambda_1(k) A_{1,1}(k) = \tilde{\mu}_1(k)\mu_1(k) B_{1,1}$;
(5) $\Pi_{j=1}^n \tilde{\lambda}_j(k) = 1 = \Pi_{j=1}^n \tilde{\mu}_j(k)$.

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Now we define sequences $C'(k)$, $C(k)$ as follows:

(1) $C'_{j,h}(k) = \lambda_j(k)A_{h,j}(k)$;

(2) $C_{j,h}(k) = \tilde{\lambda}_h(k)C'_{j,h}(k) = \tilde{\lambda}_h(k)\lambda_j(k)A_{h,j}(k)$.

Lemma 10.3 now first implies that $A(k)$ and $C'(k)$ are equivalent well-placed sequences and then it implies that $C(k)$ and $C'(k)$ are equivalent well-placed sequences. Thus, $A(k)$ and $C(k)$ are equivalent. Similarly, one verifies that for

$$D_{j,h}(k) = \tilde{\lambda}_h(k)\lambda_j(k)B_{h,j}(k)$$

the sequences $B(k)$ and $D(k)$ are equivalent. □

PROPOSITION 10.6. *Any two tame sequences in* $S = \mathrm{SL}_n(\mathbf{C})$ *are equivalent.*

PROOF. Let $A(k)$, $B(k)$ be two tame sequences. Each tame sequence is equivalent to a well-placed sequence (Proposition 10.4). Thus, we may assume that both $A(k)$ and $B(k)$ are well-placed. Due to Proposition 10.5, we may furthermore assume that for every k the first columns of the matrices $A(k)$ and $B(k)$ coincide. The map projecting each matrix A to its first column can be described as $\tau : S \to S/Q \simeq \mathbf{C}^n \setminus \{(0, \ldots, 0)\}$. By Proposition 10.5, we know that $\tau(A(k)) = \tau(B(k))$ constitutes a sequence in S/Q which is discrete in $\mathbf{C}^n$. For each $k \in \mathbf{N}$, let $g(k) \in Q$ denote the element such that $A(k) = B(k) \cdot g(k)$. Now let $F : \mathbf{C}^n \to Q$ be a holomorphic map such that $F(\tau(A(k))) = g(k)$ for all $k \in \mathbf{N}$. Then F defines an automorphism ϕ of the manifold S given by $x \mapsto x \cdot F(\tau(x))$. Via this automorphism ϕ, the two sequences $A(k)$ and $B(k)$ are equivalent. □

COROLLARY 10.7. *Let* D *be a tame discrete subset of* $S = \mathrm{SL}_n(\mathbf{C})$. *Then every permutation of* D *extends to an automorphism of the complex manifold* S.

COROLLARY 10.8. *Let* $S = \mathrm{SL}_n(\mathbf{C})$, *let* Ω *be a Stein open subset with* $\Omega \neq S$ *and let* D *be a tame discrete subset of* S. *Then there exists a holomorphic automorphism* ϕ *of* S *such that* $\phi(\Omega) \cap D = \{\}$.

PROOF. First we observe that $S \setminus \Omega$ is unbounded due to Hartogs' 'Kugelsatz', because Ω is assumed to be Stein. Furthermore, S admits a threshold sequence (cf. Proposition 9.7). Hence, Proposition 4.10 implies that S admits a tame discrete subset D' with $D' \subset S \setminus \Omega$. Due to Proposition 10.6, the discrete sets D and D' are equivalent. In particular, there is a biholomorphic self-map ϕ of S with $\phi(D) \subset S \setminus \Omega$. □

PROPOSITION 10.9. *Let* D *be a discrete subset of* $S = \mathrm{SL}_n(\mathbf{C})$. *Let* $\pi : S \to \mathbf{C}^n$ *denote the linear projection of matrices in* $S \subset \mathrm{Mat}(n \times n, \mathbf{C})$ *onto its first column. Then* D *is tame if* $\pi|_D$ *has finite fibers and its image is a discrete subset of* $\mathbf{C}^n$.

COROLLARY 10.10. *Every discrete subset* D *of* $S = \mathrm{SL}_n(\mathbf{C})$ *can be realized as the union of* n *tame discrete subsets.*

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Proof. Let π_k denote the projection onto the kth column and define

$$D_k = \{x \in D : \|\pi_k(x)\| \geq \|\pi_j(x)\| \ \forall j\}.$$

By construction,

$$\|\pi_k(x)\| \geq \frac{1}{n}\|x\| \quad \forall x \in D_k \ \forall k.$$

Since D is discrete, $\{x \in D : \|x\| < R\}$ is finite for all $R > 0$. Hence, $\{x \in D_k : \|\pi_k(x)\| < (1/n)R\}$ is finite for all $R > 0$. It follows that $\pi_k(D_k)$ is discrete and that $\pi_k|_{D_k}$ has finite fibers. Hence, each D_k is tame due to Proposition 8.4. □

Corollary 10.11. *Let T be a torus (that is, a commutative reductive complex Lie subgroup) of* $\mathrm{SL}_n(\mathbf{C})$ *and let D be a discrete subset of T. Then D is a tame discrete subset in $X = \mathrm{SL}_n(\mathbf{C})$.*

Proof. Every torus is contained in a maximal torus and all the maximal tori are conjugate. Hence, we may assume that T is the group of diagonal $n \times n$ matrices with determinant 1.

Let A denote the matrix given as

$$A = \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}.$$

Then, for each diagonal matrix M with coefficients $\lambda_1, \ldots, \lambda_n$),

$$\pi(M \cdot A) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in Z = \{v \in \mathbf{C}^n : \Pi_i v_i = 1\}.$$

Thus, $\psi : M \to M \cdot A$ defines a biholomorphic self-map of X which maps T biholomorphically onto the closed complex-analytic subset Z of $\mathbf{C}^n$. Hence, $\psi(D)$ (and therefore also D) is tame due to the proposition. □

11. Special results for $\mathrm{SL}_2(\mathbf{C})$

11.1. $\mathrm{SL}_2(\mathbf{Z}[i])$ is tame. We will see that $\mathrm{SL}_2(\mathbf{Z})$, $\mathrm{SL}_2(\mathbf{Z}[i])$ and more generally $\mathrm{SL}_2(\mathcal{O}_K)$ for every imaginary quadratic number field K are tame discrete subsets in $\mathrm{SL}_2(\mathbf{C})$.

We will need some kind of 'overshears' on $\mathrm{SL}_2(\mathbf{C})$.

Lemma 11.1. *For every holomorphic function $\lambda : \mathbf{C}^2 \to \mathbf{C}^*$ with $\lambda(0, w) = 1$, for every $w \in \mathbf{C}$ there is a biholomorphic automorphism ϕ_λ of the complex manifold* $\mathrm{SL}_2(\mathbf{C})$ *such that*

$$\phi_\lambda : \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto \begin{pmatrix} a & c\lambda(a,b) \\ b & d' \end{pmatrix}$$

with

$$d' = \frac{1 + bc\lambda(a,b)}{a}$$

if $a \neq 0$.

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If we choose a global coordinate for a fiber $F_v = \pi^{-1}(v) \simeq \mathbf{C}$, the restriction of ϕ_λ to F_v assumes the form $t \mapsto \lambda(v)t + c$ (with c depending on the choice of the global coordinate).

PROOF. First note that

$$\begin{aligned}\det\begin{pmatrix} a & c\lambda(a,b) \\ b & d' \end{pmatrix} &= ad' - bc\lambda(a,b) \\ &= a\Big(\frac{1+bc\lambda(a,b)}{a}\Big) - bc\lambda(a,b) \\ &= 1 + bc\lambda(a,b) - bc\lambda(a,b) = 1\end{aligned}$$

(whenever $a \neq 0$).

Next we claim that d' defined as above has a removable singularity along $a = 0$. Indeed, $1 + bc\lambda(a,b)$ is divisible by a: the identity $\lambda(0,w) = 1$ for all $w \in \mathbf{C}$ combined with $ad - bc = 1$ implies that

$$1 + bc\lambda(0,b) = 1 + bc = ad = 0$$

for every

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C})$$

with $a = 0$. □

PROPOSITION 11.2. *Let $\pi_i : \mathrm{SL}_2(\mathbf{C}) \to \mathbf{C}^2$ denote the projection onto the ith column. Let Γ be a discrete subset of $\mathrm{SL}_2(\mathbf{C})$ such that $\pi_1(\Gamma)$ is a discrete subset of $\mathbf{C}^2$. Then Γ is tame.*

PROOF. Define $H = \{(z_1, z_2) \in \mathbf{C}^2 : z_1 z_2 = 0\}$. Observe that $\pi_1(AB) = A \cdot \pi_1(B)$ for $A, B \in \mathrm{SL}_2(\mathbf{C})$. For each $\gamma \in \Gamma$, the set $U_\gamma = \{A : A\pi_1(\gamma) \notin H\}$ is Zariski open, because $\pi_1(\Gamma) \subset \mathbf{C}^2 \setminus \{(0,0)\}$. Since Γ is countable, $\bigcap_{\gamma\in\Gamma} U_\gamma$ is not empty. Thus, by replacing Γ with $\{A\gamma : \gamma \in \Gamma\}$ for a suitably chosen A, we may assume that $\Lambda = \pi_1(\Gamma) \cap H$ is empty. We fix a bijection $\alpha : \mathbf{N} \to \Lambda$.

The fibers of π_1 are the orbits of the principal right action given as

$$R_t : \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c + at \\ b & d + bt \end{pmatrix}.$$

Thus, there is a natural distance function on each π_1-fiber given as $d(A, B) = |t|$ if $B = R_t(A)$. Using this distance function, choose numbers ρ_k such that $\pi_1^{-1}(\alpha(k)) \setminus \Gamma$ contains a ρ_k-ball. Next we choose a map $\lambda_0 : H \cup \Lambda \to \mathbf{C}$ such that $\lambda_0|_H \equiv 1$ and

$$|\lambda(\alpha(k))|\rho_k\|\alpha(k)\| > k \quad \forall k \in \mathbf{N}.$$

Using Lemma 11.1, we may from now on assume that $\rho_k\|\alpha(k)\| > k$. Observe that for any $v, A, B \in \pi_1^{-1}(v)$ we have $\|\pi_2(A) - \pi_2(B)\| = \|v\| d(A, B)$.

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From this, we may deduce that there exists a sequence t_k such that

$$\pi_2(R_{t_k}(\Gamma \cap \pi_1^{-1}(\alpha_k))) \cap \{w \in \mathbf{C}^2 : \|w\| < k\} = \{\}.$$

Moreover, choosing t_k sufficiently generic, we may assume that after applying an automorphism ψ extending the R_{t_k} the projection map $\pi_2 : \Gamma \to \mathbf{C}^2$ becomes injective.

Then the assertion follows from Proposition 10. □

Corollary 11.3. *For $K = \mathbf{Q}$ or an imaginary quadratic number field K, let $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ denotes the ring of algebraic integers in K. Then Γ is tame.*

Proof. If π_1 denotes the projection onto the first column, then $\pi_1(\Gamma) \subset \mathcal{O}_K \times \mathcal{O}_K \subset \mathbf{C}^2$ and $\mathcal{O}_K$ is discrete in $\mathbf{C}$. □

In particular, $\mathrm{SL}_2(\mathbf{Z}[i])$ is tame.

12. Miscellenea

Proposition 12.1. *Let $G = \mathrm{SL}_n(\mathbf{C})$ with a tame infinite discrete subset D.*

Then the automorphism group $\mathrm{Aut}_{\mathcal{O}}(G \setminus D)$ has uncountably many connected components (with respect to the compact-open topology).

Proof. Since D is of codimension at least two in G, every holomorphic function on $G \setminus D$ extends to G. Now G is Stein and therefore may be realized as a closed complex submanifold in some $\mathbf{C}^N$. For every holomorphic automorphism ϕ of $G \setminus D$, both ϕ and its inverse ϕ^{-1} extend as holomorphic maps F (respectively F_1) with values in $\mathbf{C}^N$ through D. Since $G \setminus D$ is dense in G, we have $F(G) \subset G$ and $F_1(G) \subset G$. Now $F \circ F_1$ and $F_1 \circ F$ equal the identity map on $G \setminus D$ and therefore also on G. Hence, $F : G \to G$ is a biholomorphic self-map of G extending $\phi \in \mathrm{Aut}_{\mathcal{O}}(G \setminus D)$. Thus, every holomorphic automorphism of $G \setminus D$ induces a permutation of D. Conversely, every permutation of D extends to a holomorphic automorphism. Let $p, q \in D$. We choose a compact subset K of $G \setminus D$ and a Stein open subset $\Omega \subset G$ such that

$$p \in \hat{K} \subset \Omega$$

(where $\hat{K}$ denotes the holomorphically convex hull of K in G) and such that $\Omega \cap D = \{p\}$. Let ϕ_0 be an automorphism of $G \setminus D$ (respectively its extension to G) with $\phi_0(p) = q$. Now

$$W = \{\phi \in \mathrm{Aut}_{\mathcal{O}}(G \setminus D) : \phi(K) \subset \phi_0(\Omega)\}$$

is open with respect to the compact-open topology on $\mathrm{Aut}_{\mathcal{O}}(G \setminus D)$. For every $\phi \in W$, the extension to G (by abuse of notation again denoted by ϕ) has the property $\phi(\hat{K}) \subset \phi_0(\Omega)$, because $\phi_0(\Omega)$ is Stein and $\phi(K) \subset \phi_0(\Omega)$. Since

$$\phi(p) \in D \cap \phi_0(\Omega) = \phi_0(\Omega \cap D) = \{\phi_0(p)\},$$

we obtain that $\phi(p) = q$ for all $\phi \in W$. It follows that

$$\{\phi \in \mathrm{Aut}_{\mathcal{O}}(G \setminus D) : \phi(p) = q\}$$

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is open in $\mathrm{Aut}_{\mathcal{O}}(G \setminus D)$. This in turn implies that we have a continuous surjective map from $\mathrm{Aut}_{\mathcal{O}}(G \setminus D)$ to $\mathrm{Perm}(D)$, where $\mathrm{Perm}(D)$ is endowed with a totally disconnected topology, namely the topology which has

$$W_{p,q} = \{\phi : \phi(p) = q\}$$

as a basis of topology. Because $\mathrm{Perm}(D)$ is uncountable for an infinite countable set D, the assertion follows. □

It is easily verified that a discrete subset D in $\mathbf{C}^n$ is tame if and only if there is a biholomorphic self-map ψ of $\mathbf{C}^n$ such that $\psi(D)$ is contained in a complex line. The complex lines through the origin in $\mathbf{C}^n$ are precisely the one-dimensional connected complex Lie subgroups of the additive group $(\mathbf{C}^n, +)$. Below we present a statement in the same spirit for semisimple complex Lie groups.

Proposition 12.2. *An infinite discrete subset of $S = \mathrm{SL}_n(\mathbf{C})$ is tame if and only if there exist a connected one-dimensional complex algebraic subgroup A and an automorphism of the complex manifold S such that $\phi(D) \subset A$.*

Proof. Since A is unbounded, it contains a tame discrete subset D' (Propositions 9.7 and 4.10). This proves one direction, because any two tame discrete subsets of S are equivalent (Proposition 10.6).

For the opposite direction, let $D \subset A$ be a discrete subset.

If A is reductive, that is, if $A \simeq \mathbf{C}^*$, then D is tame due to Corollary 10.11.

It remains to discuss the case where $A \simeq \mathbf{C}$. Let $p_k : S \to \mathbf{C}^n$ denote the map which associates to each matrix its kth column. Choose a k such that p_k is not constant on A. Then $p_k : A \simeq \mathbf{C} \to \mathbf{C}^n$ is a nonconstant algebraic morphism. Note that every algebraic morphism from $\mathbf{C}$ to $\mathbf{C}^n$ is given by polynomials and therefore is a proper map. Hence, D is tame due to Proposition 10. □

13. Some open questions

- Is every discrete subgroup in a Stein complex Lie group tame?
- To what extent do our results for $\mathrm{SL}_n(\mathbf{C})$ extend to arbitrary Stein complex Lie groups and, more generally, to arbitrary homogeneous manifolds, perhaps presuming 'flexibility' or a 'density property'?
- Does every noncompact complex manifold admit a nontame infinite discrete subset?

Instead of considering the full automorphism group, one may also discuss certain subgroups, for example the subgroup preserving some volume form.

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https://doi.org/10.1017/S1446788718000241 Published online by Cambridge University Press