COMPOSITIO MATHEMATICA

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Compositio Math. **146** (2010), 638–682.

doi:10.1112/S0010437X09004485

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Abstract

In this paper, we define Swan conductors for unit-root overconvergent F-isocrystals using the theory of arithmetic $\mathscr{D}$-modules due to Berthelot. Our Swan conductors are compared with the Swan conductors for ℓ-adic sheaves constructed by Kato and Saito using a geometric method. As an application, we prove the integrality of Swan conductors in the sense of Kato and Saito under the 'resolution of singularities' assumption.

Introduction

Let k be a perfect field and let U be a connected smooth separated scheme of finite type over k. Take a $\overline{\mathbb{Q}}_\ell$-lisse sheaf $\mathcal{F}$ on U. When the characteristic of k is zero, it is well-known that the Euler–Poincaré characteristic

$$\chi_c(U,\mathcal{F}) := \sum (-1)^i \dim_{\overline{\mathbb{Q}}_\ell} H^i_c(U_{\overline{k}},\mathcal{F})$$

is equal to $\mathrm{rank}(\mathcal{F})\cdot\chi_c(U)$ where $\chi_c(U):=\chi_c(U,\overline{\mathbb{Q}}_\ell)$. However, the situation is completely different when k is a field of characteristic $p>0$ and U is not proper over k, even if $\ell\neq p$. Calculation of the difference $\chi_c(U,\mathcal{F})-\mathrm{rk}(\mathcal{F})\cdot\chi_c(U)$ is one of the main subjects of ramification theory. In what follows, let k be a field of characteristic $p>0$.

When U is a curve, the above difference can be expressed by means of Swan conductors according to the Grothendieck–Ogg–Shafarevich formula. Kato and Saito [KS08] recently succeeded in extending the Grothendieck–Ogg–Shafarevich formula to higher dimensions, based on works by S. Bloch, K. Kato, G. Laumon, S. Saito, and others. The higher-dimensional analog of the Grothendieck–Ogg–Shafarevich formula is as follows. Take a connected proper smooth scheme X over k which contains U as an open dense subscheme such that $X\backslash U$ is a simple normal crossing divisor. Let $\ell\neq p$ and take a $\overline{\mathbb{Q}}_\ell$-lisse sheaf $\mathcal{F}$ on U. Then we have

$$\chi_c(U,\mathcal{F}) = \mathrm{rk}(\mathcal{F})\chi_c(U,\overline{\mathbb{Q}}_\ell) - \deg \mathrm{Sw}^{\mathrm{KS}}_X(\mathcal{F}),$$

where $\mathrm{Sw}^{\mathrm{KS}}_X(\mathcal{F})$ is the Swan conductor of Kato and Saito, defined as a 0-cycle on X with rational coefficients (i.e. $\mathrm{Sw}^{\mathrm{KS}}_X(\mathcal{F})\in \mathrm{CH}_0(X)_{\mathbb{Q}}$). We remark that Kato and Saito actually dealt with a more general situation, but we shall only consider this particular case. This Swan conductor was defined using geometric methods. More precisely, Kato and Saito first defined $\mathrm{Sw}^{\mathrm{KS}}_X(f_*\overline{\mathbb{Q}}_\ell)$ for a finite étale morphism $f: V\to U$ using log blow-ups (see Definition 3.0.1) and then extended it to the general case, using the open Lefschetz trace formula, so that the analog of the Grothendieck–Ogg–Shafarevich formula holds. However, the integrality of this Swan conductor,

Received 21 February 2008, accepted in final form 28 July 2009, published online 10 March 2010.
2000 Mathematics Subject Classification 14F20, 14F30 (primary).
Keywords: arithmetic $\mathscr{D}$-modules, overholonomic modules, ramification theory, Swan conductors.

which can be seen as an analog of the classical Hasse–Arf theorem, has been left as a conjecture as follows.

Conjecture (Integrality conjecture [KS08, Conjecture 4.3.7]). Let X be a connected smooth proper scheme over k and let U be an open subscheme of X whose complement is a simple normal crossing divisor. For a $\overline{\mathbb{Q}}_\ell$-lisse sheaf $\mathcal{F}$, $\mathrm{Sw}_X^{\mathrm{KS}}(\mathcal{F})$ is in the image of

$$\mathrm{CH}_0(X) \to \mathrm{CH}_0(X)_{\mathbb{Q}}.$$

In this paper, we use a different method to define the Swan conductor $\mathrm{Sw}_X^{\mathscr{D}}(\mathscr{E})$ as an element of $\mathrm{CH}_0(X)$ (the Chow group with integral coefficients) for a unit-root overconvergent F-isocrystal $\mathscr{E}$ over U. Our definition of $\mathrm{Sw}_X^{\mathscr{D}}$ is based on the theory of arithmetic $\mathscr{D}$-modules due to Berthelot, particularly the theory of characteristic cycles. By the Kashiwara–Dubson formula for arithmetic $\mathscr{D}$-modules, also due to Berthelot, we get an analog of the Grothendieck–Ogg–Shafarevich formula:

$$\chi_{\mathrm{DR}}(U, \mathscr{E}) = \mathrm{rk}(\mathscr{E})\chi_{\mathrm{DR}}(U) - \deg \mathrm{Sw}_X^{\mathscr{D}}(\mathscr{E}).$$

Thus, it is natural to ask whether there exists a relation between $\mathrm{Sw}_X^{\mathrm{KS}}$ and $\mathrm{Sw}_X^{\mathscr{D}}$. We propose the following conjecture.

Conjecture. Let X be a projective smooth scheme and let U be an open subscheme whose complement in X is a normal crossing divisor. Let $\chi : \pi_1(U) \to \mathbb{C}^\times$ be a character factoring through a finite group. We denote by $\mathcal{F}(\chi)$ and $\mathscr{E}(\chi)$ the corresponding $\overline{\mathbb{Q}}_\ell$-lisse sheaf and unit-root overconvergent F-isocrystal arising from χ via fixed isomorphisms $\overline{\mathbb{Q}}_\ell \cong \mathbb{C} \cong \overline{\mathbb{Q}}_p$ (see § 4). Then we have

$$\mathrm{Sw}_X^{\mathrm{KS}}\mathcal{F}(\chi) = \mathrm{Sw}_X^{\mathscr{D}}\mathscr{E}(\chi)$$

in $\mathrm{CH}_0(X)_{\mathbb{Q}}$.

Before stating the main result, we introduce some terminology.

Definition 0.0.1. Let X be a smooth scheme over k, U an open subscheme of X, and $f : V \to U$ a finite étale morphism. We say that (U, X, V) is resolvable if there exists a cartesian diagram

$$\begin{array}{ccc} V & \xrightarrow{j} & Y \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle f'} \\ U & \longrightarrow & X \end{array}$$

such that Y is a smooth scheme over k, j is an open immersion, the complement $Y \setminus V$ is a simple normal crossing divisor, and f' is projective. We say that (U, X) is resolvable if (U, X, V) is resolvable for any finite étale scheme V over U.

If we assume the following resolution of singularities, then any pair (U, X) such that X is smooth and U is its open subscheme is resolvable.

Resolution of singularities. Let X be a scheme of finite type over k and let U be a dense open subscheme that is smooth over k. Then there exists a projective morphism $f : X' \to X$ such that the scheme X' is smooth, the morphism $f^{-1}(U) \to U$ is an isomorphism, and the complement of $f^{-1}(U)$ in X' is a simple normal crossing divisor.

The main result of this paper is the following.

Theorem 4.0.5. *With the notation in the previous conjecture, suppose that (U, X) is resolvable. Then the conjecture is true for χ factoring through a finite group $\mathbb{Z}/p^i\mathbb{Z}$ for some $i > 0$.*

As a corollary, we prove the integrality conjecture under the same assumption.

Corollary 4.0.7. *Let X be a projective smooth variety over k. Under the assumption of resolution of singularities, the integrality conjecture holds for X.*

The integrality conjecture was proved in the case where $\dim X \leqslant 2$ in [KS08, Corollary 5.1.7]. Although not written down explicitly in [KS08], it can be proved with the same method that the integrality conjecture is true under assumption of the cleanness conjecture [Kat94, Definition 5.3]. However, in general the cleanness conjecture is stronger than the assumption of resolution of singularities.

Now let us go into more detail. We fix a complete discrete valuation ring R such that the residue field is k and the characteristic of the fractional field K is zero. To construct a theory of Swan conductors for a smooth scheme X over k, we first have to construct a category of arithmetic $\mathscr{D}$-modules on X. However, Berthelot's theory of arithmetic $\mathscr{D}$-modules deals only with smooth formal schemes over $\mathscr{S} := \mathrm{Spf}(R)$. There are two main approaches to the problem.

(i) Since X is smooth, we can take smooth liftings over $\mathscr{S}$ locally. We construct the desired category by gluing sheaves.

(ii) Suppose X can be embedded into a proper smooth formal scheme $\mathscr{X}$ over $\mathscr{S}$. We define the sheaves on X to be sheaves on $\mathscr{X}$ whose support is contained in X.

We believe that a systematic treatment is needed for the first method. Since this paper is not intended to provide a thorough treatment of the theory of arithmetic $\mathscr{D}$-modules, we do not adopt the first approach. Instead, we take the second approach by restricting ourselves to schemes over k which can be embedded into smooth proper formal schemes (e.g. quasi-projective schemes).

There are two major obstacles that may be encountered in the construction of Swan conductors for overconvergent F-isocrystals.

- We do not know whether proper push-forwards preserve holonomicity.
- We do not know whether specializations of overconvergent F-isocrystals are holonomic.

These issues are known as part of Berthelot's conjecture, and they prevent us from defining Swan conductors for general overconvergent F-isocrystals. However, we may use the theory of overholonomic modules due to Caro to remedy these problems. The main properties of overholonomic modules are the following.

- Overholonomic modules with Frobenius structures are holonomic.
- Push-forwards by proper morphisms preserve overholonomicity.
- Specializations of unit-root overconvergent F-isocrystals are overholonomic.

These properties enable us to define Swan conductors at least for unit-root overconvergent F-isocrystals. For a general definition, it seems that we will have to wait for the Berthelot conjecture to be resolved.

To compare Swan conductors $\mathrm{Sw}_X^{\mathrm{KS}}$ and $\mathrm{Sw}_X^{\mathscr{D}}$, we first compare them in the case where sheaves can be written as the push-forwards of constant sheaves (or structure sheaves) by finite étale morphism. We call this 'the geometric case'. In this case, both Swan conductors can be calculated explicitly. We calculate $\mathrm{Sw}_X^{\mathrm{KS}}$ directly using Fulton's intersection theory. For the calculation of $\mathrm{Sw}_X^{\mathscr{D}}$ in the geometric case, we need the relative Kashiwara–Dubson formula, which is a generalization of the Kashiwara–Dubson formula proved by Berthelot.

Now let us outline the contents of this paper. In § 1, we introduce some basic terminology that will be used in the paper. In this section, we also attach overholonomic modules to some representations of the fundamental group of a smooth scheme over k.

In § 2, we define characteristic homomorphisms and characteristic cycles, and prove the following relative Kashiwara–Dubson formula for arithmetic $\mathscr{D}$-modules.

Corollary 2.3.17 (Relative Kashiwara–Dubson formula). *Let $f : X \to Y$ be a proper morphism between projective smooth schemes over k. Let $\sigma_X : X \to T^*X$ and $\sigma_Y : Y \to T^*Y$ be zero-sections. Let $\mathscr{E}$ be an overholonomic $\mathscr{D}^\dagger_{X,\mathbb{Q}}$-module with a Frobenius structure, and suppose that $f_+\mathscr{E}$ is an overholonomic module (or that all the cohomology sheaves of $f_+\mathscr{E}$ are overholonomic).*[1] *Then*

$$f_*(\sigma_X^*(\mathrm{ZCar}^\dagger(\mathscr{E}))) = \sigma_Y^*(\mathrm{ZCar}^\dagger(f_+\mathscr{E}))$$

in $\mathrm{CH}_0(Y)_{\mathbb{Q}}$.

On the way to proving Corollary 2.3.17, we give a complete proof of the Kashiwara–Dubson–Berthelot formula, whose proof is only sketched in [Ber02]. By using the relative Kashiwara–Dubson formula, we are able to calculate $\mathrm{Sw}_X^{\mathscr{D}}$ in the geometric case. This is done at the end of § 2.

In § 3, we calculate $\mathrm{Sw}_X^{\mathrm{KS}}$ using Fulton's theory of intersection. The comparison of Swan conductors in the geometric case is done at this point.

In § 4, we prove the main comparison theorem and its corollary.

Notation

Fix a perfect field k of characteristic $p > 0$. Let R be a complete discrete valuation ring of mixed characteristic $(0, p)$, with maximal ideal $\mathfrak{m} = (\pi)$, whose residue field is k. Let K be a fractional field and let e be the absolute ramification index of K. Unless otherwise stated, we also fix R and K.

Each scheme is assumed to be noetherian separated of finite type over its base scheme. All smooth schemes are assumed to be equidimensional. In principle, we use block letters (e.g. X) for schemes and script letters (e.g. $\mathscr{X}$) for formal schemes. Let $\mathscr{X}$ be a formal scheme over $\mathscr{S}$, and let X be its special fiber. With an abuse of notation, we say that Z is a divisor in $\mathscr{X}$ if Z is a divisor in X.

1. Overholonomic arithmetic $\mathscr{D}$-modules

To construct Swan conductors for arithmetic $\mathscr{D}$-modules on a smooth scheme X over k, we have to construct a category of arithmetic $\mathscr{D}$-modules on X; but we cannot use the theory of Berthelot directly since we should take a formally smooth lifting of X over $\mathscr{S}$, which is not possible in general. The standard method is to take an embedding into a proper smooth formal scheme locally and glue locally constructed $\mathscr{D}$-modules, but this is not easy since we do not know whether holonomicity will be preserved (see [Ber02, 5.3.6]). Caro defined the categories of overcoherent

[1] Using Caro and Tsuzuki, *Surholonomie des F-isocristaux surconvergents*, Preprint, this condition always holds under the assumption of Shiho's conjecture, which was announced by Kedlaya.

modules and overholonomic modules, and proved an analog of the conjecture by Berthelot. This has enabled us to construct, at least, the category of overcoherent (overholonomic) $\mathscr{D}$-modules over X. In this section, we define the categories and functors needed in this paper using the work of Caro on overcoherent and overholonomic $\mathscr{D}^\dagger_{X,\mathbb{Q}}$-modules.

1.1 Preliminaries

1.1.1 Let R be a complete discrete valuation ring of mixed characteristic $(0,p)$ as in the notation above, let $\mathfrak{m}$ be its maximal ideal, and let $\mathscr{X}$ be a smooth formal scheme over $\mathscr{S}:=\mathrm{Spf}(R)$. For $i\geqslant 0$, let X_i be the scheme $\mathscr{X}\otimes_R R/\mathfrak{m}^{i+1}$. We shall sometimes write X_0 simply as X. Let $m\geqslant 0$ be an integer such that $p^m>e/(p-1)$. Then we may endow R with a canonical nilpotent m-PD structure. We shall use freely the notation of [Ber96b, Ber00, Ber02, Car05a, Car05c]. To deal with holonomic modules, we suppose that there exists a lifting of absolute Frobenius automorphisms $\sigma:\mathscr{S}\to\mathscr{S}$ and fix one such lifting. For short, we denote the category of holonomic F-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-complexes with bounded cohomology sheaves F-$D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$ by $D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$, and write holonomic $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module instead of holonomic F-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module. We denote the full subcategory of F-$D^b(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$ consisting of overcoherent (respectively, overholonomic) complexes by $D^b_{\mathrm{overcoh}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$ (respectively, $D^b_{\mathrm{overhol}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$). Throughout this paper, all overcoherent and overholonomic modules are considered with Frobenius structures. Let Z be a closed subscheme of $\mathscr{X}$. Then we denote by $\mathbb{R}\underline{\Gamma}^\dagger_Z$ (respectively, $({}^\dagger Z)$) the local cohomology functor (respectively, the restriction functor) defined in [Car05a, Définition 2.2.6]. (Note that the definition of the local cohomology functor may differ from that of Berthelot in [Ber02, 4.4.4].)

Definition 1.1.2.

(i) We call $(U,X,Z,\mathscr{P})$ a quadruple over $\mathscr{S}$ if X is a scheme over k, $\mathscr{P}$ is a smooth formal scheme over $\mathscr{S}$ that contains X as a closed subscheme, Z is a closed subscheme of $\mathscr{P}$, and $U=X\backslash Z$. In addition, we say that the quadruple is a d-quadruple (respectively, proper quadruple) if Z is a divisor (respectively, if $\mathscr{P}$ is proper). A morphism of quadruples over $\mathscr{S}$,

$$f:(U,X,Z,\mathscr{P})\to(U',X',Z',\mathscr{P}'),$$

is a morphism of formal schemes $f_{\mathscr{P}}:\mathscr{P}\to\mathscr{P}'$ over $\mathscr{S}$ with $f_{\mathscr{P}}(U)\subset U'$. We call $f_{\mathscr{P}}$ the realization of f.

(ii) We call $(X,Z,\mathscr{P})$ a (d-)triple over $\mathscr{S}$ if $(X\backslash Z,X,Z,\mathscr{P})$ is a (d-)quadruple over $\mathscr{S}$. A morphism of triples $(X,Z,\mathscr{P})\to(X',Z',\mathscr{P}')$ is a morphism of quadruples $(X\backslash Z,X,Z,\mathscr{P})\to(X'\backslash Z',X',Z',\mathscr{P}')$.

(iii) For a scheme U, we say that U is a scheme with quadruples (respectively, d-quadruples) over $\mathscr{S}$ if there exists a proper quadruple (respectively, a proper d-quadruple) of the form $(U,X,Z,\mathscr{P})$ over $\mathscr{S}$.

Example 1.1.3. If X is a quasi-projective scheme over k, then X is always a scheme with quadruples. If X is a projective or affine scheme over k, then X is a scheme with d-quadruples. For an affine scheme X, take a closed immersion $X\hookrightarrow\mathbb{A}^n_k$ for some $n\geqslant 0$ and define Z to be the hyperplane section of $\mathbb{P}^n_k$. Consider X as a subscheme of $\mathbb{P}^n_k$ via the composition $X\hookrightarrow\mathbb{A}^n_k\hookrightarrow\mathbb{P}^n_k$. Then $(X,\overline{X},Z,\widehat{\mathbb{P}}^n_{\mathscr{S}})$ is a d-quadruple.

Definition 1.1.4. For a triple $(X,Z,\mathscr{P})$, let $\mathfrak{M}_{(X,Z,\mathscr{P})}$ (respectively, $\mathfrak{M}^+_{(X,Z,\mathscr{P})}$) be the category of overcoherent (respectively, overholonomic) $\mathscr{D}^\dagger_{\mathscr{P},\mathbb{Q}}$-modules (see [Car05a, 3.1.1] and

[Car05c, Définition 2.1]) $\mathscr{E}$ with $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathscr{E})=0$ and whose support is contained in X. We denote by $D^b(\mathfrak{M}_{(X,Z,\mathscr{P})})$ (respectively, $D^b(\mathfrak{M}^{+}_{(X,Z,\mathscr{P})})$) the full subcategory of $D^b(\mathscr{D}^{\dagger}_{\mathscr{P},\mathbb{Q}})$ consisting of overcoherent (respectively, overholonomic) $\mathscr{D}^{\dagger}_{\mathscr{P},\mathbb{Q}}$-complexes $\mathscr{E}$ that satisfy $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathscr{E})=\mathscr{E}$ and $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathscr{E})=0$. (Note that these do not have the same meaning as the derived category of $\mathfrak{M}^{(+)}_{(X,Z,\mathscr{P})}$.)

Definition 1.1.5. Let $f:(X,Z,\mathscr{P})\to(X',Z',\mathscr{P}')$ be a morphism of triples. Let $f_{\mathscr{P}}:\mathscr{P}\to\mathscr{P}'$ be the realization of f. Let $\mathscr{F}\in D^b(\mathfrak{M}^{(+)}_{(X',Z',\mathscr{P}')})$. We define

$$f^{!}(\mathscr{F}):=(\mathbb{R}\underline{\Gamma}^{\dagger}_{X}\circ({}^{\dagger}Z)\circ f^{!}_{\mathscr{P}})(\mathscr{F}).$$

This is contained in $D^b(\mathfrak{M}^{(+)}_{(X',Z',\mathscr{P}')})$. Moreover, we suppose that $f_{\mathscr{P}}$ is proper. Let $\mathscr{E}\in D^b(\mathfrak{M}^{(+)}_{(X,Z,\mathscr{P})})$. We define

$$f_{+}(\mathscr{E}):=f_{\mathscr{P}+}(\mathscr{E}).$$

This is contained in $D^b(\mathfrak{M}^{(+)}_{(X,Z,\mathscr{P})})$.

Theorem 1.1.6 [Car05c, Théorème 3.8, 3.12]. *Let $f:(U,X,Z,\mathscr{P})\to(U,X',Z',\mathscr{P}')$ be a morphism of proper quadruples over $\mathscr{S}$ whose restriction to U is the identity. Then f_{+} and $f^{!}$ give an equivalence between $D^b(\mathfrak{M}^{(+)}_{(X,Z,\mathscr{P})})$ and $D^b(\mathfrak{M}^{(+)}_{(X',Z',\mathscr{P}')})$. If, moreover, quadruples are d-quadruples and the realization $f_{\mathscr{P}}$ of f is smooth, this equivalence induces an equivalence of categories between $\mathfrak{M}^{(+)}_{(X,Z,\mathscr{P})}$ and $\mathfrak{M}^{(+)}_{(X',Z',\mathscr{P}')}$.*

Definition 1.1.7.

(i) Let U be a scheme with quadruples over $\mathscr{S}$. Take a proper quadruple $(U,X,Z,\mathscr{P})$. We define $D^b(\mathfrak{M}_{U/\mathscr{S}})$ to be $D^b(\mathfrak{M}_{(X,Z,\mathscr{P})})$ and call it *the derived category of bounded overcoherent complexes of $\mathscr{D}^{\dagger}_{U,\mathbb{Q}}$-modules*. By Theorem 1.1.6 above, we can see that this category does not depend on the choice of proper quadruples, except for the canonical equivalences of categories. This justifies the notation.

(ii) Suppose that U is a scheme with d-quadruples. Taking a proper d-quadruple $(U,X,Z,\mathscr{P})$, $\mathfrak{M}_{(X,Z,\mathscr{P})}$ also does not depend on the choice of proper d-quadruple. We denote this by $\mathfrak{M}_{U/\mathscr{S}}$ and call it *the category of overcoherent $\mathscr{D}^{\dagger}_{U,\mathbb{Q}}$-modules*.

(iii) Let $\mathscr{E}$ be an overcoherent $\mathscr{D}^{\dagger}_{U,\mathbb{Q}}$-module and $(U,X,Z,\mathscr{P})$ a proper d-quadruple. By definition, this gives an element $\tilde{\mathscr{E}}$ of $\mathfrak{M}_{(X,Z,\mathscr{P})}$. We call $\tilde{\mathscr{E}}$ the realization of $\mathscr{E}$ in the quadruple $(U,X,Z,\mathscr{P})$ (or triple $(X,Z,\mathscr{P})$). We define realizations of elements of $D^b(\mathfrak{M}_{U/\mathscr{S}})$ in a quadruple (or triple) in the same way.

(iv) In the same way, we define the overholonomic counterparts $D^b(\mathfrak{M}^{+}_{U/\mathscr{S}})$ and $\mathfrak{M}^{+}_{U/\mathscr{S}}$.

For more details, see [Car05a, 3.2.9] and [Car05c, 3.17].

Remark 1.1.8.

(i) Let U be a scheme with d-quadruples. Then, for $i\in\mathbb{Z}$, we have ith cohomology functors $\mathscr{H}^i:D^b(\mathfrak{M}_U)\to\mathfrak{M}_U$. Taking the realization in a proper d-quadruple $(U,X,Z,\mathscr{P})$, these are the usual ith cohomology functors $\mathscr{H}^i:D^b(\mathfrak{M}_{(X,Z,\mathscr{P})})\to\mathfrak{M}_{(X,Z,\mathscr{P})}$. We can see that this does not depend on the choice of the d-quadruples (cf. [Car05a, Remarque 3.2.2.2]).

https://doi.org/10.1112/S0010437X09004485 Published online by Cambridge University Press

(ii) Even for schemes which cannot be embedded into proper smooth formal schemes, we can define $\mathfrak{M}_{U/\mathscr{S}}$ by taking d-triples locally and gluing the locally constructed ones. However, we will not go into the details here, since in this paper we only consider schemes with d-quadruples. Interested readers can consult [Car05c, 3.17].

1.1.9 Now let $f: U \to V$ be a proper morphism between smooth schemes over k with quadruples over $\mathscr{S}$. In this situation, we get a morphism of proper quadruples $\tilde{f}: (U, X, Z, \mathscr{P}) \to (V, Y, W, \mathscr{P}')$ whose restriction to the special fibers is f. Indeed, take proper quadruples $(U, X, Z, \mathscr{P})$ and $(V, Y, W, \mathscr{P}')$, and let $i: U \hookrightarrow \mathscr{P}$ and $f': V \to Y \hookrightarrow \mathscr{P}'$. Now consider the proper quadruple $(U, \overline{U}, Z', \mathscr{P} \times \mathscr{P}')$, where U is thought of as a subscheme of $\mathscr{P} \times \mathscr{P}'$ by the immersion $(i, f'): U \hookrightarrow \mathscr{P} \times \mathscr{P}'$ and $Z' = Z \times \mathscr{P}' \cup \mathscr{P} \times W$. Then define $\tilde{f}: (U, \overline{U}, Z', \mathscr{P} \times \mathscr{P}') \to (V, Y, W, \mathscr{P}')$ to be the morphism induced by the canonical projection $\mathscr{P} \times \mathscr{P}' \to \mathscr{P}'$. The restriction of $\tilde{f}$ to the special fiber is f, and this is what we wanted.

Definition 1.1.10. Let $f: U \to V$ be a proper morphism between smooth schemes with quadruples over k. Preserving the previous notation, we define a functor $f_+: D^b(\mathfrak{M}_{U/\mathscr{S}}) \to D^b(\mathfrak{M}_{V/\mathscr{S}})$ to be $\tilde{f}_+$, and similarly for $f^!$. One can easily show, using Theorem 1.1.6, that this definition is independent of the choice of quadruples and morphisms of quadruples. This justifies our notation.

1.2 Some constructions

Let X be a proper scheme with d-quadruples and let U be an open subscheme such that its complement in X is a divisor. In this case, there may not exist a d-quadruple $(U, X, Z, \mathscr{P})$, and we are not able to use the category $\mathfrak{M}_U$ defined in the previous section. We do, however, have the following.

Lemma 1.2.1. *Let X be a smooth scheme with d-quadruples and let U be an open subscheme such that $Z := X \backslash U$ is a divisor in X. Let $(U, \overline{X}, \overline{Z}, \mathscr{P})$ be a proper quadruple (not necessarily a d-quadruple) that satisfies the following condition:*

there exists a d-quadruple of the form $(X, \overline{X}, D, \mathscr{P})$.

Then $\mathfrak{M}^{(+)}_{(\overline{X}, \overline{Z}, \mathscr{P})}$ does not depend on the choice of proper quadruples.

Proof. We shall treat only the overcoherent case, since the proof for the other case is the same. Let $(U, \overline{X}', \overline{Z}', \mathscr{P}')$ be another quadruple, where $\overline{X}'$ is the closure of X in $\mathscr{P}'$ and $\overline{Z}'$ also satisfies the condition. Replacing $\mathscr{P}'$ by $\mathscr{P} \times \mathscr{P}'$ if needed, we may suppose that there exists a smooth proper morphism of triples $f: (\overline{X}, \overline{Z}, \mathscr{P}) \to (\overline{X}', \overline{Z}', \mathscr{P}')$. Since $\mathfrak{M}_{(\overline{X}, \overline{Z}, \mathscr{P})}$ is the subcategory of $D^b(\mathfrak{M}_{(\overline{X}, \overline{Z}, \mathscr{P})})$ consisting of complexes whose cohomology is 0 except for $\mathscr{H}^0$ and there exists a canonical equivalence between $D^b(\mathfrak{M}_{(\overline{X}, \overline{Z}, \mathscr{P})})$ and $D^b(\mathfrak{M}_{(\overline{X}', \overline{Z}', \mathscr{P}')})$, all we have to show is that for $\mathscr{E} \in \mathfrak{M}_{(\overline{X}, \overline{Z}, \mathscr{P})}$, $\mathscr{H}^k(f_+\mathscr{E}) = 0$ for $k \neq 0$, and that for $\mathscr{E}' \in \mathfrak{M}_{(\overline{X}', \overline{Z}', \mathscr{P}')}$, $\mathscr{H}^k(f^!\mathscr{E}') = 0$ for $k \neq 0$. The push-forward part is easy to verify using Theorem 1.1.6, since X is a scheme with d-quadruples. We will show the extraordinary pull-back part.

Put $\mathscr{U} := \mathscr{P} \backslash D$, $D' := f^{-1}(D)$, and $\mathscr{U}' := f^{-1}(\mathscr{U})$. Since D is a divisor, we are reduced to proving the assertion for the restriction

$$f|_{\mathscr{U}}: (U, X, \overline{Z} \cap \mathscr{U}, \mathscr{U}) \to (U, X, \overline{Z}' \cap \mathscr{U}', \mathscr{U}'),$$

by [Ber96b, 4.3.12.]. Owing to [Car05a, 3.2.4.], we can suppose that $Z = \overline{Z} \cap \mathscr{U}$ and $Z = \overline{Z}' \cap \mathscr{U}'$.

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There exists an open covering $\{\mathscr{U}'_i\}_{i\in I}$ of $\mathscr{U}'$ and divisors W'_i of $\mathscr{U}'_i$ such that $W'_i \cap X = Z \cap \mathscr{U}'_i$. Let $\mathscr{U}_i := f'^{-1}(\mathscr{U}'_i)$, $W_i := f'^{-1}(W'_i)$, $U_i := U \cap \mathscr{U}_i$, and $X_i := X \cap \mathscr{U}_i$; then the $(U_i, X_i, W_i, \mathscr{U}_i)$ are d-quadruples for each $i \in I$. Let $f'_i : (X_i, W_i, \mathscr{U}_i) \to (X_i, W'_i, \mathscr{U}'_i)$ be the restriction of f. This is a proper smooth morphism. It suffices to show that $\mathscr{H}^k(f'^!_i \mathscr{E}|_{\mathscr{U}_i}) = 0$ for each $i \in I$. Note that $\mathscr{E}|_{\mathscr{U}_i}$ is an element of $\mathfrak{M}_{(X_i,W_i,\mathscr{U}_i)}$. Thus $\mathscr{H}^k(f'^! \mathscr{E}|_{\mathscr{U}'_i}) = \mathscr{H}^k(f'^!_i(\mathscr{E}|_{\mathscr{U}_i})) = 0$, where the second equality holds by Theorem 1.1.6. □

The following lemma is used in the coming sections.

Lemma 1.2.2. *With the above notation, let $\mathscr{E}$ be an element in $D^b(\mathfrak{M}_{(U,\overline{X},\overline{Z},\mathscr{P})})$ such that $\mathscr{E}|_{\mathscr{P}\backslash\overline{Z}}$ is contained in $\mathfrak{M}_{(U,U,\emptyset,\mathscr{P}\backslash\overline{Z})}$. Then $\mathscr{E}$ is contained in $\mathfrak{M}_{(U,\overline{X},\overline{Z},\mathscr{P})}$.*

Proof. Take an open covering $\{\mathscr{P}_i\}_{i\in I}$ such that there exist d-quadruples $\{(U_i, \overline{X}_i, D_i, \mathscr{P}_i)\}$ where $U_i := U \cap \mathscr{P}_i$ and $\overline{X}_i := \overline{X} \cap \mathscr{P}_i$. Since the verification is local, it suffices to show that $\mathscr{H}^k(\mathscr{E}|_{\mathscr{P}_i}) = 0$ for all $i \in I$ and $k \neq 0$. Let $\overline{Z}_i := \overline{Z} \cap \mathscr{P}_i$. Note that $\mathscr{H}^k(\mathscr{E}) \in \mathfrak{M}_{(\overline{X},\overline{Z},\mathscr{P})}$. By [Car05a, 3.2.4], we get that

$$\mathfrak{M}_{(\overline{X}_i,\overline{Z}_i,\mathscr{P}_i)} = \mathfrak{M}_{(\overline{X}_i,\overline{Z}_i\cup D_i,\mathscr{P}_i)} \subset \mathfrak{M}_{(\overline{X}_i,D_i,\mathscr{P}_i)}.$$

By [Ber96b, 4.3.12(ii)], it is enough to show that $\mathscr{H}^k(\mathscr{E})|_{\mathscr{P}_i\backslash D_i} = 0$ for $k \neq 0$. Since this is local in X, which is smooth, we can conclude by invoking Berthelot and Kashiwara's theorem [Ber02, 5.3.3]. □

Definition 1.2.3. Let X be a smooth scheme with d-quadruples and let U be an open subscheme such that the complement $X\backslash U$ is a divisor. Let $(U, \overline{X}, Z, \mathscr{P})$ be a quadruple satisfying the condition in Lemma 1.2.1. We write $\mathfrak{M}^{(+)}_{(\overline{X},Z,\mathscr{P})}$ as $\mathfrak{M}^{(+)}_{(U,X)}$ (Lemma 1.2.1 justifies this notation) and call it the category of overcoherent (or overholonomic) $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-modules if no confusion can arise. We also denote $D^b(\mathfrak{M}^{(+)}_{(\overline{X},Z,\mathscr{P})})$ by $D^b(\mathfrak{M}^{(+)}_{(U,X)})$.

1.2.4 Note that given a family of d-quadruples $\{(U_i, X_i, W_i, \mathscr{P}_i)\}_{i\in I}$ such that $\{\mathscr{P}_i\}$ is an open covering of $\mathscr{P}$ and $W_i \cap X = Z \cap \mathscr{P}_i$, and given $\mathscr{E}_i \in \mathfrak{M}_{(X_i,W_i,\mathscr{P}_i)}$ for each $i \in I$ and $\mathscr{E}_i|_{\mathscr{P}_i\cap\mathscr{P}_j} \cong \mathscr{E}_j|_{\mathscr{P}_i\cap\mathscr{P}_j}$ for every $i, j \in I$ satisfying the cocycle condition, we have an element of $\mathscr{E} \in \mathfrak{M}_{(U,X)}$ such that $\mathscr{E}|_{\mathscr{P}_i} = \mathscr{E}_i$.

Definition 1.2.5.

(i) Let $(U, X, W, \mathscr{P})$ be a proper quadruple. Let $Z := X\backslash U$. We define an overholonomic complex in $D^b(\mathfrak{M}^+_U)$ by

$$\mathcal{O}_{U,\mathbb{Q}} := \mathbb{R}\underline{\Gamma}^\dagger_X \circ ({}^\dagger W)(\mathcal{O}_{\mathscr{P},\mathbb{Q}})[-d_{X/\mathscr{P}}],$$

where $d_{X/\mathscr{P}} := \dim(X) - \dim(\mathscr{P})$. We also write $\mathcal{O}_{X,\mathbb{Q}}({}^\dagger Z)$ for $\mathcal{O}_{U,\mathbb{Q}}$. It can easily be seen that the definition depends only on U, which justifies our notation. This complex is called the structure complex of U. (Caro also defined this sheaf and called it the constant coefficient in [Car05c].)

(ii) Suppose that X is a smooth scheme with d-quadruples and let U be an open subscheme whose complement is a divisor. Then $\mathcal{O}_{X,\mathbb{Q}}({}^\dagger Z)$ is an overholonomic module in $\mathfrak{M}^+_{(U,X)}$, as verified in [Car05c, 3.16]. With an abuse of notation, we shall call this the structure sheaf of U.

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1.3 Arithmetic $\mathscr{D}$-modules associated to overconvergent isocrystals

1.3.1 For a scheme U of finite type over k, Berthelot defined the category of (F-)overconvergent isocrystals, denoted by (F-)$\mathrm{Isoc}^{\dagger}(U)$ (see [Ber96a]). Let $(U, X, Z, \mathscr{P})$ be a d-quadruple. We denote by (F-)$\mathrm{Isoc}^{\dagger}(U, X, \mathscr{P}/K)$ the category of realizations of (F-)overconvergent isocrystals in this quadruple, i.e. (F-)isocrystals on X that are overconvergent along $X\backslash U$ in $\mathscr{P}$. Suppose, in addition, that X is smooth. In [Car05b, Théorème 2.5.10], Caro defined a fully faithful functor

$$\mathrm{sp}_{X\hookrightarrow\mathscr{P},Z,+} : F\text{-}\mathrm{Isoc}^{\dagger}(U, X, \mathscr{P}) \to F\text{-}\mathrm{Coh}(X, Z, \mathscr{P}),$$

where $F\text{-}\mathrm{Coh}(X, Z, \mathscr{P})$ denotes the category of coherent $F\text{-}\mathscr{D}^{\dagger}_{\mathscr{P},\mathbb{Q}}({}^{\dagger}Z)$-modules whose support is contained in X.

The idea of the construction of this functor is the following. Assume $X \hookrightarrow \mathscr{P}$ has a smooth lifting $i : \mathscr{X} \hookrightarrow \mathscr{P}$. Taking an F-isocrystal E on X that is overconvergent along $X\backslash U$, $\mathrm{sp}_{+}(E)$ is by definition $i_{+}(\mathrm{sp}_{*}(E))$, where sp is the specialization map from the rigid space of generic fibers into the formal scheme $\mathscr{X}_K \to \mathscr{X}$. If X does not have a smooth lifting, then we glue together locally constructed ones.

1.3.2 Now, by the fact that the image of a unit-root overconvergent isocrystal under the specialization map is overholonomic [Car05c, Théorème 5.3], we get

$$\mathrm{sp}_{X\hookrightarrow\mathscr{P},Z,+} : F\text{-}\mathrm{Isoc}^{\dagger}(U, X, \mathscr{P})^{0} \to \mathfrak{M}^{+}_{(X,Z,\mathscr{P})},$$

where $F\text{-}\mathrm{Isoc}^{\dagger}(U, X, \mathscr{P})^{0}$ denotes the category of unit-root F-isocrystals on U that are overconvergent along $X\backslash U$ in $\mathscr{P}$. We will generalize this functor slightly as follows.

Lemma 1.3.3. *Let X be a smooth scheme with d-triples and let U be an open subscheme of X such that $Z := X\backslash U$ is a divisor. Then we have a canonical fully faithful functor*

$$\mathrm{sp}_{U,X,+} : F\text{-}\mathrm{Isoc}^{\dagger}(U, X/K)^{0} \to \mathfrak{M}^{+}_{(U,X)/\mathscr{S}}.$$

We shall often write $\mathrm{sp}_{U,X,+}$ as sp_{+} for short, if this is unlikely to cause confusion.

Proof. Take a smooth formal scheme $\mathscr{P}$ with closed embedding $X \hookrightarrow \mathscr{P}$. Since Z is a divisor in X, there exists an open affine covering $\{\mathscr{P}_i\}_{i\in I}$ of $\mathscr{P}$ and a family of divisors Z_i of $\mathscr{P}_i$ for $i \in I$ such that $(U_i, X_i, Z_i, \mathscr{P}_i)$ are d-quadruples for all $i \in I$, where $U_i := U \cap \mathscr{P}_i$ and $X_i := X \cap \mathscr{P}_i$ is affine. Given $E \in F\text{-}\mathrm{Isoc}^{\dagger}(U, X, \mathscr{P})^{0}$, we get $\mathrm{sp}_{X_i\hookrightarrow\mathscr{P}_i,Z_i,+}(E_i) \in \mathfrak{M}_{(X_i,Z_i,\mathscr{P}_i)}$, where E_i is the restriction of E to U_i. Moreover, we have canonical isomorphisms

$$\mathrm{sp}_{X_i\hookrightarrow\mathscr{P}_i,Z_i,+}(E_i)|_{\mathscr{P}_i\cap\mathscr{P}_j} \cong \mathrm{sp}_{X_j\hookrightarrow\mathscr{P}_j,Z_j,+}(E_j)|_{\mathscr{P}_i\cap\mathscr{P}_j} \qquad (*)$$

for every $i, j \in I$ satisfying the cocycle condition. Indeed, since $X_i \cap X_j$ is affine, it can be lifted to a smooth formal scheme $\mathscr{X}_{ij}$, and $X_i \cap X_j \hookrightarrow \mathscr{P}_{ij}$ can be lifted to $i : \mathscr{X}_{ij} \to \mathscr{P}_{ij}$ where $\mathscr{P}_{ij} := \mathscr{P}_i \cap \mathscr{P}_j$. By the construction of $\mathrm{sp}_{X_\bullet\hookrightarrow\mathscr{P}_\bullet,Z_\bullet,+}$, both sides of $(*)$ are $i_{+}\mathrm{sp}_{*}(E_{ij})$, where $\mathrm{sp} : \mathscr{X}_{ij_K} \to \mathscr{X}_{ij}$ is the specialization map and E_{ij} is the restriction of E to $X_i \cap X_j$. The cocycle condition comes from the independence of the construction up to canonical isomorphism in the choice of liftings. This shows that the family $\{\mathrm{sp}_{X_i\hookrightarrow\mathscr{P}_i,Z_i,+}(E_i)\}_{i\in I}$ defines an element of $\mathfrak{M}_{(U,X)}$.

Now we have to show that this definition does not depend on the choice of $\mathscr{P}$ and its open coverings. Given two open coverings $\{\mathscr{P}_i\}$ and $\{\mathscr{P}'_i\}$, there exists a refinement $\{\mathscr{P}''_i\}$ of the two, so we may suppose that $\{\mathscr{P}'_i\}$ is a refinement of $\{\mathscr{P}_i\}$. All we have to show is that for all $\mathscr{P}'_j \subset \mathscr{P}_i$, $\mathrm{sp}_{X_i\hookrightarrow\mathscr{P}_i,Z_i,+}(E_i)|_{\mathscr{P}'_i} = \mathrm{sp}_{X_j\hookrightarrow\mathscr{P}'_j,Z_j,+}(E_j)$, which is easy.

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Let us show that the construction does not depend on the choice of $\mathscr{P}$. Suppose we are given two smooth formal schemes $\mathscr{P}$ and $\mathscr{P}'$. Since $X \hookrightarrow \mathscr{P} \times \mathscr{P}'$ is a closed immersion, we may suppose that there exists a smooth morphism of d-quadruples

$$f : (U, X, Z, \mathscr{P}) \to (U, X, Z, \mathscr{P}').$$

Take a family of d-triples $\{(X_i, Z'_i, \mathscr{P}'_i)\}_{i\in I}$ where $\{\mathscr{P}'_i\}$ is an open covering of $\mathscr{P}'$ and Z'_i is a divisor of $\mathscr{P}_i$ for all $i \in I$. Since f is smooth, $\{(X_i, f^{-1}(Z_i), f^{-1}(\mathscr{P}'_i))\}_{i\in I}$ is the family of d-triples satisfying the same condition. Now, to show that $\{\mathrm{sp}_{X_i\hookrightarrow\mathscr{P}_i,Z_i,+}(E_i)\}$ and $\{\mathrm{sp}_{X_i\hookrightarrow f^{-1}(\mathscr{P}_i),f^{-1}(Z_i),+}(E_i)\}$ define the same element in $\mathfrak{M}_{(U,X)}$, we have to show that the following diagram is commutative for all $i \in I$.

$$\begin{array}{ccc} F\text{-Isoc}^\dagger(U_i, X_i, \mathscr{P}'_i)^0 & \xrightarrow{\mathrm{sp}_{X_i\hookrightarrow\mathscr{P}_i,Z_i,+}} & \mathfrak{M}_{(X_i,Z_i,\mathscr{P}'_i)} \\ \downarrow f^* & & \downarrow f^! \\ F\text{-Isoc}^\dagger(U_i, X_i, f^{-1}(\mathscr{P}'_i))^0 & \xrightarrow{\mathrm{sp}_{X_i\hookrightarrow f^{-1}(\mathscr{P}_i),f^{-1}(Z_i),+}} & \mathfrak{M}_{(X_i,f^{-1}(Z_i),f^{-1}(\mathscr{P}'_i))} \end{array}$$

But this is just [Car05b, Proposition 4.1.8]. The fully faithful property follows from the fact that $\mathrm{sp}_{X_i\hookrightarrow\mathscr{P}_i,Z_i,+}$ is fully faithful. □

We can write the essential image of sp_+ in the following way.

Lemma 1.3.4. *With the above notation, let $\mathscr{E}$ be an element of $\mathfrak{M}^+_{(U,X)}$. Take d-quadruples $(U_i, X_i, Z_i, \mathscr{P}_i)$ as in the proof of the previous lemma such that the U_i are affine schemes. Fix smooth liftings $\mathscr{U}_i$ of U_i. Then $\mathscr{E}$ is in the essential image of sp_+ if and only if, for each i, $\mathscr{E}_i|_{\mathscr{U}_i}$ is the specialization of a unit-root convergent isocrystal. Here $\mathscr{E}_i$ denotes the restriction of $\mathscr{E}$ to $\mathscr{P}_i$ and $\mathscr{E}_i|_{\mathscr{U}_i}$ denotes the pull-back to $\mathscr{U}_i$.*

Proof. This easily follows from using the fully faithful property of sp_+ and [Car05a, Théorème 2.5.10 and Remarques 2.5.11]. □

1.4 Construction of overconvergent *F*-isocrystals associated to representations

1.4.1 Let R, K and σ be as in 1.1.1. In this subsection, we further assume that there exists a lifting of Frobenius automorphisms $\sigma : R \to R$ such that $\sigma(\pi) = \pi$ and fix one such lifting. Let Λ be the subfield of K fixed by σ and let $K_0 := \mathrm{Frac}(W(k))$. Note that Λ is finite over $\mathbb{Q}_p = \mathrm{Frac}(W(\mathbb{F}_p))$ since $\sigma(\pi) = \pi$, and that $\Lambda \otimes_{\mathbb{Q}_p} K_0 \xrightarrow{\sim} K$.

Let X be a smooth scheme over k and let $\pi_1(X)$ be its algebraic fundamental group. We denote by $\mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(X))$ the category of continuous $\pi_1(X)$-representations on finite-dimensional Λ-vector spaces factoring through a finite group.

Before proceeding to the proposition, we recall the following theorem concerning the fully faithful property for unit-root F-isocrystals, which will be used in the proofs in this subsection.

Theorem 1.4.2 [Tsu02, Theorem 1.2.2]. *Let Y be a smooth scheme over k and let $X \hookrightarrow Y$ be an open immersion such that X is dense in Y. Let U be an open dense subscheme of X. Then the natural functor*

$$F\text{-Isoc}^\dagger(U, Y/K)^0 \to F\text{-Isoc}^\dagger(U, X/K)^0$$

is fully faithful. Here, $F\text{-Isoc}^\dagger(U, X/K)^0$ denotes the category of unit-root F-isocrystals on U that are overconvergent along $X\backslash U$.

Now we obtain the following proposition.

Proposition 1.4.3. *Let X be a smooth quasi-projective scheme with d-quadruples over k, and let U be an open subscheme whose complement in X is a divisor. Then there exists a unique fully faithful functor*

$$G^\dagger : \mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(U)) \to F\text{-Isoc}^\dagger(U, X/K)^0$$

such that the following diagram commutes.

$$\begin{array}{ccc} & \xrightarrow{G^\dagger} & F\text{-Isoc}^\dagger(U, X/K)^0 \\ \mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(U)) & & \downarrow \\ & \xrightarrow[G]{} & F\text{-Isoc}(U/K)^0 \end{array}$$

Here, $F\text{-Isoc}(U/K)^0$ denotes the category of unit-root convergent F-isocrystals on U, the vertical functor is the forgetful functor, and G denotes the fully faithful functor defined by Crew and Katz (see [Cre87, Theorem 2.1]).

Proof. Let $\mathscr{S}_0 := \mathrm{Spf}(W(k))$. Take $\rho \in \mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(U))$ and let

$$\rho : \pi_1(U) \to \mathrm{GL}_\Lambda(A).$$

Let H be the image of ρ. By assumption, this is a finite group. Let $f : V \to U$ be the finite étale covering corresponding to $\pi_1(U) \twoheadrightarrow H$. We first define the push-forward $f_*\mathcal{O}_{V/K_0}$ of the trivial overconvergent isocrystal $\mathcal{O}_{V/K_0}$ on V by using arithmetic $\mathscr{D}$-module theory.

Since f is a finite morphism and U is quasi-projective, V is also quasi-projective and, in particular, a scheme with quadruples since U is a scheme with quadruples. Thus, we may consider the push-forward

$$f_+ : D^b(\mathfrak{M}^+_{V/\mathscr{S}_0}) \to D^b(\mathfrak{M}^+_{U/\mathscr{S}_0}).$$

Since $\mathcal{O}_{V,\mathbb{Q}}$ is an overholonomic complex, $f_+\mathcal{O}_{V,\mathbb{Q}}$ is an overholonomic complex. Let

$$r : D^b(\mathfrak{M}^+_{U/\mathscr{S}_0}) \to D^b(\mathfrak{M}^+_{(U,X)/\mathscr{S}_0})$$

be the restriction functor. We put $f_{(U,X)+} := r \circ f_+$.

Claim. $f_{(U,X)+}\mathcal{O}_{V,\mathbb{Q}}$ is in $\mathfrak{M}^+_{(U,X)/\mathscr{S}_0}$, and this is contained in the image of sp_+.

To prove the claim, take a quadruple $(U, \overline{X}, W, \mathscr{P})$ such that there exists a divisor D in $\mathscr{P}$ with $X = \overline{X} \backslash D$. Let $(f_{(U,X)+}\mathcal{O}_{V,\mathbb{Q}})_{\mathscr{P}}$ be the realization. To verify the first part of the claim, it suffices to show that $\mathscr{H}^k(f_{(U,X)+}\mathcal{O}_{V,\mathbb{Q}})_{\mathscr{P}} = 0$ for $k \neq 0$, since $f_{(U,X)+}\mathcal{O}_{V,\mathbb{Q}}$ is an overholonomic complex. By Lemma 1.2.2, we are reduced to showing that

$$\mathscr{H}^i(f_{(U,X)+}\mathcal{O}_{V,\mathbb{Q}})_{\mathscr{P}}|_{\mathscr{P}\backslash W} = 0$$

for $i \neq 0$.

Since the statement is local, we may assume that U and, thus, V can be lifted to smooth formal schemes over $\mathscr{S}_0$, which we denote by $\mathscr{U}$ and $\mathscr{V}$, respectively, and that f can be lifted to $\tilde{f} : \mathscr{V} \to \mathscr{U}$. Note that $\tilde{f}$ is a finite étale morphism. All we have to show is that $\mathscr{H}^i(\tilde{f}_+\mathcal{O}_{\mathscr{V},\mathbb{Q}}) = 0$ for $i \neq 0$. This is easy by the definition of the functor $\tilde{f}_+$ (or use [Ber02, 4.3.6.3]). The second part of the claim follows from Lemma 1.3.4.

Let us denote by $f_{(U,X)*}\mathcal{O}_{V/K_0}$ this F-isocrystal on U that is overconvergent along $X\backslash U$. We denote by $f_{(U,X)*}\mathcal{O}_{V/K_0}|_U$ the convergent F-isocrystal on U obtained by forgetting the overconvergent structure on $f_{(U,X)*}\mathcal{O}_{V/K_0}$. We have an action of H on $f_{(U,X)*}\mathcal{O}_{V/K_0}|_U$ induced

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by the action of H. By Theorem 1.4.2, we can extend the action of H on $f_{(U,X)*}\mathcal{O}_{V/K_0}|_U$ uniquely to $f_{(U,X)*}\mathcal{O}_{V/K_0}$. Now we define

$$G^\dagger(\rho) := f_{(U,X)*}\mathcal{O}_{V/K_0} \otimes_{K_0[H]} (A \otimes_{\mathbb{Q}_p} K_0). \tag{1.4.3.1}$$

This is a convergent F-isocrystal on U that is overconvergent along $X\backslash U$ over K or, in other words, contained in the image of $\mathrm{sp}_{U,X,+}$ of Lemma 1.3.3.

The isocrystal on U obtained by forgetting the overconvergent structure of $G^\dagger(\rho)$ coincides with $G(\rho)$. Indeed, we may suppose that U is affine. In this case, U, V and f can be lifted, and we can verify what we want by using the explicit description of the construction of $G(\rho)$ by Crew in [Cre87, 2.1]. We can attach a homomorphism of overconvergent isocrystals to a homomorphism of representations by using Theorem 1.4.2 again. The uniqueness of the functor $G^\dagger$ follows from the faithfulness of G. □

1.4.4 Let X be a projective smooth scheme with d-quadruples and let U be an open subscheme of X whose complement $Z := X\backslash U$ is a divisor. Combining Proposition 1.4.3 with the previous result, we have a functor

$$\mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(U)) \xrightarrow{G^\dagger} F\text{-}\mathrm{Isoc}^\dagger(U, X/K)^0 \xrightarrow{\mathrm{sp}_+} \mathfrak{M}_{(U,X)/\mathscr{S}}$$

that attaches an overholonomic $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-module to a representation of $\pi_1(U)$.

1.4.5 Now, let us consider the compatibility of induced representations with $\mathscr{D}^\dagger$ push-forwards. Let $f : V \to U$ be a finite étale morphism of schemes over k. Then we have the injective homomorphism $\pi_1(V) \to \pi_1(U)$. Let ρ be an object of $\mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(V))$. The homomorphism yields the induced $\pi_1(U)$-representation $\mathrm{Ind}_f(\rho)$, and defines a functor

$$\mathrm{Ind}_f : \mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(V)) \to \mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(U)).$$

The following lemma relates Ind_f and f_+.

Lemma 1.4.6. *Consider the cartesian diagram*

$$\begin{array}{ccc} V & \longrightarrow & Y \\ {\scriptstyle f_U}\downarrow & \square & \downarrow{\scriptstyle f} \\ U & \longrightarrow & X \end{array}$$

where all the schemes are smooth, X and Y are projective over k with d-quadruples, f is a proper morphism, f_U is a finite étale morphism, and $D := X\backslash U$, $E := Y\backslash V$ are divisors. Then the diagram of categories

$$\begin{array}{ccccc} \mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(V)) & \xrightarrow{G^\dagger} & F\text{-}\mathrm{Isoc}^\dagger(V, Y/K)^0 & \xrightarrow{\mathrm{sp}_+} & \mathfrak{M}^+_{(V,Y)/\mathscr{S}} \\ {\scriptstyle \mathrm{Ind}_f}\downarrow & & & & \downarrow{\scriptstyle f_+} \\ \mathrm{Rep}^{\mathrm{fin}}_\Lambda(\pi_1(U)) & \xrightarrow{G^\dagger} & F\text{-}\mathrm{Isoc}^\dagger(U, X/K)^0 & \xrightarrow{\mathrm{sp}_+} & \mathfrak{M}^+_{(U,X)/\mathscr{S}} \end{array}$$

is commutative up to canonical isomorphism.

Proof. By assumption, we may find a morphism of proper quadruples

$$(V, Y, W, \mathscr{Q}) \to (U, X, Z, \mathscr{P})$$

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which induces f on the special fiber. We denote this morphism also by f. Consider the convergent isocrystal case, i.e. we shall show that the following diagram is commutative.

$$\begin{array}{ccccc} \mathrm{Rep}^{\mathrm{fin}}_{\Lambda}(\pi_1(V)) & \xrightarrow{G} & F\text{-Isoc}(V/K)^0 & \xrightarrow{\mathrm{sp}_+} & (\mathrm{coh}\ \mathscr{D}^{\dagger}_{\mathscr{Q}\backslash W,\mathbb{Q}}\text{-mod}) \\ {\scriptstyle \mathrm{Ind}_f}\downarrow & & & & \downarrow{\scriptstyle f_+} \\ \mathrm{Rep}^{\mathrm{fin}}_{\Lambda}(\pi_1(U)) & \xrightarrow{G} & F\text{-Isoc}(U/K)^0 & \xrightarrow{\mathrm{sp}_+} & (\mathrm{coh}\ \mathscr{D}^{\dagger}_{\mathscr{P}\backslash Z,\mathbb{Q}}\text{-mod}) \end{array}$$

Now, by [Tsu02, Theorem 4.1.1], checking commutativity is local, and it suffices to treat the case where U can be lifted to a smooth formal scheme $\mathscr{U}$ over $\mathscr{S}$. Since V is étale over U, V can also be lifted to a smooth formal scheme $\mathscr{V}$. In this case, we are reduced to showing that the following square is commutative, by using Theorem 1.1.6.

$$\begin{array}{ccccc} \mathrm{Rep}^{\mathrm{fin}}_{\Lambda}(\pi_1(V)) & \xrightarrow{G} & F\text{-Isoc}(V,\mathscr{V}/K)^0 & \xrightarrow{\mathrm{sp}_*} & (\mathrm{coh}\ \mathscr{D}^{\dagger}_{\mathscr{V},\mathbb{Q}}\text{-mod}) \\ {\scriptstyle \mathrm{Ind}_f}\downarrow & & {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_+} \\ \mathrm{Rep}^{\mathrm{fin}}_{\Lambda}(\pi_1(U)) & \xrightarrow{G} & F\text{-Isoc}(U,\mathscr{U}/K)^0 & \xrightarrow{\mathrm{sp}_*} & (\mathrm{coh}\ \mathscr{D}^{\dagger}_{\mathscr{U},\mathbb{Q}}\text{-mod}) \end{array}$$

where sp denotes the usual specialization map. It is easy to see the commutativity of the square on the right. For the commutativity of the square on the left, take

$$\rho : \pi_1(V) \to \mathrm{GL}_{\Lambda}(A(\rho)) \ \in \mathrm{Rep}^{\mathrm{fin}}_{\Lambda}(\pi_1(V))$$

and let $\mathrm{Ind}_f(\rho) : \pi_1(U) \to \mathrm{GL}_{\Lambda}(A(\mathrm{Ind}_f(\rho))$. Let W be the finite étale scheme over V corresponding to $\mathrm{Im}(\rho)$, and let $g : W \to V$. Now we get

$$\begin{aligned} f_*(G(\rho)) &= f_*(g_*\mathcal{O}_{W/K_0} \otimes_{K_0[\pi_1(V)]} (A(\rho) \otimes_{\mathbb{Q}_p} K_0)) \\ &= (f \circ g)_*\mathcal{O}_{W/K_0} \otimes_{K_0[\pi_1(U)]} K_0[\pi_1(U)] \otimes_{K_0[\pi_1(V)]} (A(\rho) \otimes_{\mathbb{Q}_p} K_0) \\ &= (f \circ g)_*\mathcal{O}_{W/K_0} \otimes_{K_0[\pi_1(U)]} (A(\mathrm{Ind}_f(\rho)) \otimes_{\mathbb{Q}_p} K_0) \\ &= G(\mathrm{Ind}_f(\rho)), \end{aligned}$$

and the commutativity of the square on the left follows. By using Theorem 1.4.2, we conclude the proof. □

2. Characteristic cycles and the Kashiwara–Dubson formula

The aim of this section is to present the theory of characteristic cycles. Let X be a smooth algebraic variety over $\mathbb{C}$, and let $\mathscr{E}$ be a coherent $\mathscr{D}_X$-module. It is well-known that we can attach a cycle in T^*X, denoted by $\mathrm{ZCar}(\mathscr{E})$, called the characteristic cycle. When $\mathscr{E}$ is a non-zero holonomic module, $\mathrm{ZCar}(\mathscr{E})$ is a cycle purely of dimension $\dim X$, and the support coincides with the characteristic variety of $\mathscr{E}$. Now let $f : X \to Y$ be the proper morphism between smooth schemes. Then we can write the relation between $\mathrm{ZCar}(\mathscr{E})$ and $\mathrm{ZCar}(f_+\mathscr{E})$ in terms of intersection theory. This is the so-called Riemann–Roch theorem for $\mathscr{D}$-modules. Using this, we are able to prove the index theorem of Kashiwara and Dubson. In this section, we prove an analogous result for arithmetic $\mathscr{D}$-modules.

In § 2.1, we define characteristic homomorphisms and characteristic cycles for $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules. In § 2.2, we prove the Riemann–Roch theorem for formal schemes (Theorem 2.2.6). As an application, we give a complete proof of the Kashiwara–Dubson–Berthelot formula

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(Corollary 2.2.9), which was announced in [Ber02, 5.4.4]. In § 2.3, we start by defining characteristic homomorphisms and characteristic cycles for overcoherent $\mathscr{D}^{\dagger}_{X,\mathbb{Q}}$-modules, and then prove the Riemann–Roch theorem for overcoherent modules on schemes, which is one of the main theorems in this paper. In the final subsection, § 2.4, we define Swan conductors for overholonomic $\mathscr{D}^{\dagger}_{X,\mathbb{Q}}$-modules and calculate them in the geometric case using the Riemann–Roch theorem for overcoherent modules.

2.1 Characteristic homomorphisms and characteristic cycles

In this section, we define characteristic homomorphisms and characteristic cycles for arithmetic $\mathscr{D}$-modules. The basic ideas of the construction of characteristic cycles can be found in [Ber02, § 5]. Although, to simplify the description, only the unramified case (i.e. $K = W(k)$) was treated there, the ramified case can be dealt with similarly.

Characteristic homomorphisms

Definition 2.1.1.

(i) Let $\mathscr{A}$ be a triangulated category and let $K(\mathscr{A})$ denote the Grothendieck group of $\mathscr{A}$. Let X be a smooth scheme over k and $\mathscr{X}$ a smooth formal scheme over $\mathscr{S}$. We define

$$K(\mathcal{O}_{T^*X}) := K(\text{coh } \mathcal{O}_{T^*X}\text{-mod}), \qquad K(\mathscr{D}^{(m)}_X) := K(\text{coh } \mathscr{D}^{(m)}_X\text{-mod}),$$
$$K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}) := K(\text{coh } \widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}\text{-mod}), \qquad K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}) := K(\text{coh } \widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}\text{-mod}),$$

where (coh $\mathscr{A}$-mod) for a sheaf of rings $\mathscr{A}$ denotes the category of coherent $\mathscr{A}$-modules.

(ii) Let X be a smooth scheme over k. Let $m \geqslant 0$ be an integer. We have the canonical filtration of sub-$\mathcal{O}_X$-modules $\{\mathscr{D}^{(m)}_{X,i}\}_{i\in\mathbb{Z}}$ on $\mathscr{D}^{(m)}_X$ by orders of differential operators. We write $T^{(m)*}X := \mathbf{Spec}(\text{gr}(\mathscr{D}^{(m)}_X))$. Note that in [Ber02, 5.2.1], the same notation $T^{(m)*}X$ is used for the reduced scheme $\mathbf{Spec}(\text{gr}(\mathscr{D}^{(m)}_X))_{\text{red}}$. There is a canonical isomorphism $T^*X \cong T^{(0)*}X$, so we identify the two.

Remark 2.1.2. Let $\mathscr{A}$ be one of the four sheaves of rings $\mathcal{O}_{T^*X}$, $\mathscr{D}^{(m)}_X$, $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$ and $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$. Then, since the cohomological dimensions of $\mathscr{A}$ are finite by [Ber02, 2.1.6, 3.1.1 and 4.1.5], $K(\mathscr{A})$ coincides with the Grothendieck group of perfect complexes of $\mathscr{A}$-modules $K(D_{\text{parf}}(\mathscr{A}))$ or $K(D^b_{\text{coh}}(\mathscr{A}))$.

As in [Lau83, 6.1], we will construct the homomorphism

$$\text{Car}^{(m)} : K(\mathscr{D}^{(m)}_X) \to K(\mathcal{O}_{T^{(m)*}X}),$$

called the characteristic homomorphism, as follows.

Lemma 2.1.3. *Let $\mathscr{E}$ be a coherent $\mathscr{D}^{(m)}_X$-module, and suppose we are given two good filtrations F_n and G_n (see [Ber02, 5.2.3]). Then $\text{gr}^F(\mathscr{E})$ and $\text{gr}^G(\mathscr{E})$ define the same element in $K(\mathcal{O}_{T^{(m)*}X})$.*

Proof. The proof is exactly the same as that of [Lau83, Lemme 6.1.2]. □

Let $\mathscr{E}$ be a coherent $\mathscr{D}^{(m)}_X$-module. By [Ber02, 5.2.3(iv)], we can take a good filtration $F_n\mathscr{E}$ on $\mathscr{E}$. By Lemma 2.1.3 above, the class of $\text{gr}^F\mathscr{E}$ in $K(\mathcal{O}_{T^{(m)*}X})$ does not depend on the choice

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of good filtration. This defines a functor

$$\mathrm{Car}'^{(m)} : \mathrm{Ob}(\mathrm{coh}\ \mathscr{D}^{(m)}_X\text{-mod}) \to K(\mathcal{O}_{T^{(m)*}X}).$$

By the next lemma, the functor $\mathrm{Car}'^{(m)}$ induces $\mathrm{Car}^{(m)}$.

Lemma 2.1.4. *Given a short exact sequence $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ of coherent $\mathscr{D}^{(m)}_X$-modules, we have $\mathrm{Car}'^{(m)}(\mathscr{E}) = \mathrm{Car}'^{(m)}(\mathscr{E}') + \mathrm{Car}'^{(m)}(\mathscr{E}'')$.*

Proof. This follows directly from [Ber02, 5.2.3(iii)]. □

2.1.5 Now, we will clarify the relation between $\mathrm{Car}^{(m)}$ and $\mathrm{Car}^{(m+s)}$. Let $F^s : X \to X$ be the sth absolute Frobenius morphism (i.e. $F^s := \underbrace{F \circ \cdots \circ F}_{s \text{ times}}$). Berthelot proved the following result.

Theorem 2.1.6 [Ber00, 2.3.6]. *For a $\mathscr{D}^{(m)}_X$-module $\mathscr{E}$, there exists a canonical $\mathscr{D}^{(m+s)}_X$-module structure on $F^{s*}\mathscr{E}$. We denote this $\mathscr{D}^{(m+s)}_X$-module by $F^{s*}_{\mathscr{D}}\mathscr{E}$. Then we have an equivalence of categories*

$$F^{s*}_{\mathscr{D}} : (\mathscr{D}^{(m)}_X\text{-modules}) \xrightarrow{\sim} (\mathscr{D}^{(m+s)}_X\text{-modules}).$$

With an abuse of notation, we sometimes write $F^{s*}_{\mathscr{D}}$ as F^{s*}.

Now, we have a morphism

$$f : T^{(m)*}X \times_{X \swarrow F^s} X \to T^{(m+s)*}X$$

by [Ber02, 5.2.2]. We also define

$$p : T^{(m)*}X \times_{X \swarrow F^s} X \to T^{(m)*}X$$

to be the canonical projection. Then we have the following lemma.

Lemma 2.1.7. *The following diagram is commutative.*

$$\begin{array}{ccc} K(\mathscr{D}^{(m)}_X) & \xrightarrow{F^{s*}} & K(\mathscr{D}^{(m+s)}_X) \\ {\scriptstyle \mathrm{Car}^{(m)}}\downarrow & & \downarrow{\scriptstyle \mathrm{Car}^{(m+s)}} \\ K(\mathcal{O}_{T^{(m)*}X}) & \xrightarrow[\Psi]{} & K(\mathcal{O}_{T^{(m+s)*}X}) \end{array}$$

Here, F^{*s} denotes the homomorphism of Grothendieck groups defined by the functor $F^{*s}_{\mathscr{D}}$, and $\Psi := f_* p^*$.

Proof. To prove the commutativity, it suffices to check commutativity for classes of coherent $\mathscr{D}^{(m)}_X$-modules, since the functor $\mathrm{Car}^{(m)}$ is additive and the class of coherent sheaves generates $K(\mathscr{D}^{(m)}_X)$. Let $\mathscr{L}$ be a coherent $\mathscr{D}^{(m)}_X$-module. Take a good filtration F on $\mathscr{L}$. Then $\Psi\mathrm{Car}^{(m)}(\mathscr{L}) = f_* p^* \mathrm{gr}^F(\mathscr{L})$. Note here that since p is flat and f is affine, we do not need to take derived functors. On the other hand, the $\mathscr{D}^{(m+s)}_X$-module $F^{s*}_{\mathscr{D}}(\mathscr{L})$ is $F^{s*}(\mathscr{L})$ as a $\mathcal{O}_X$-module, and since F^s is flat, $G_i F^{s*}_{\mathscr{D}}\mathscr{L} := F^{s*}(F_i\mathscr{L})$ is a filtration on $F^{s*}\mathscr{L}$ satisfying [Ber02, 5.2.3(a)] and, moreover, is a good filtration of $F^{s*}_{\mathscr{D}}(\mathscr{L})$. Indeed, we can check that the filtration satisfies [Ber02, 5.2.3(b)] by the concrete description [Ber00, 2.2.4]. We have that $\mathrm{gr}^G(F^{s*}\mathscr{L}) = f_* p^* \mathrm{gr}^F(\mathscr{L})$ is a coherent $\mathscr{O}_{T^{*(m+s)}X}$-module, since f is finite, and we have also checked [Ber02, 5.2.3(ii)]. Thus, the commutativity follows. □

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Since the exact functor $F^{s*}_{\mathscr{D}}$ defines an equivalence of categories, we know that $F^{s*}\colon K(\mathscr{D}^{(m)}_X)\to K(\mathscr{D}^{(m+s)}_X)$ is an isomorphism.

Definition 2.1.8. We define $\mathrm{Car}\colon K(\mathscr{D}^{(m)}_X)\to K(\mathcal{O}_{T^*X})$ to be

$$K(\mathscr{D}^{(m)}_X)\xrightarrow[(F^{*m})^{-1}]{\sim}K(\mathscr{D}^{(0)}_X)\xrightarrow[\mathrm{Car}^{(0)}]{}K(\mathcal{O}_{T^*X}).$$

Now we move on to the definition of characteristic homomorphisms for arithmetic $\mathscr{D}$-modules on formal schemes. We use the theory of Frobenius descent, so we fix m such that $p^m>e/(p-1)$.

Definition 2.1.9. The categories of coherent F-$\mathscr{D}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules and coherent F-$\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules are abelian categories. We denote by $K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}})$ and $K(F\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}})$ the Grothendieck groups of coherent F-$\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules and coherent F-$\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules, respectively.

2.1.10 Let $\mathscr{X}$ be a smooth formal scheme over $\mathscr{S}$. Let $\mathscr{E}$ be a coherent F-$\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module. By the theorem of Frobenius descent [Ber00, Théorème 4.5.4], there exists a unique $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{E}^{(m)}$, up to isomorphism, such that $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}\otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}}\mathscr{E}^{(m)}\cong\mathscr{E}$ compatible with Frobenius structures. Let $\mathscr{E}'^{(m)}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module without p^∞-torsion such that $\mathscr{E}^{(m)}=\mathscr{E}'^{(m)}\otimes\mathbb{Q}$. (We can take such a lifting by [Ber96b, 3.4.5].) We define $\mathrm{Car}(\mathscr{E})$ by $\mathrm{Car}(\mathscr{E}'^{(m)}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathcal{O}_X)$ in $K(\mathcal{O}_{T^*X})$. This construction does not depend on the choice of $\mathscr{E}'^{(m)}$, and can be passed to Grothendieck groups by the following lemma.

Lemma 2.1.11. *The above definition does not depend on the choice of* $\mathscr{E}'^{(m)}$ *and yields a homomorphism of groups*

$$\mathrm{Car}^{\dagger}\colon K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}})\longrightarrow K(\mathcal{O}_{T^*X}).$$

Moreover, $\mathrm{Car}^{\dagger}$ *does not depend on the choice of* m *such that* $p^m>e/(p-1)$. *We call this homomorphism the characteristic homomorphism.*

Proof. First, we define a homomorphism $\delta\colon K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}})\to K(\mathscr{D}^{(m)}_X)$. Let $\mathscr{E}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module. We can take a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module $\mathscr{E}'$ such that $\mathscr{E}'\otimes\mathbb{Q}\cong\mathscr{E}$. Then we define $\delta(\mathscr{E})$ to be

$$\delta'(\mathscr{E}'):=[\mathscr{H}_0(\mathscr{E}'\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathscr{X}}}\mathcal{O}_X)]-[\mathscr{H}_1(\mathscr{E}'\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathscr{X}}}\mathcal{O}_X)].$$

We will show that this defines a homomorphism of Grothendieck groups. First, let us show that this does not depend on the choice of liftings. We may suppose that $\mathscr{E}'$ has no torsion. Indeed, let $\mathscr{E}'_p$ the p^∞-torsion part of $\mathscr{E}'$. Then we get the following exact sequence:

$$0\to\mathscr{E}'_p\to\mathscr{E}'\to\mathscr{E}'/\mathscr{E}'_p\to 0.$$

This shows that $\delta'(\mathscr{E}')=\delta'(\mathscr{E}'_p)+\delta'(\mathscr{E}'/\mathscr{E}'_p)$, since $\mathscr{H}_i(\mathscr{F}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}}}\mathcal{O}_X)=0$ for any coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module $\mathscr{F}$ and $i\geqslant 2$. Since $\mathscr{E}'/\mathscr{E}'_p$ is p-torsion free, it suffices to show that $\delta'(\mathscr{E}'_p)=0$. Since $\mathscr{E}'_p$ is also coherent, there exists an integer a such that $\mathscr{E}'_p$ is p^a-torsion. When $a=1$, the claim is clear. To finish the reduction, we just use induction on a.

Since $\mathscr{E}'$ has no torsion, the canonical homomorphism $\mathscr{E}'\hookrightarrow\mathscr{E}$ is an inclusion. Suppose we are given another lifting $\mathscr{E}''\hookrightarrow\mathscr{E}$. Since these are isomorphic after tensoring with $\mathbb{Q}$, we can find homomorphisms $\phi\colon\mathscr{E}'\to\mathscr{E}''$ and $\psi\colon\mathscr{E}''\to\mathscr{E}'$ such that $\psi\circ\phi=p^n$ and $\phi\circ\psi=p^n$. Note here

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that ϕ and ψ are injective since $\mathscr{E}'$ and $\mathscr{E}''$ are torsion-free modules. We have to show that the images of $\mathscr{E}'$ and $\mathscr{E}''$ under δ' are the same. Let us define C by the exact sequence

$$0 \to \mathscr{E}' \xrightarrow{\phi} \mathscr{E}'' \to C \to 0. \tag{2.1.11.1}$$

Since $\phi \circ \psi = p^n$, we know that $p^n C = 0$. Thus, by the above argument, we get that $\delta'(C) = 0$. By the exact sequence, we obtain

$$\delta'(\mathscr{E}') + \delta'(C) = \delta'(\mathscr{E}''),$$

and we see that the image of $\mathscr{E}'$ and $\mathscr{E}''$ in $K(\mathcal{O}_X)$ are the same.

For the definition of δ, we have to show that for an exact sequence of $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0,$$

we have $\delta(\mathscr{F}) = \delta(\mathscr{E}) + \delta(\mathscr{G})$. We can take a homomorphism of $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-modules $\mathscr{E}' \hookrightarrow \mathscr{F}'$ with $\mathscr{E}' \otimes \mathbb{Q} \cong \mathscr{E}$ and $\mathscr{F}' \otimes \mathbb{Q} \cong \mathscr{F}$ which coincides with $\mathscr{E} \hookrightarrow \mathscr{F}$ after tensoring with $\mathbb{Q}$. Then take $\mathscr{G}'$ to be $\mathscr{F}'/\mathscr{E}'$. Since $\otimes\mathbb{Q}$ is an exact functor, we have that $\mathscr{G}' \otimes \mathbb{Q} \cong \mathscr{G}$. Thus the above exact sequence has a lifting $0 \to \mathscr{E}' \to \mathscr{F}' \to \mathscr{G}' \to 0$, and the additivity follows.

Now, we define the homomorphism $\mathrm{Car}^\dagger$ by the diagram

$$\begin{array}{ccccccc} K(F\text{-}\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}) & \xrightarrow{\sim} & K(F\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}) & \longrightarrow & K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}) & \xrightarrow{\delta} & K(\mathscr{D}^{(m)}_X) \\ & \searrow^{\mathrm{Car}^\dagger} & & & & & \downarrow \mathrm{Car} \\ & & & & & & K(\mathcal{O}_{T^*X}) \end{array} \tag{$*$}$$

where the first horizontal isomorphism is induced by the theorem of Frobenius descent and the second by the forgetful functor of Frobenius structure. The independence of m can be seen from the compatibility of Frobenius pull-backs and base changes [Ber00, 2.2.6]. □

Characteristic cycles

2.1.12 Let X be a scheme over k. Let $Z_n(X)$ be the group of cycles of dimension n. For an integer $n \geqslant 0$, let $Z_n K(\mathcal{O}_X)$ be the Grothendieck group of the category consisting of coherent $\mathcal{O}_X$-modules $\mathcal{M}$ such that $\dim \mathrm{Supp}(\mathcal{M}) \leqslant n$. Then we can define the multiplicity homomorphism

$$\mathrm{mult}_n : Z_n K(\mathcal{O}_X) \to Z_n(X)$$

such that for $[\mathcal{M}] \in Z_n K(\mathcal{O}_X)$, we have

$$\mathrm{mult}_n(\mathcal{M}) := \sum_{\dim \xi = n} m_\xi(\mathcal{M}) \overline{\{\xi\}} \in Z_n(X)$$

where ξ ranges over all points of dimension n in X and m_ξ denotes the length of $\mathcal{M}_\xi$ as a $(\mathcal{O}_{X,\xi})_{\mathrm{red}}$-module. (For a ring A, we denote by A_{red} the maximal reduced ring in A.)

Lemma 2.1.13. *We use the same notation as in Lemma 2.1.7. Then the following diagram is commutative.*

$$\begin{array}{ccc} Z_n K(\mathcal{O}_{T^{(m)*}X}) & \xrightarrow{\Psi} & Z_n K(\mathcal{O}_{T^{(m+s)*}X}) \\ \downarrow \mathrm{mult}_n & & \downarrow \mathrm{mult}_n \\ Z_n(T^{(m)*}X) & \xrightarrow[\Phi]{} & Z_n(T^{(m+s)*}X) \end{array}$$

where $\Phi := f_* \circ p^*$.

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Proof. We only note here that for a flat morphism of schemes $g : X \to Y$ of relative dimension r and a coherent $\mathcal{O}_Y$-module $\mathcal{M}$ such that $\dim \mathrm{Supp}(\mathscr{M}) \leqslant n$, we have that $\mathrm{mult}_{(n+r)}(g^*(\mathcal{M})) = g^*\mathrm{mult}_n(\mathcal{M})$. □

2.1.14 Let $(n\text{-}\mathscr{D}^{(m)}_{X,\mathbb{Q}}\text{-mod})$ be the category of coherent $\mathscr{D}^{(m)}_{X,\mathbb{Q}}$-modules with dimension less than n. We denote by $K(n\text{-}\mathscr{D}^{(m)}_X)$ the Grothendieck group of $(n\text{-}\mathscr{D}^{(m)}_{X,\mathbb{Q}}\text{-mod})$. We will define a homomorphism

$$Z_n\mathrm{Car}^{(m)} : K(n\text{-}\mathscr{D}^{(m)}_X) \to Z_nK(\mathcal{O}_{T^{(m)*}X})$$

as in the construction of Car. The only thing we need to verify is the following lemma.

Lemma 2.1.15. *Let $\mathscr{E}$ be a coherent $\mathscr{D}^{(m)}_X$-module of dimension less than or equal to n, and suppose we are given two good filtrations F_r and G_r. Then $\mathrm{gr}^F(\mathscr{E})$ and $\mathrm{gr}^G(\mathscr{E})$ define the same element in $Z_nK(\mathcal{O}_{T^{(m)*}X})$.*

Proof. The proof is the same as that of Lemma 2.1.3. □

We also define

$$Z_n\mathrm{Car} : K(n\text{-}\mathscr{D}^{(m)}_X) \to Z_nK(\mathcal{O}_{T^*X})$$

in the same way.

2.1.16 Let $\mathscr{X}$ be a smooth formal scheme, and let X be its special fiber. Let $d := \dim X$. We define a homomorphism in the following way:

$$\delta_d : K(d\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}) \to K(d\text{-}\mathscr{D}^{(m)}_X),$$

where $K(d\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}})$ denotes the Grothendieck group of the category of $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules of dimension less than or equal to d. Let $\mathscr{E}$ be a $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module of dimension less than or equal to d. We may take a $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module $\mathscr{E}'$ such that $\mathscr{E}'$ is p-torsion free and $\mathscr{E}' \otimes \mathbb{Q} = \mathscr{E}$. Then we define $\delta_d([\mathscr{E}]) := [\mathscr{E}' \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_X]$. In the same way as in Lemma 2.1.11, we can prove that the definition does not depend on the choice of liftings. Here we note only that $C \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_X$, where C denotes the module in (2.1.11.1), is a $\mathscr{D}^{(m)}_X$-module of dimension less than or equal to d, since $\mathscr{E}' \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_X$ is of dimension less than or equal to d.

Let $K(\mathrm{hol}\ \mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$ be the Grothendieck group of the category of holonomic $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules. By [Ber02, 5.3.4], $K(\mathrm{hol}\ \mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}) = K(d\text{-}\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$. We define

$$Z_d\mathrm{Car}^\dagger : K(\mathrm{hol}\ \mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}) \to Z_dK(\mathcal{O}_{T^*X})$$

in the same way as in Lemma 2.1.11 but using δ_d instead of δ. The well-definedness can be seen from Lemma 2.1.13. Now we are able to define the characteristic cycles as follows.

Definition 2.1.17. Let $n \geqslant 0$ be an integer and $d := \dim X$. We define

$$\mathrm{ZCar}^{(m)}_n : K(n\text{-}\mathscr{D}^{(m)}_X) \xrightarrow{Z_n\mathrm{Car}^{(m)}} Z_nK(\mathcal{O}_{T^{(m)*}X}) \xrightarrow{\mathrm{mult}_n} Z_n(T^{(m)*}X),$$

$$\mathrm{ZCar}_n : K(n\text{-}\mathscr{D}^{(m)}_X) \xrightarrow{Z_n\mathrm{Car}} Z_nK(\mathcal{O}_{T^*X}) \xrightarrow{\mathrm{mult}_n} Z_n(T^*X),$$

$$\mathrm{ZCar}^\dagger : K(\mathrm{hol}\ \mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}) \xrightarrow{Z_d\mathrm{Car}^\dagger} Z_dK(\mathcal{O}_{T^*X}) \xrightarrow{\mathrm{mult}_d} Z_d(T^*X).$$

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Combining Lemmas 2.1.7 and 2.1.13 gives the following lemma.

Lemma 2.1.18. *With notation as in Lemmas 2.1.7 and 2.1.13, we have the following commutative diagram.*

$$\begin{array}{ccc} K(n\text{-}\mathscr{D}_X^{(m)}) & \xrightarrow{F^{s*}} & K(n\text{-}\mathscr{D}_X^{(m+s)}) \\ {\scriptstyle \mathrm{ZCar}_n^{(m)}}\downarrow & & \downarrow{\scriptstyle \mathrm{ZCar}_n^{(m+s)}} \\ Z_n(T^{(m)*}X) & \xrightarrow[\Phi]{} & Z_n(T^{(m+s)*}X) \end{array}$$

Note that since Φ is injective, $\mathrm{ZCar}_n^{(m)}$ is determined by $\mathrm{ZCar}_n^{(m+s)}$.

2.1.19 Before concluding this subsection, let us explain the relation between characteristic homomorphisms and characteristic cycles. Let S be a scheme over k. We denote by $\mathrm{CH}_*(S)$ the Chow group (or cycle class group) of S and write $\mathrm{CH}_*(S)_{\mathbb{Q}} := \mathrm{CH}_*(S) \otimes \mathbb{Q}$. Let $\tau_S : K(\mathcal{O}_S) \to \mathrm{CH}_*(S)_{\mathbb{Q}}$ be the Riemann–Roch homomorphism. For details see [Ful98, ch. 15 and 18]. The homomorphisms $\mathrm{Car}^{(\dagger)}$ and $\mathrm{ZCar}^{(\dagger)}$ are related to each other by the following lemma.

Lemma 2.1.20. *Let $\mathscr{E}$ be a coherent $\mathscr{D}_X^{(m)}$-module of dimension less than or equal to $\dim(X)$ (respectively, a holonomic $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module). Then $\tau_{T^*X} \circ \mathrm{Car}(\mathscr{E}) = \mathrm{ZCar}_d(\mathscr{E})$ (respectively, $\tau_{T^*X} \circ \mathrm{Car}^{\dagger}(\mathscr{E}) = \mathrm{ZCar}^{\dagger}(\mathscr{E})$) in $\mathrm{CH}_*(T^*X)_{\mathbb{Q}}$.*

Proof. By definition, we only have to deal with $\mathscr{D}_X^{(0)}$-modules. In this case the proof is exactly the same as that of [Lau83, Lemme 6.6.1]. Note that since we are also dealing with proper schemes which may not be quasi-projective, we use the corresponding properties in [Ful98, Theorem 18.3]. □

2.2 Relative Kashiwara–Dubson formula for $\mathscr{D}_X^{(m)}$-modules and $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules

The scheme case

2.2.1 In this section, we prove the Riemann–Roch theorem for arithmetic $\mathscr{D}$-modules for schemes. Let X be a smooth scheme and let $\mathscr{E}$ be a $\mathscr{D}_X^{(0)}$-complex. We can attach $\sum(-1)^i[\mathscr{H}^i(\mathscr{E})] \in K(\mathscr{D}_X^{(0)})$ to $\mathscr{E}$. For a proper morphism of smooth schemes $f : X \to Y$, this induces a homomorphism $f_+ : K(\mathscr{D}_X^{(0)}) \to K(\mathscr{D}_Y^{(0)})$.

Theorem 2.2.2 (Laumon). *Let $f : X \to Y$ be a proper morphism between smooth schemes over k. Consider the following diagram.*

$$\begin{array}{ccccc} X & \longleftarrow & T^*X & & \\ & & & \nwarrow{\scriptstyle g} & \\ {\scriptstyle f}\downarrow & & & & T^*Y \times_Y X \\ & & & \swarrow{\scriptstyle \bar{f}} & \\ Y & \longleftarrow & T^*Y & & \end{array} \qquad (\star)$$

$$\pi_f : T^*Y \times_Y X \to X$$

where g is induced by the canonical homomorphism of sheaves $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$. Let $\mathscr{T}_f$ be the virtual tangent bundle (see, e.g., [Ful98, Appendix B.7.6]), let Todd *denote the Todd class (see, e.g., [Ful98, Example 3.2.4]), and let τ_X be the Riemann–Roch homomorphism*

$K(\mathcal{O}_X) \to \mathrm{CH}_*(X)_{\mathbb{Q}}$. *Then the following diagram is commutative.*

$$\begin{array}{ccc}
K(\mathscr{D}_X^{(0)}) & \xrightarrow{\ f_+\ } & K(\mathscr{D}_Y^{(0)}) \\
{\scriptstyle \tau_{T^*X}\circ\mathrm{Car}}\downarrow & & \downarrow{\scriptstyle \tau_{T^*Y}\circ\mathrm{Car}} \\
\mathrm{CH}_*(T^*X)_{\mathbb{Q}} & \xrightarrow[\overline{f}_*(\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\text{-}))]{} & \mathrm{CH}_*(T^*Y)_{\mathbb{Q}}
\end{array}$$

Proof. Since the proof is exactly the same as that of [Lau83, Corollaire 6.3.3], as pointed out in [Ber02, 5.4.4], here we only prove that the following diagram is commutative.

$$\begin{array}{ccc}
K(\mathscr{D}_X^{(0)}) & \xrightarrow{\ f_+\ } & K(\mathscr{D}_Y^{(0)}) \\
{\scriptstyle \mathrm{Car}}\downarrow & & \downarrow{\scriptstyle \mathrm{Car}} \\
K(\mathcal{O}_{T^*X}) & \xrightarrow{(-1)^{d_f}\overline{f}_*\circ g^!} & K(\mathcal{O}_{T^*Y}) \\
{\scriptstyle \tau_{T^*X}}\downarrow & & \downarrow{\scriptstyle \tau_{T^*Y}} \\
\mathrm{CH}_*(T^*X)_{\mathbb{Q}} & \xrightarrow[\overline{f}_*(\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\text{-}))]{} & \mathrm{CH}_*(T^*Y)_{\mathbb{Q}}
\end{array}$$

The commutativity of the upper square follows from arguments involving filtered modules, and the commutativity of the lower square follows from the classic Riemann–Roch theorem [Ful98, Theorem 18.3]. □

2.2.3 For arithmetic $\mathscr{D}$-modules of level m, the following diagram is also commutative.

$$\begin{array}{ccc}
K(\mathscr{D}_X^{(m)}) & \xrightarrow{\ f_+\ } & K(\mathscr{D}_Y^{(m)}) \\
{\scriptstyle (F^{*m})^{-1}}\downarrow & & \downarrow{\scriptstyle (F^{*m})^{-1}} \\
K(\mathscr{D}_X^{(0)}) & \xrightarrow{\ f_+\ } & K(\mathscr{D}_Y^{(0)}) \\
{\scriptstyle \tau_{T^*Y}\circ\mathrm{Car}}\downarrow & & \downarrow{\scriptstyle \tau_{T^*Y}\circ\mathrm{Car}} \\
\mathrm{CH}_*(T^*X)_{\mathbb{Q}} & \xrightarrow[\overline{f}_*(\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\text{-}))]{} & \mathrm{CH}_*(T^*Y)_{\mathbb{Q}}
\end{array}$$

The commutativity of the upper square can be seen from the fact that f_+ and F^* are commutative, by [Ber00, Théorème 3.4.4]. Thus, by the definition of Car, we can use arithmetic $\mathscr{D}$-modules of level $m \geqslant 0$ instead of level zero in Theorem 2.2.2 above.

Remark 2.2.4. When f is a closed immersion, we have a more precise result than Theorem 2.2.2. Let $n \geqslant 0$ be an integer and let $d_f := \dim X - \dim Y$. In this case, note that g is flat. Thus we get a homomorphism $g^* : Z_n(T^*X) \to Z_{n-d_f}(T^*Y \times_Y X)$. Now, the following diagram is commutative.

$$\begin{array}{ccc}
K(n\text{-}\mathscr{D}_X^{(m)}) & \xrightarrow{\ f_+\ } & K((n-d_f)\text{-}\mathscr{D}_Y^{(m)}) \\
{\scriptstyle \mathrm{ZCar}_n}\downarrow & & \downarrow{\scriptstyle \mathrm{ZCar}_{(n-d_f)}} \\
Z_n(T^*X) & \xrightarrow[\ \overline{f}_*g^*\ ]{} & Z_{n-d_f}(T^*Y)
\end{array}$$

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Proof. To show this, we need only deal with the case where $m=0$. First of all, we will show that the following diagram is commutative.

$$\begin{array}{ccccc}
Z_nK(\mathcal{O}_{T^*X}) & \xrightarrow{(-1)^{d_f}g^!} & Z_{n-d_f}K(\mathcal{O}_{T^*Y\times_Y X}) & \xrightarrow{\overline{f}_*} & Z_{n-d_f}K(\mathcal{O}_{T^*Y}) \\
\downarrow{\scriptstyle \mathrm{mult}_n} & & \downarrow{\scriptstyle \mathrm{mult}_{(n-d_f)}} & & \downarrow{\scriptstyle \mathrm{mult}_{(n-d_f)}} \\
Z_n(T^*X) & \xrightarrow[g^*]{} & Z_{n-d_f}(T^*Y\times_Y X) & \xrightarrow[\overline{f}_*]{} & Z_{n-d_f}(T^*Y)
\end{array}$$

Since $\overline{f}$ is a closed immersion, the commutativity of the square on the right is easily seen. To show commutativity of the square on the left, we need only check commutativity on generators of $Z_n(K(\mathcal{O}_{T^*X}))$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_{T^*X}$-module with $\dim \mathrm{Supp}(\mathcal{F})\leqslant n$. Since g is flat, we have $(-1)^{d_f}g^!(\mathcal{F})=g^*(\mathcal{F})\otimes_{\mathcal{O}_{T^*Y\times_Y X}}\omega$, where $\omega:=\omega_{T^*Y\times_Y X}\otimes_{\mathcal{O}_{T^*Y\times_Y X}}\omega^{-1}_{T^*X}$. Since ω is an invertible sheaf and mult_{n-d_f} can be calculated locally, we have

$$\begin{aligned}\mathrm{mult}_{n-d_f}((-1)^{d_f}g^!(\mathcal{F})) &= \mathrm{mult}_{n-d_f}(g^*(\mathcal{F})\otimes_{\mathcal{O}_{T^*Y\times_Y X}}\omega)\\ &= \mathrm{mult}_{n-d_f}(g^*(\mathcal{F}))=g^*\mathrm{mult}_n(\mathcal{F}),\end{aligned}$$

and the claim follows.

Now, from the above commutative diagram, we see that the image of $Z_nK(\mathcal{O}_{T^*X})\subset K(\mathcal{O}_{T^*X})$ under the homomorphism $(-1)^{d_f}\overline{f}_*\circ g^!:K(\mathcal{O}_{T^*X})\to K(\mathcal{O}_{T^*Y})$ is contained in $Z_{n-d_f}K(\mathcal{O}_{T^*Y})$. Thus, by the proof of Theorem 2.2.2, we get the following commutative diagram.

$$\begin{array}{ccc}
K(n\text{-}\mathscr{D}^{(0)}_X) & \xrightarrow{f_+} & K((n-d_f)\text{-}\mathscr{D}^{(0)}_Y) \\
\downarrow{\scriptstyle \mathrm{Car}} & & \downarrow{\scriptstyle \mathrm{Car}} \\
Z_nK(\mathcal{O}_{T^*X}) & \xrightarrow[(-1)^{d_f}\overline{f}_*\circ g^!]{} & Z_{n-d_f}K(\mathcal{O}_{T^*Y})
\end{array}$$

The remark follows upon combining the above two commutative diagrams. □

The formal scheme case

In the rest of this subsection, we fix m so that $p^m>e/(p-1)$.

2.2.5 Let $\mathscr{X}$ be a smooth formal scheme. The Riemann–Roch theorem is also valid in the context of $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules. Let $\mathscr{E}$ be an F-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-complex. We can attach

$$\sum(-1)^i[\mathscr{H}^i(\mathscr{E})]\in K(F\text{-}\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$$

to $\mathscr{E}$. For a proper morphism of smooth formal schemes $f:\mathscr{X}\to\mathscr{Y}$, this induces a homomorphism

$$f_+:K(F\text{-}\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})\to K(F\text{-}\mathscr{D}^\dagger_{\mathscr{Y},\mathbb{Q}}).$$

Theorem 2.2.6 (Riemann–Roch theorem). *Let $f:\mathscr{X}\to\mathscr{Y}$ be a proper morphism between smooth formal schemes over $\mathscr{S}$, and let X and Y be their special fibers. We use the notation of*

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($\star$) *in Theorem 2.2.2. Then*

$$\begin{array}{ccc} K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}) & \xrightarrow{\quad f_+ \quad} & K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}) \\ \downarrow{\scriptstyle \tau_{T^*X}\circ \mathrm{Car}^{\dagger}} & & \downarrow{\scriptstyle \tau_{T^*Y}\circ \mathrm{Car}^{\dagger}} \\ \mathrm{CH}_*(T^*X)_{\mathbb{Q}} & \xrightarrow[\overline{f}_*(\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\text{-}))]{} & \mathrm{CH}_*(T^*Y)_{\mathbb{Q}} \end{array}$$

is commutative.

Proof. We define a homomorphism Red to be the composition of homomorphisms in the proof of Lemma 2.1.11:

$$\mathrm{Red}: K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}) \xrightarrow{\sim} K(F\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}) \to K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}) \to K(\mathscr{D}^{(m)}_{X}).$$

We will show that the following diagram is commutative.

$$\begin{array}{ccc} K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}) & \xrightarrow{\mathrm{Red}} & K(\mathscr{D}^{(m)}_{X}) \\ \downarrow{\scriptstyle f_+} & & \downarrow{\scriptstyle f_+} \\ K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}) & \xrightarrow[\mathrm{Red}]{} & K(\mathscr{D}^{(m)}_{Y}) \end{array}$$

To do this, consider the following diagram of functors.

$$\begin{array}{ccccccc} D^b_{\mathrm{coh}}(\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}) & \xleftarrow{\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}\otimes} & D^b_{\mathrm{coh}}(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}) & \xleftarrow{\otimes\mathbb{Q}} & D^b_{\mathrm{coh}}(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}) & \xrightarrow{\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathscr{X}}}\mathcal{O}_X} & D^b_{\mathrm{coh}}(\mathscr{D}^{(m)}_{X}) \\ \downarrow{\scriptstyle f_+} & ① & \downarrow{\scriptstyle f_+^{(m)}} & ② & \downarrow{\scriptstyle f_+^{(m)}} & ③ & \downarrow{\scriptstyle f_+^{(m)}} \\ D^b_{\mathrm{coh}}(\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}) & \xleftarrow[\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}\otimes]{} & D^b_{\mathrm{coh}}(\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y},\mathbb{Q}}) & \xleftarrow[\otimes\mathbb{Q}]{} & D^b_{\mathrm{coh}}(\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y}}) & \xrightarrow[\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathscr{X}}}\mathcal{O}_X]{} & D^b_{\mathrm{coh}}(\mathscr{D}^{(m)}_{Y}) \end{array}$$

① This diagram is commutative by [Ber02, 4.3.8]. Moreover, the functors commute with Frobenius morphism by [Ber02, 4.3.9].

② Since $\otimes\mathbb{Q}$ is an exact functor, one can define $D^b_{\mathrm{coh}}(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}) \to D^b_{\mathrm{coh}}(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}})$ without taking left derivations. This diagram is commutative. Indeed, we have

$$\begin{aligned} f_+^{(m)}(\mathscr{E}\otimes\mathbb{Q}) &= \mathbb{R}f_*(\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y}\leftarrow\mathscr{X},\mathbb{Q}} \overset{\mathbb{L}}{\otimes}_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} (\mathscr{E}\otimes_{\mathbb{Z}}\mathbb{Q})) = \mathbb{R}f_*((\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y}\leftarrow\mathscr{X}} \overset{\mathbb{L}}{\otimes}_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}} \mathscr{E})\otimes_{\mathbb{Z}}\mathbb{Q}) \\ &= (\mathbb{R}f_*(\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y}\leftarrow\mathscr{X}} \overset{\mathbb{L}}{\otimes}_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}} \mathscr{E}))\otimes_{\mathbb{Z}}\mathbb{Q} = f_+^{(m)}(\mathscr{E})\otimes_{\mathbb{Z}}\mathbb{Q}. \end{aligned}$$

The third equality holds because $\otimes_{\mathbb{Z}}\mathbb{Q}$ preserves flasque sheaves (see [Har77]).

③ This diagram is commutative by [Ber02, 2.4.2].

From the above we can deduce that the diagram of K-groups is commutative. Now, using the Riemann–Roch theorem for schemes, i.e. Theorem 2.2.2, we obtain

$$\begin{aligned} (\tau_{T^*Y}\circ\mathrm{Car}^{\dagger})(f_+(\mathscr{E})) &= (\tau_{T^*Y}\circ\mathrm{Car})(\mathrm{Red}(f_+\mathscr{E})) = (\tau_{T^*Y}\circ\mathrm{Car})(f_+\mathrm{Red}(\mathscr{E})) \\ &= \Phi(\tau_{T^*X}\circ\mathrm{Car})(\mathrm{Red}(\mathscr{E})) = \Phi(\tau_{T^*X}\circ\mathrm{Car}^{\dagger})(\mathscr{E}), \end{aligned}$$

where we have set $\Phi := \overline{f}_*(\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\text{-}))$. □

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Corollary 2.2.7 (Relative Kashiwara–Dubson formula). *Let $f:\mathscr{X}\to\mathscr{Y}$ be a proper morphism between smooth formal schemes over $\mathscr{S}$, let X and Y be their special fibers, and let $\sigma_X : X \to T^*X$ and $\sigma_Y : Y \to T^*Y$ be the zero-sections. Let $\mathscr{E}$ be a holonomic $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module. Then*

$$f_*(\sigma_X^*(\mathrm{ZCar}^\dagger(\mathscr{E}))) = \sigma_Y^*(\tau_{T^*Y}\circ \mathrm{Car}^\dagger(f_+\mathscr{E})).$$

In particular, if $f_+(\mathscr{E})$ is, moreover, holonomic, then

$$f_*(\sigma_X^*(\mathrm{ZCar}^\dagger(\mathscr{E}))) = \sigma_Y^*(\mathrm{ZCar}^\dagger(f_+\mathscr{E})).$$

Proof. By the Riemann–Roch theorem for formal schemes, i.e. Theorem 2.2.6, and Lemma 2.1.20, we have

$$\sigma_Y^*(\tau_{T^*Y}\circ \mathrm{Car}^\dagger(f_+\mathscr{E})) = \sigma_Y^*(\overline{f}_*(\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\mathrm{ZCar}^\dagger(\mathscr{E})))).$$

Now consider the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{\ \overline{\sigma}_Y\ } & T^*Y\times_Y X \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle \overline{f}} \\ Y & \xrightarrow[\ \sigma_Y\ ]{} & T^*Y \end{array}$$

Since this diagram is cartesian and σ_Y and $\overline{\sigma}_Y$ are regular closed immersions with the same codimension, we have

$$\sigma_Y^*\overline{f}_* = f_*\overline{\sigma}_Y^*. \tag{2.2.7.1}$$

(Indeed, $\sigma_Y^! = \overline{\sigma}_Y^* : \mathrm{CH}_*(T^*Y\times_Y X)\to \mathrm{CH}_*(X)$ in the notation of Fulton, by [Ful98, Remark 6.2.1], and we have $\sigma_Y^*\overline{f}_* = f_*\sigma_Y^! = f_*\overline{\sigma}_Y^*$ where the first equality follows from [Ful98, Theorem 6.2].)

By definition, we get $g^*(\mathrm{ZCar}^\dagger(\mathscr{E}))\in \mathrm{CH}_{d_Y}(T^*Y\times_Y X)$ (where $d_Y=\dim(Y)$). By the definition of Todd class, we can write

$$\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1} = 1 + (\text{degree} \geqslant 1).$$

These show that there exists an α in $\mathrm{CH}_*(T^*Y\times_Y X)$ whose dimension is less than d_Y such that we may write

$$\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\mathrm{ZCar}^\dagger(\mathscr{E})) = g^*(\mathrm{ZCar}^\dagger(\mathscr{E})) + \alpha.$$

Since $\mathrm{CH}_i(T^*X)=0$ for $i<d_Y$, we get $\overline{\sigma}_Y^*(\alpha)=0$ and

$$\overline{\sigma}_Y^*(\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\mathrm{ZCar}^\dagger(\mathscr{E}))) = \sigma_X^*(\mathrm{ZCar}^\dagger(\mathscr{E})).$$

Upon combining this with (2.2.7.1), the corollary follows. □

2.2.8 Let $f:\mathscr{X}\to\mathscr{S}$ be the structure morphism for a proper smooth formal scheme $\mathscr{X}$ over $\mathscr{S}$, and let $\mathscr{E}$ be a coherent $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module. For $i\in\mathbb{Z}$, we define its ith de Rham cohomology group, denoted by $H^i_{\mathrm{DR}}(\mathscr{X},\mathscr{E})$, to be $\Gamma(\mathscr{S},\mathscr{H}^i(f_+\mathscr{E}[-d]))$. We know that this is a finite-dimensional K-vector space since f_+ preserves coherence; we also know by [Ber02, 4.3.6.3] that $H^i_{\mathrm{DR}}(\mathscr{X},\mathscr{E})$ is zero except for $0\leqslant i\leqslant 2\dim X$. Note that any coherent $\mathscr{D}^\dagger_{\mathscr{S},\mathbb{Q}}$-module is holonomic. We have the following absolute case as a corollary.

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Corollary 2.2.9 (Kashiwara–Dubson–Berthelot formula [Ber02, 5.4.4]). *Let $\mathscr{X}$ be a proper smooth scheme over $\mathscr{S}$ of dimension d, and let $\mathscr{E}$ be a holonomic $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module. Then*

$$\chi_{\mathrm{DR}}(\mathscr{E}) = (-1)^d \deg([X]\cdot \mathrm{ZCar}^{\dagger}(\mathscr{E})),$$

where the intersection product is taken in $\mathrm{CH}_*(T^*X)$ and

$$\chi_{\mathrm{DR}}(\mathscr{E}) := \sum_{i=0}^{2d} (-1)^i \dim_K H^i_{\mathrm{DR}}(\mathscr{X},\mathscr{E})$$

is the Euler–Poincaré characteristic of $\mathscr{E}$.

The sign here differs from that in [Ber02], because the definition of $H^i_{\mathrm{DR}}(\mathscr{X},\mathscr{E})$ is a little different.

We also have the following $\mathscr{D}^{\dagger}$ version of Remark 2.2.4, whose proof is the same as that of Theorem 2.2.6, using Remark 2.2.4.

Proposition 2.2.10. *Using the notation of ($\star$) in Theorem 2.2.2, suppose that f is a closed immersion. Recall that, in this case, f_+ preserves holonomicity (see [Ber02, 5.3.5]). Then we have the following commutative diagram.*

$$\begin{array}{ccc} K(\mathrm{hol}\ \mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}) & \xrightarrow{f_+} & K(\mathrm{hol}\ \mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}) \\ \downarrow{\scriptstyle \mathrm{ZCar}^{\dagger}} & & \downarrow{\scriptstyle \mathrm{ZCar}^{\dagger}} \\ Z_{d_X}(T^*X) & \xrightarrow[\overline{f}_* g^*]{} & Z_{d_Y}(T^*Y) \end{array}$$

Remark 2.2.11. The above proposition is a refined version of the result just before [Ber02, 5.3.4] for holonomic modules.

Before concluding this subsection, we prove a lemma which will be used in the proof of the main theorem of this paper.

Lemma 2.2.12. *Let $\iota:\mathscr{S}\to\mathscr{S}$ be an automorphism compatible with Frobenius structure (i.e. $\sigma\circ\iota=\iota\circ\sigma$). Let $\mathscr{X}$ be a smooth formal scheme over $\mathscr{S}$. Assume that we have the following cartesian diagram.*

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{f} & \mathscr{X} \\ \downarrow & \square & \downarrow \\ \mathscr{S} & \xrightarrow[\iota]{} & \mathscr{S} \end{array}$$

Let $\overline{f}:T^*X' \xrightarrow{\sim} T^*X$ denote the canonical isomorphism of the cotangent bundles of the special fibers of $\mathscr{X}$ and $\mathscr{X}'$. Let $\mathscr{E}$ be a coherent F-$\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module, and let $\mathscr{E}^{\iota}$ be the coherent F-$\mathscr{D}^{\dagger}_{\mathscr{X}',\mathbb{Q}}$-module induced by the base change ι. Then we have

$$\mathrm{ZCar}^{\dagger}(\mathscr{E}^{\iota}) = \overline{f}^*(\mathrm{ZCar}^{\dagger}(\mathscr{E})).$$

Proof. Let the $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{E}^{(m)}$ be the Frobenius descent of $\mathscr{E}$. Then $\mathscr{E}^{(m)\iota}$ is the Frobenius descent of the $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{E}^{\iota}$. Let $\mathscr{E}'^{(m)}$ be a $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module without torsion such that $\mathscr{E}'^{(m)}\otimes\mathbb{Q}\cong\mathscr{E}^{(m)}$. Then we also have that

$$(\mathscr{E}'^{(m)})^{\iota}\otimes\mathbb{Q}\cong\mathscr{E}^{(m)\iota} \quad\text{and}\quad (\mathscr{E}'^{(m)}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathcal{O}_X)^{\iota}\cong(\mathscr{E}'^{(m)})^{\iota}\otimes_{\mathcal{O}_{\mathscr{X}'}}\mathcal{O}_{X'}.$$

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Thus,

$$\mathrm{ZCar}^\dagger(\mathscr{E}) \cong \mathrm{ZCar}(\mathscr{E}'^{(m)} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_X), \qquad \mathrm{ZCar}^\dagger(\mathscr{E}^\iota) \cong \mathrm{ZCar}((\mathscr{E}'^{(m)} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_X)^\iota),$$

and we are reduced to showing that given a $\mathscr{D}^{(m)}_X$-module $\mathscr{F}$, $\mathrm{ZCar}(\mathscr{F}^\iota) = \overline{f}^*(\mathrm{ZCar}(\mathscr{F}))$; but this is easy. □

2.3 Characteristic cycles for $\mathscr{D}^\dagger_{X,\mathbb{Q}}$-modules

In this subsection, we fix m so that $p^m > e/(p-1)$ unless otherwise stated.

2.3.1 Let X be a smooth scheme over k. Let $\{X_i\}_{i\in I}$ be a finite affine open covering of X. Note that for each i, X_i can be lifted to a smooth formal scheme over $\mathscr{S}$. For $J \subset I$, we denote $\bigcap_{i\in J} X_i$ by X_J. We fix smooth liftings $\mathscr{X}_J$ of X_J. For $I \supset J \supset J'$, let $\alpha_{J,J'} : X_J \to X_{J'}$ be the canonical inclusion. Note that by [Ber00, 2.1.6], we have the canonical functor

$$\alpha^!_{J,J'} : (\mathrm{coh}\ \mathscr{D}^\dagger_{\mathscr{X}_{J'},\mathbb{Q}}\text{-mod}) \to (\mathrm{coh}\ \mathscr{D}^\dagger_{\mathscr{X}_J,\mathbb{Q}}\text{-mod})$$

without lifting $\alpha_{J,J'}$ over $\mathscr{S}$, and it satisfies the following associativity property.

For $J \supset J' \supset J''$, there exists a canonical isomorphism of functors

$$\alpha^!_{J,J''} \cong \alpha^!_{J,J'} \circ \alpha^!_{J',J''}. \tag{$\sharp$}$$

The same also holds for $\mathscr{D}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules and $\mathscr{D}^{(m)}_{\mathscr{X}}$-modules instead of $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules.

Definition 2.3.2. With the notation introduced above, we define the following.

(i) A coherent $\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-module is a set $\{\mathscr{E}_i\}_{i\in I}$ such that $\mathscr{E}_i$ is a coherent $\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}$-module with gluing data as follows.

For any $(i,j) \in I \times I$, there are gluing isomorphisms $c_{j,i} : \alpha^!_{\{i,j\},i}(\mathscr{E}_i) \to \alpha^!_{\{i,j\},j}(\mathscr{E}_j)$ satisfying the cocycle condition

$$\alpha^!_{\{i,j,k\},\{j,k\}}(c_{k,j}) \circ \alpha^!_{\{i,j,k\},\{i,j\}}(c_{j,i}) = \alpha^!_{\{i,j,k\},\{i,k\}}(c_{k,i}).$$

(ii) Let $\{\mathscr{E}_i\}_{i\in I}$ and $\{\mathscr{F}_i\}_{i\in I}$ be coherent $\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-modules. We define a homomorphism $f : \{\mathscr{E}_i\}_{i\in I} \to \{\mathscr{F}_i\}_{i\in I}$ of $\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-modules to be a set of homomorphisms of $\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}$-modules $\{f_i : \mathscr{E}_i \to \mathscr{F}_i\}_{i\in I}$ such that the following diagram is commutative for any $(i,j) \in I \times I$.

$$\begin{array}{ccc} \alpha^!_{\{i,j\},i}(\mathscr{E}_i) & \xrightarrow{\alpha^!_{\{i,j\},i}(f_i)} & \alpha^!_{\{i,j\},i}(\mathscr{F}_i) \\ {\scriptstyle c^{\mathscr{E}}_{j,i}}\downarrow & & \downarrow{\scriptstyle c^{\mathscr{F}}_{j,i}} \\ \alpha^!_{\{i,j\},j}(\mathscr{E}_j) & \xrightarrow[\alpha^!_{\{i,j\},j}(f_j)]{} & \alpha^!_{\{i,j\},j}(\mathscr{F}_j) \end{array}$$

where $c^{\mathscr{E}}$ (respectively, $c^{\mathscr{F}}$) denotes the gluing isomorphisms for $\{\mathscr{E}_i\}$ (respectively, $\{\mathscr{F}_i\}$).

(iii) Let $\{\mathscr{E}_i\}_{i\in I}$ be a coherent $\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-module. We define $F^*\{\mathscr{E}_i\} := \{F^*\mathscr{E}_i\}$, which then defines a $\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-module. We define a coherent F-$\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-module to be a pair $(\{\mathscr{E}_i\}, \Phi)$ where $\{\mathscr{E}_i\}$ is a $\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-module and $\Phi : \{\mathscr{E}_i\} \xrightarrow{\sim} F^*\{\mathscr{E}_i\}$ is an isomorphism of $\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-modules. We also define homomorphisms of coherent F-$\{\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}$-modules in the obvious way.

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(iv) In the same way, we define coherent $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}$-modules, coherent $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i}\}$-modules, and coherent F-$\{\mathscr{D}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}$-modules, as well as homomorphisms of these modules.

It is easy to verify the following lemma.

Lemma 2.3.3. *The category of coherent $\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\}$-modules forms an abelian category. The same is true for the category of coherent F-$\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\}$-modules, $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}$-modules, or $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i}\}$-modules.*

Construction of characteristic homomorphisms and characteristic cycles

2.3.4 We keep the same notation as before. Moreover, we assume that X is a smooth scheme over k with d-quadruples. Now let us define characteristic homomorphisms for overcoherent $\mathscr{D}^{\dagger}_{X,\mathbb{Q}}$-modules. As usual, let $K(F\text{-}\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\})$, $K(F\text{-}\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\})$ and $K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i}\})$ be the Grothendieck groups of coherent F-$\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\}$-modules, coherent F-$\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}$-modules and coherent $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i}\}$-modules, respectively. We also define $K(\mathscr{D}^{\dagger}_{X,\mathbb{Q}})$ to be the Grothendieck group of the category of *overcoherent* $\mathscr{D}^{\dagger}_{X,\mathbb{Q}}$-modules.

Now we are going to construct $\mathrm{Car}^{\dagger} : K(\mathscr{D}^{\dagger}_{X,\mathbb{Q}}) \to K(\mathscr{O}_{T^*X})$. The idea of the definition is $(*)$ in Lemma 2.1.11. Thus, we define the characteristic homomorphism $\mathrm{Car}^{\dagger}$ to be the composition

$$\begin{array}{ccccccc} K(\mathscr{D}^{\dagger}_{X,\mathbb{Q}}) & \xrightarrow{a} & K(F\text{-}\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\}) & \xrightarrow{b} & K(F\text{-}\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}) & \xrightarrow{c} & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}) \\ & & & & & & \downarrow d \\ & & & & & & K(\mathscr{D}^{(m)}_{X}) \\ & & & \searrow^{\mathrm{Car}^{\dagger}} & & & \downarrow \mathrm{Car}^{(m)} \\ & & & & & & K(\mathscr{O}_{T^*X}) \end{array}$$

where the homomorphisms a, b, c and d will be defined next.

2.3.5 *Definition of* $a : K(\mathscr{D}^{\dagger}_{X,\mathbb{Q}}) \to K(F\text{-}\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\})$. Let $(X, \overline{X}, Z, \mathscr{P})$ be a d-quadruple. There exists a finite open covering $\{\mathscr{P}_i\}_{i\in I}$ of $\mathscr{P}\backslash Z$ such that $\mathscr{P}_i \cap X = X_i$. Let $g_i : X_i \to P_i$ be the closed immersions for $i \in I$. Let $\mathscr{E} \in \mathfrak{M}_{(\overline{X},Z,\mathscr{P})}$. Consider $\{g_i^!(\mathscr{E}|_{\mathscr{P}_i})\}$. Since $\mathscr{E}$ is an element of $\mathfrak{M}_{(\overline{X},Z,\mathscr{P})}$, note that $g_i^!(\mathscr{E}|_{\mathscr{P}_i})$ are coherent F-$\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}$-*modules*. This defines a coherent F-$\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\}$-module by the associativity $(\sharp)$ of extraordinary pull-backs. Thus, we get an exact functor

$$\tilde{a} : (\text{overcoh } \mathscr{D}^{\dagger}_{X,\mathbb{Q}}\text{-mod}) \to (F\text{-coh}\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\text{-mod}\}),$$

which induces the homomorphism of Grothendieck groups a.

Lemma 2.3.6. *This construction does not depend on the choice of d-quadruples.*

Proof. First, we note that the construction does not depend on the choice of open coverings $\{\mathscr{P}_i\}$ such that $\mathscr{P}_i \cap X = X_i$. Let $(X, \overline{X}', W, \mathscr{Q})$ be another d-quadruple. Then, upon taking fiber products if needed, we may suppose that there exists a smooth proper morphism of d-quadruples $f : (X, \overline{X}', W, \mathscr{Q}) \to (X, \overline{X}, Z, \mathscr{P})$. Take an open covering of $\mathscr{Q}$ by $\{\mathscr{Q}_i := f^{-1}(\mathscr{P}_i)\}$. There is a

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commutative diagram as follows.

$$\begin{array}{ccc} & \xrightarrow{h_i} & Q_i \\ X_i & & \downarrow f_i \\ & \xrightarrow{g_i} & P_i \end{array}$$

Let $\mathscr{E}$ be a overcoherent $\mathscr{D}^{\dagger}_{X,\mathbb{Q}}$-module, and let $\mathscr{E}_{\mathscr{P}}$ and $\mathscr{E}_{\mathscr{Q}}$ be the realizations in $\mathscr{P}$ and $\mathscr{Q}$, respectively. Then we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{X_i} f_i^!(\mathscr{E}_{\mathscr{P}}|_{\mathscr{P}_i}) \cong \mathscr{E}_{\mathscr{Q}}|_{\mathscr{Q}_i}$. It suffices to show that $h_i^!(\mathscr{Q}|_{\mathscr{Q}}) \cong g_i^!(\mathscr{P}|_{\mathscr{P}})$, but this follows from the result of Caro, Theorem 1.1.6. □

2.3.7 *Definition of* $b : K(\{F\text{-}\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\}) \to K(\{F\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\})$. We define this homomorphism to be the homomorphism induced by Frobenius descent. To be precise, let $\{\mathscr{E}_i\}$ be a coherent $F\text{-}\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\}$-module. Then, for each i, take the Frobenius descent $\mathscr{E}'_i$. By the uniqueness of Frobenius descent, $\{\mathscr{E}'_i\}$ defines a coherent $F\text{-}\{\mathscr{D}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}$-module, and we get the functor

$$\tilde{b} : (F\text{-coh}\{\mathscr{D}^{\dagger}_{\mathscr{X}_i,\mathbb{Q}}\}\text{-mod}) \to (F\text{-coh}\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}\text{-mod}).$$

We define b to be the homomorphism of Grothendieck groups defined by this exact functor.

2.3.8 *Definition of* $c : K(\{F\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}) \to K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\})$. We define this homomorphism to be the homomorphism induced by the forgetful functor

$$\tilde{c} : (F\text{-coh}\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}\text{-mod}) \to (\text{coh}\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}\text{-mod}).$$

2.3.9 *Definition of* $d : K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}) \to K(\mathscr{D}^{(m)}_X)$. First, we prove the following lemma.

Lemma 2.3.10. *Let $\{\mathscr{E}_i\}$ be a coherent $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}$-module. Then there exists a coherent $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i}\}$-module $\{\mathscr{E}'_i\}$ such that the $\mathscr{E}'_i$ are p^{∞}-torsion free and $\mathscr{E}'_i \otimes \mathbb{Q} = \mathscr{E}_i$.*

Proof. The proof is the same as that of [Ber96b, 3.4.5]. Let $J \subset I$ and $X' := \bigcup_{j\in J} X_j$, and take $k \in I$ such that $k \notin J$. Let $\{\mathscr{M}_j\}_{j\in J}$ (respectively, $\mathscr{M}_k$) be a p-torsion-free coherent $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i}\}_{i\in J}$-module (respectively, $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_k}$-module). Let $\alpha : X' \cap X_k \to X'$, $\alpha_k : X' \cap X_k \to X_k$. Then we define $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_{j,k}}\}_{j\in J}$-modules by $\alpha^!(\{\mathscr{M}_j\}_{j\in J}) := \{\alpha^!_{k,\{j,k\}}(\mathscr{M}_j)\}_{j\in J}$ and $\alpha^!_k(\mathscr{M}_k) := \{\alpha^!_{k,\{j,k\}}(\mathscr{M}_k)\}_{j\in J}$. The lemma follows easily from the claim below, using induction on the number of open coverings.

Claim. With the above notation, suppose we are given an isomorphism $\epsilon : \alpha^!(\{\mathscr{M}_j \otimes \mathbb{Q}\}_{j\in J}) \xrightarrow{\sim} \alpha^!_k(\mathscr{M}_k \otimes \mathbb{Q})$. Then there exist a $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_k}$-module $\mathscr{M}'_k$ and $\epsilon' : \alpha^!(\{\mathscr{M}_j\}) \xrightarrow{\sim} \alpha^!_k(\mathscr{M}'_k)$ such that $\mathscr{M}'_j \otimes \mathbb{Q} \cong \mathscr{M}_j \otimes \mathbb{Q}$ and $\epsilon' \otimes \mathbb{Q} \cong \epsilon$.

To prove the claim, we note first that by multiplying ϵ by a power of p, if necessary, we may assume that $\epsilon(\alpha^!\{(\mathscr{M}_j)\}_J) \subset \alpha^!_k(\mathscr{M}_k)$. There exists an $n \geqslant 0$ such that $p^{n+1}(\alpha^!(\{\mathscr{M}_j\}_J)/\epsilon(\alpha^!_k(\mathscr{M}_k))) = 0$. Let $\mathscr{M}_{j,n} := \mathscr{M}_j/p^{n+1}\mathscr{M}_j$, and let $\overline{\mathscr{M}}_{j,n}$ be the image of $\alpha^!_i(\mathscr{M}_j)$ in $\alpha^!_k(\mathscr{M}_{k,n})$. Note here that $\{\overline{\mathscr{M}}_{j,n}\}_J$ is a coherent $\{\mathscr{D}^{(m)}_{(X_{\{j,k\}})_n}\}_J$-module. Now, fix liftings $\widetilde{\alpha}_j : (X_{\{j,k\}})_n \to (X_k)_n$ of $\alpha_{k,\{j,k\}} : X_{\{j,k\}} \to X_k$. (We can take such liftings since X is smooth.) Then there exists a sub-$\mathcal{O}_{(X_j)_n}$-module of $\mathscr{M}_{j,n}$ $\mathscr{F}$ such that the image of the homomorphism

$$\widetilde{\alpha}^*_j(\mathscr{F}) \to \widetilde{\alpha}^*_j(\mathscr{M}_{j,n}) \cong \alpha^!_{k,\{j,k\}}(\mathscr{M}_{j,n}) \to \overline{\mathscr{M}}_{j,n}$$

generates $\overline{\mathscr{M}}_{j,n}$ as a $\mathscr{D}^{(m)}_{(X_{\{j,k\}})_n}$-module. Let $\overline{\mathscr{N}}_n := \mathscr{D}^{(m)}_{(X_k)_n}\mathscr{F}$ be a sub-$\mathscr{D}^{(m)}_{(X_k)_n}$-module of $\mathscr{M}_{k,n}$ and define $\mathscr{M}'_k \subset \mathscr{M}_k$ to be the inverse image of $\overline{\mathscr{N}}_n$. It can easily be seen that this is what was asserted in the claim.

Applying induction then completes the proof of Lemma 2.3.10. □

Let $\{\mathscr{E}_i\}$ be a coherent $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}$-module. By Lemma 2.3.10, we can take a $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i}\}$-module $\{\mathscr{E}'_i\}$ such that $\{\mathscr{E}'_i \otimes \mathbb{Q}\} = \{\mathscr{E}_i\}$. Now, consider the cohomologies of its reduction to the special fibers $\mathscr{H}_0(\mathscr{E}'_i \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}_i}} \mathcal{O}_{X_i})$ and $\mathscr{H}_1(\mathscr{E}'_i \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}_i}} \mathcal{O}_{X_i})$ for $i \in I$. Note here that $\mathscr{H}_n(\mathscr{E}'_i \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}_i}} \mathcal{O}_{X_i})$ $(n = 0, 1)$ are coherent $\mathscr{D}^{(m)}_{X_i}$-modules. Since $\alpha^!_{\{i,j\},i}$ is exact for $(i, j) \in I \times I$, we have an isomorphism

$$\begin{aligned} c^X_{j,i} : \alpha^!_{\{i,j\},i}(\mathscr{H}_n(\mathscr{E}'_i \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}_i}} \mathcal{O}_{X_i})) &\cong \mathscr{H}_n(\alpha^!_{\{i,j\},i}(\mathscr{E}'_i) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}_{\{i,j\}}}} \mathcal{O}_{X_{\{i,j\}}}) \\ &\xrightarrow[\mathscr{H}_n(c^{\mathscr{X}}_{j,i}\otimes \mathcal{O}_{X_{\{i,j\}}})]{\sim} \mathscr{H}_n(\alpha^!_{\{i,j\},j}(\mathscr{E}'_j) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}_{\{i,j\}}}} \mathcal{O}_{X_{\{i,j\}}}) \cong \alpha^!_{\{i,j\},j}(\mathscr{H}_n(\mathscr{E}'_j \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}_j}} \mathcal{O}_{X_j})) \end{aligned}$$

for $n = 0, 1$, where $c^{\mathscr{X}}_{i,j}$ denotes the gluing isomorphism of $\{\mathscr{E}'_i\}$. Moreover, it is easy to see that the $c^X_{i,j}$ satisfy the cocycle condition. Thus, $\{\mathscr{H}_n(\mathscr{E}'_i \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{X}_i}} \mathcal{O}_{X_i})\}$ $(n = 0, 1)$ satisfy the gluing condition, and we may glue them together to get coherent $\mathscr{D}^{(m)}_X$-modules $\mathscr{E}_n$ for $n = 0, 1$. We define the image of $\mathscr{E}$ under the homomorphism d to be $[\mathscr{E}_0] - [\mathscr{E}_1]$ in $K(\mathscr{D}^{(m)}_X)$. In exactly the same way as we argued in the proof of Lemma 2.1.11, we can show that this definition does not depend on the choice of liftings and defines a homomorphism of Grothendieck groups.

2.3.11 This homomorphism does not depend on the choice of covering of X and liftings $\mathscr{X}_J$. Indeed, take another covering $\{X'_{i'}\}_{i'\in I'}$. We have a refinement $\{X''_j\}_{j\in J}$ of $\{X_i\}_{i\in I}$ and $\{X'_{i'}\}_{i'\in I'}$. Take liftings $\{\mathscr{X}'_{i'}\}$ and $\{\mathscr{X}''_j\}$. Then we have the pull-back homomorphisms

$$K(\{\mathscr{D}^\dagger_{\mathscr{X}^{(\prime)}_{i^{(\prime)}},\mathbb{Q}}\}_{i^{(\prime)}\in I^{(\prime)}}) \to K(\{\mathscr{D}^\dagger_{\mathscr{X}''_j,\mathbb{Q}}\}_{j\in J})$$

and so on. Now the independence can be seen from the following commutative diagram.

$$\begin{array}{ccccccccc}
 & \nearrow & K(\{F\text{-}\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}\}_{i\in I}) & \to & K(\{F\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}_{i\in I}) & \to & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_i,\mathbb{Q}}\}_{i\in I}) & \searrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
K(F\text{-}\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}) & \to & K(\{F\text{-}\mathscr{D}^\dagger_{\mathscr{X}''_j,\mathbb{Q}}\}_{j\in J}) & \to & K(\{F\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}''_j,\mathbb{Q}}\}_{j\in J}) & \to & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}''_j,\mathbb{Q}}\}_{j\in J}) & \to & K(\mathscr{D}^{(m)}_X) \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & \searrow & K(\{F\text{-}\mathscr{D}^\dagger_{\mathscr{X}'_{i'},\mathbb{Q}}\}_{i'\in I'}) & \to & K(\{F\text{-}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}'_{i'},\mathbb{Q}}\}_{i'\in I'}) & \to & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}'_{i'},\mathbb{Q}}\}_{i'\in I'}) & \nearrow &
\end{array}$$

The independence of m can be seen from the lifted case, Lemma 2.1.11.

2.3.12 Now we will define the characteristic cycles. Let $\mathscr{E}$ be an overholonomic $\mathscr{D}^\dagger_{X,\mathbb{Q}}$-module. Let $(X, \overline{X}, Z, \mathscr{P})$ be a d-quadruple, and denote also by $\mathscr{E}$ the realization of $\mathscr{E}$ in the d-quadruple $(X, \overline{X}, Z, \mathscr{P})$. Let $\tilde{a}(\mathscr{E}) = \{\mathscr{E}_i\}$. The modules $\mathscr{E}_i$ are overholonomic $\mathscr{D}^\dagger_{\mathscr{X}_i,\mathbb{Q}}$-modules and, in particular, holonomic modules. Let $\tilde{c} \circ \tilde{b}(\{\mathscr{E}_i\}) = \{\mathscr{E}^{(m)}_i\}$. Now, using Lemma 2.3.10, there exists a $\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}\}$-module $\{\mathscr{E}'_i\}$ which is p-torsion free and such that $\{\mathscr{E}'_i\} \otimes \mathbb{Q} = \{\mathscr{E}^{(m)}_i\}$. Then $\mathscr{E}'' := \{\mathscr{E}'_i \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_X\}$ defines a $\mathscr{D}^{(m)}_X$-module as in the construction of the homomorphism d.

Since $\mathscr{E}_i^{(m)}$ are modules of dimension less than or equal to d, $\mathscr{E}''$ is also a module of dimension less than or equal to$d$. By the same argument as in § 2.1.16, it can easily be seen that this defines an element $[\mathscr{E}'']$ in $K(d\text{-}\mathscr{D}_X^{(m)})$ and that the definition does not depend on the choice of lifting $\{\mathscr{E}'_i\}$. By the same argument as in § 2.3.11, the definition does not depend on the choice of d-quadruples. We define $Z_d\mathrm{Car}^\dagger(\mathscr{E}) := [\mathscr{E}''] \in K(d\text{-}\mathscr{D}_X^{(m)})$.

Definition 2.3.13. Let $\mathscr{E}$ be an overholonomic $\mathscr{D}^\dagger_{X,\mathbb{Q}}$-module. We define the characteristic cycle by

$$\mathrm{ZCar}^\dagger(\mathscr{E}) := \mathrm{mult}_d(Z_d\mathrm{Car}^\dagger(\mathscr{E})).$$

Remark 2.3.14. Let $(X, \overline{X}, Z, \mathscr{P})$ be a d-quadruple, let P be the special fiber of $\mathscr{P}$, and let $T^*X \xleftarrow{f} T^*(P\backslash Z) \times_{(P\backslash Z)} X \xrightarrow{k} T^*(P\backslash Z)$. Let $\mathscr{E}_{\mathscr{P}}$ be the realization of $\mathscr{E}$ in the quadruple $(X, \overline{X}, Z, \mathscr{P})$. Let $k_* : Z_*(T^*(P\backslash Z) \times_{(P\backslash Z)} X) \to Z_*(T^*(P\backslash Z))$ denote the push-forward and $f^* : Z_*(T^*X) \to Z_*(T^*(P\backslash Z) \times_{(P\backslash Z)} X)$ the pull-back. Then $\mathrm{ZCar}^\dagger(\mathscr{E})$ is the unique cycle $\alpha \in Z_{\dim X}(T^*X)$ such that

$$k_* f^*(\alpha) = \mathrm{ZCar}^\dagger(\mathscr{E}_{\mathscr{P}}|_{(\mathscr{P}\backslash Z)}).$$

Proof. The uniqueness can be seen from the fact that k_* and f^* are injective homomorphisms. All that remains to do is prove that $k_* f^*(\mathrm{ZCar}^\dagger(\mathscr{E})) = \mathrm{ZCar}^\dagger(\mathscr{E}_{\mathscr{P}}|_{(\mathscr{P}\backslash Z)})$. Since $\mathrm{ZCar}^\dagger(\mathscr{E})$ is in $Z_d(T^*X)$, we may calculate locally and assume that there exist a smooth formal lifting $\mathscr{X}$ of X and $i : \mathscr{X} \to \mathscr{P}$. Now let $\mathscr{E}_{\mathscr{X}}$ be the realization of $\mathscr{E}$ in $\mathscr{X}$; then we have $i_+(\mathscr{E}_{\mathscr{X}}) \cong \mathscr{E}_{\mathscr{P}}$. Thus, by Proposition 2.2.10, we conclude the proof. □

The Riemann–Roch theorem

2.3.15 Let $f : X \to Y$ be a proper morphism between smooth schemes over k with d-quadruples. Then there exists a push-forward functor $f_+ : D^b(\mathfrak{M}_X^+) \to D^b(\mathfrak{M}_Y^+)$. Using Remark 1.1.8, we get the homomorphism $K(\mathscr{D}^\dagger_{X,\mathbb{Q}}) \to K(\mathscr{D}^\dagger_{Y,\mathbb{Q}})$.

Theorem 2.3.16 (Riemann–Roch theorem). *Let $f : X \to Y$ be a proper morphism between smooth schemes over k with d-quadruples. Let d_X and d_Y be the dimensions of X and Y, respectively. We use the notation of $(\star)$ in Theorem 2.2.2. Then, the diagram*

$$\begin{array}{ccc}
K(\mathscr{D}^\dagger_{X,\mathbb{Q}}) & \xrightarrow{\quad f_+ \quad} & K(\mathscr{D}^\dagger_{Y,\mathbb{Q}}) \\
{\scriptstyle \tau_{T^*X}\circ\mathrm{Car}^\dagger}\downarrow & & \downarrow{\scriptstyle \tau_{T^*Y}\circ\mathrm{Car}^\dagger} \\
\mathrm{CH}_*(T^*X)_{\mathbb{Q}} & \xrightarrow[\overline{f}_*(\mathrm{Todd}(\pi_f^*\mathscr{T}_f)^{-1}\cdot g^*(\text{-}))]{} & \mathrm{CH}_*(T^*Y)_{\mathbb{Q}}
\end{array}$$

is commutative.

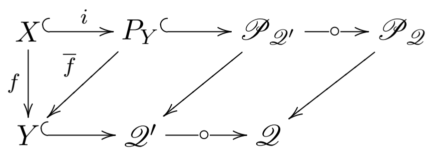
Proof. The idea of the proof is the same as that of Theorem 2.2.6. First, we need to prepare some ground. Let $f : P \to Q$ be a proper morphism of smooth schemes such that there exist coverings $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$ with $f^{-1}(Q_i) = P_i$, where P_i and Q_i have smooth liftings $\mathscr{P}_i$ and $\mathscr{Q}_i$, respectively, and there exists $f_i : \mathscr{P}_i \to \mathscr{Q}_i$ which is a lifting of f. Then we define, for $k \in \mathbb{Z}$, the kth push-forward

$$\mathscr{H}^k f_+ : (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{P}_i,\mathbb{Q}}\}\text{-mod}) \to (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{Q}_i,\mathbb{Q}}\}\text{-mod})$$

https://doi.org/10.1112/S0010437X09004485 Published online by Cambridge University Press

as follows. Let $\{\mathscr{E}_i\}$ be a coherent $\{\mathscr{D}^\dagger_{\mathscr{P}_i,\mathbb{Q}}\}$-module. Since f_i is proper, $\{\mathscr{H}^k(f_{i+}\mathscr{E}_i)\}$ defines a coherent $\{\mathscr{D}^\dagger_{\mathscr{Q}_i,\mathbb{Q}}\}$-module. Then we define $(\mathscr{H}^k f_+)\{\mathscr{E}_i\}$ to be $\{\mathscr{H}^k(f_{i+}\mathscr{E}_i)\}$.

Now we are ready to prove the theorem. Let $(X, \overline{X}, Z, \mathscr{P})$ and $(Y, \overline{Y}, W, \mathscr{Q})$ be d-quadruples. Let P be the special fiber of $\mathscr{P}$. We write $\mathscr{Q}' := \mathscr{Q}\backslash W$, $\mathscr{P}_{\mathscr{Q}} := \mathscr{P} \times \mathscr{Q}$, $\mathscr{P}_{\mathscr{Q}'} := \mathscr{P} \times \mathscr{Q}'$ and $P_Y := P \times Y$. Then we have the following diagram.



where the $\hookrightarrow$ denote closed immersions and the $\circ\!\!\rightarrow$ denote open immersions. Take a finite open affine covering $\{\mathscr{Q}'_j\}_{j\in J}$ of $\mathscr{Q}'$. We write $Y_j := \mathscr{Q}'_j \cap Y$ and fix liftings of Y_j which we denote by $\mathscr{Y}_j$. Let $X_j := f^{-1}(Y_j)$. Let $\mathscr{P}_{\mathscr{Y}_j} := \mathscr{P} \times \mathscr{Y}_i$. Note that $X_j \hookrightarrow \mathscr{P}_{\mathscr{Y}_j}$ are closed immersions. Take a finite open affine covering $\{\mathscr{P}_{j,i}\}_{i\in I_j}$ of $\mathscr{P}_{\mathscr{Y}_j}$. Let $X_{j,i}$ be an open affine scheme $X_j \cap \mathscr{P}_{j,i}$ of X, and fix its smooth liftings $\mathscr{X}_{j,i}$ as well as the lifting $\mathscr{X}_{j,i} \hookrightarrow \mathscr{P}_{j,i}$ of the closed immersion. Then we define

$$\mathscr{H}^k f_+ : (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{X}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}\text{-mod}) \to (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{Y}_j,\mathbb{Q}}\}_{j\in J}\text{-mod})$$

in the following way. The preparations above give functors

$$\mathscr{H}^{k'} i_+ : (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{X}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}\text{-mod}) \to (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{P}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}\text{-mod}),$$
$$\mathscr{H}^{k''} \overline{f}_+ : (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{P}_{\mathscr{Y}_j},\mathbb{Q}}\}_{j\in J}\text{-mod}) \to (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{Y}_j,\mathbb{Q}}\}_{j\in J}\text{-mod})$$

for $k', k'' \in \mathbb{Z}$. But since i is a closed immersion, we have $\mathscr{H}^{k'} i_+ = 0$ for $k' \neq 0$, and we get

$$i_+ := \mathscr{H}^0 i_+ : (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{X}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}\text{-mod}) \to (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{P}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}\text{-mod}).$$

Since $\{\mathscr{P}_{j,i}\}_{i\in I_j}$ is an open covering of $\mathscr{P}_{\mathscr{Y}_j}$, we get the canonical equivalence of categories

$$\iota : (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{P}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}\text{-mod}) \xrightarrow{\sim} (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{P}_{\mathscr{Y}_j},\mathbb{Q}}\}_{j\in J}\text{-mod}).$$

Then, we define the functor $\mathscr{H}^k f_+$ by

$$\mathscr{H}^k f_+ := \mathscr{H}^k \overline{f}_+ \circ \iota \circ i_+ : (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{X}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}\text{-mod}) \to (\mathrm{coh}\{\mathscr{D}^\dagger_{\mathscr{Y}_j,\mathbb{Q}}\}_{j\in J}\text{-mod}).$$

These functors define a homomorphism of Grothendieck groups

$$f_+ : K(\{\mathscr{D}^\dagger_{\mathscr{X}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}) \to K(\{\mathscr{D}^\dagger_{\mathscr{Y}_j,\mathbb{Q}}\}_{j\in J}),$$

sending a class of a coherent $\{\mathscr{D}^\dagger_{\mathscr{X}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}$-modules $[\mathscr{E}]$ to $\sum_{k\in\mathbb{Z}}(-1)^k[\mathscr{H}^k f_+\mathscr{E}]$. We may construct $\mathscr{H}^k f_+$ and f_+ for coherent F-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules, coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules, and coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-modules in the same way. Now, by Theorem 2.2.2, the proof is reduced to showing commutativity of the following diagram.

$$\begin{array}{ccccccc}
K(\mathscr{D}^\dagger_{X,\mathbb{Q}}) & \xrightarrow{a} & K(\{F\text{-}\mathscr{D}^\dagger_{\mathscr{X}_{j,i},\mathbb{Q}}\}) & \xrightarrow{cob} & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_{j,i},\mathbb{Q}}\}) & \xrightarrow{d} & K(\mathscr{D}^{(m)}_X) \\
\downarrow f_+ & ① & \downarrow f_+ & ② & \downarrow f_+ & ③ & \downarrow f_+ \\
K(\mathscr{D}^\dagger_{Y,\mathbb{Q}}) & \xrightarrow{a} & K(\{F\text{-}\mathscr{D}^\dagger_{\mathscr{Y}_j}\}) & \xrightarrow{cob} & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y}_j,\mathbb{Q}}\}) & \xrightarrow{d} & K(\mathscr{D}^{(m)}_Y)
\end{array}$$

Let us show the commutativity of each square in the diagram.

https://doi.org/10.1112/S0010437X09004485 Published online by Cambridge University Press

① Let $\mathscr{E}$ be an overholonomic $\mathscr{D}^{\dagger}_{X,\mathbb{Q}}$-module. Let $\mathscr{E}'$ be a overholonomic $\mathscr{D}^{\dagger}_{\mathscr{P}_{\mathscr{Q}'},\mathbb{Q}}$-module obtained by restricting the realization of $\mathscr{E}$ in $\mathscr{P}_{\mathscr{Q}}$ to $\mathscr{P}_{\mathscr{Q}'}$. For $j \in J$, we denote by $\mathscr{E}'_j$ the restriction of $\mathscr{E}'$ to $\mathscr{P}_{\mathscr{Q}'_j} := \mathscr{P} \times \mathscr{Q}'_j$, and for a $\{\mathscr{D}^{\dagger}_{\mathscr{Y}_j,\mathbb{Q}}\}$-module $\mathscr{F} = \{\mathscr{F}_j\}$, we write $\mathscr{F}_j$ as $\{\mathscr{F}\}_j$. It suffices to show that $\{\tilde{a}\mathscr{H}^k f_+(\mathscr{E}')\}_j \cong \{\mathscr{H}^k f_+ \tilde{a}(\mathscr{E}')\}_j$ for each $k \in \mathbb{Z}$ and $j \in J$ and to verify the cocycle conditions. (For the definition of $\tilde{a}$, see the construction of the homomorphism a.) Now, consider the following commutative diagram.

$$\begin{array}{ccc} \mathscr{P}_{\mathscr{Y}_j} & \xhookrightarrow{\;i'_j\;} & \mathscr{P}_{\mathscr{Q}'_j} \\ {\scriptstyle f_{\mathscr{Y}_j}}\big\downarrow & & \big\downarrow{\scriptstyle f_{\mathscr{Q}'_j}} \\ \mathscr{Y}_j & \xhookrightarrow[\;i_j\;]{} & \mathscr{Q}'_j \end{array}$$

By Kashiwara's theorem [Car05c, Théorème 2.11], it suffices to show that $i_{j+}\{\tilde{a}\mathscr{H}^k f_+(\mathscr{E}')\}_j \cong i_{j+}\{\mathscr{H}^k f_+\tilde{a}(\mathscr{E}')\}_j$. For the left-hand side we get

$$i_{j+}\{\tilde{a}\mathscr{H}^k f_+(\mathscr{E}')\}_j \cong i_{j+}i_j^!\{\mathscr{H}^k f_+(\mathscr{E}')\}|_{\mathscr{Q}'_j} \cong i_{j+}i_j^!\mathscr{H}^k f_{\mathscr{Q}'_j+}(\mathscr{E}'_j) \cong \mathscr{H}^k f_{\mathscr{Q}'_j+}(\mathscr{E}'_j),$$

where the third equality holds by the fact that $\mathscr{H}^k f_{\mathscr{Q}'_j+}(\mathscr{E}'_j)$ is supported in $\mathscr{Y}_j$ since $\mathscr{E}'$ is supported in X. For the right-hand side we get

$$i_{j+}\{\mathscr{H}^k f_+\tilde{a}(\mathscr{E}')\}_j \cong i_{j+}\mathscr{H}^k f_{\mathscr{Y}_j+}\{\iota \circ i_+ \circ \tilde{a}(\mathscr{E}')\}_j.$$

Since $\{\iota \circ i_+ \circ \tilde{a}(\mathscr{E}')\}_j \cong i'^!_j(\mathscr{E}'_j)$, we have

$$i_{j+}\{\mathscr{H}^k f_+\tilde{a}(\mathscr{E}')\}_j \cong i_{j+}\mathscr{H}^k f_{\mathscr{Y}_j+}i'^!_j(\mathscr{E}'_j) \cong \mathscr{H}^k f_{\mathscr{Q}'_j+}i'_{j+}i'^!_j(\mathscr{E}'_j) \cong \mathscr{H}^k f_{\mathscr{Q}'_j+}(\mathscr{E}'_j).$$

The cocycle condition is easily seen from the definition, and the commutativity follows.

② We have to prove that the following diagram is commutative.

$$\begin{array}{ccccccc} K(\{F\text{-}\mathscr{D}^{\dagger}_{\mathscr{X}_{j,i},\mathbb{Q}}\}) & \xrightarrow{\;i_+\;} & K(\{F\text{-}\mathscr{D}^{\dagger}_{\mathscr{P}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}) & \xrightarrow[\sim]{\;\iota\;} & K(\{F\text{-}\mathscr{D}^{\dagger}_{\mathscr{P}_{\mathscr{Y}_j},\mathbb{Q}}\}_{j\in J}) & \xrightarrow{\;f_+\;} & K(\{F\text{-}\mathscr{D}^{\dagger}_{\mathscr{Y}_j,\mathbb{Q}}\}_{j\in J}) \\ {\scriptstyle cob}\big\downarrow & & {\scriptstyle cob}\big\downarrow & & {\scriptstyle cob}\big\downarrow & & \big\downarrow{\scriptstyle cob} \\ K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_{j,i},\mathbb{Q}}\}) & \xrightarrow[\;i_+\;]{} & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{P}_{j,i},\mathbb{Q}}\}_{j\in J, i\in I_j}) & \xrightarrow[\;\iota\;]{\sim} & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{P}_{\mathscr{Y}_j},\mathbb{Q}}\}_{j\in J}) & \xrightarrow[\;f_+\;]{} & K(\{\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y}_j,\mathbb{Q}}\}_{j\in J}) \end{array}$$

For the squares on the left and right, it suffices to check commutativity of the following diagrams for each $j \in J$ and $i \in I_j$.

$$\begin{array}{ccc} K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{X}_{j,i},\mathbb{Q}}) & \xrightarrow{\;i_+\;} & K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{P}_{j,i},\mathbb{Q}}) \\ {\scriptstyle cob}\big\downarrow & & {\scriptstyle cob}\big\downarrow \\ K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}_{j,i},\mathbb{Q}}) & \xrightarrow[\;i_+\;]{} & K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{P}_{j,i},\mathbb{Q}}), \end{array} \qquad \begin{array}{ccc} K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{P}_{\mathscr{Y}_j},\mathbb{Q}}) & \xrightarrow{\;f_+\;} & K(F\text{-}\mathscr{D}^{\dagger}_{\mathscr{Y}_j,\mathbb{Q}}) \\ {\scriptstyle cob}\big\downarrow & & \big\downarrow{\scriptstyle cob} \\ K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{P}_{\mathscr{Y}_j},\mathbb{Q}}) & \xrightarrow[\;f_+\;]{} & K(\widehat{\mathscr{D}}^{(m)}_{\mathscr{Y}_j,\mathbb{Q}}) \end{array}$$

The verification is the same as the proof of Theorem 2.2.6; for the middle square, the verification is easy.

③ The verification is the same as that for ②, using the proof of Theorem 2.2.6. □

The proof of the following corollary is the same as that of Corollary 2.2.7.

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COROLLARY 2.3.17 (Relative Kashiwara–Dubson formula). *Let $f : X \to Y$ be a proper morphism between smooth schemes over k with d-quadruples. Let $\sigma_X : X \to T^*X$ and $\sigma_Y : Y \to T^*Y$ be zero-sections. Let $\mathscr{E}$ be an overholonomic $\mathscr{D}^\dagger_{X,\mathbb{Q}}$-module, and suppose that $f_+\mathscr{E}$ is an overholonomic module (or that all the cohomology sheaves of $f_+\mathscr{E}$ are overholonomic; see Remark 1.1.8). Then*

$$f_*(\sigma_X^*(\mathrm{ZCar}^\dagger(\mathscr{E}))) = \sigma_Y^*(\mathrm{ZCar}^\dagger(f_+\mathscr{E}))$$

in $\mathrm{CH}_0(Y)_{\mathbb{Q}}$.

2.4 Swan conductors for overholonomic $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-modules

DEFINITION 2.4.1. Let U be a smooth scheme over k with d-quadruples, and let $\mathscr{E}$ be an overholonomic $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-module. We say that $\mathscr{E}$ is a locally projective $\mathcal{O}_{U,\mathbb{Q}}$-module of rank r if

$$\mathrm{ZCar}^\dagger(\mathscr{E}) \cap T^*U = r \cdot [U],$$

where $[U]$ denotes the zero-section. We write $\mathrm{rk}(\mathscr{E})$ for r.

Now we are in a good position to define Swan conductors for $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-modules.

DEFINITION 2.4.2. Let X be a proper smooth scheme with d-quadruples over k, $U \hookrightarrow X$ an open immersion whose complement is a divisor Z, and $\mathscr{E}$ an overholonomic $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-module. We assume that $\mathscr{E}$ is a locally projective $\mathcal{O}_{U,\mathbb{Q}}$-module of rank n. Then we define

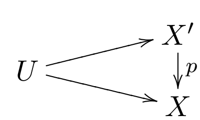
$$\mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E}) := (-1)^d\{n \cdot (\mathrm{ZCar}^\dagger(j_+\mathcal{O}_{X,\mathbb{Q}}(^\dagger Z)) \cdot [X]) - (\mathrm{ZCar}^\dagger(j_+\mathscr{E}) \cdot [X])\} \in \mathrm{CH}_0(X),$$

where $j_+\mathcal{O}_{X,\mathbb{Q}}(^\dagger Z)$ and $j_+\mathscr{E}$ are defined as follows. Let $\mathscr{E}'$ be the realization of $\mathscr{E}$ in a quadruple $(U, X, Z, \mathscr{X})$. Then, by definition, $\mathscr{E}'$ is a $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module satisfying some support conditions. By forgetting the condition $\mathbb{R}\underline{\Gamma}^\dagger_Z\mathscr{E}' = 0$, we get an overholonomic $\mathscr{D}^\dagger_{X,\mathbb{Q}}$-module defined by $\mathscr{E}'$. This overholonomic $\mathscr{D}^\dagger_{X,\mathbb{Q}}$-module is, by definition, $j_+\mathscr{E}$.

Remark 2.4.3.

(i) The functor j_+ is a notation of [Car05c].

(ii) This Swan conductor does not depend on the choice of X in the following sense. Consider the diagram



where X' is a smooth proper scheme with d-quadruples and the two morphisms from U to X and to X' are open immersions whose complement is a divisor. Then for an overholonomic $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-module $\mathscr{E}$, we have $p_*\mathrm{Sw}^{\mathscr{D}}_{X'}(\mathscr{E}) = \mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E})$. This can easily be seen from Lemma 2.4.5 below, upon taking V to be U and Y to be X'.

We collect some properties of $\mathrm{Sw}^{\mathscr{D}}_X$ and compare these properties with those of the Swan conductor defined in [KS08]. We keep the same notation.

LEMMA 2.4.4. *For an exact sequence*

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$$

of overholonomic $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-modules which are locally projective $\mathcal{O}_{U,\mathbb{Q}}$-modules of finite rank, we have $\mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E}) = \mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E}') + \mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E}'')$.

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Proof. This follows easily from the facts that

$$\mathrm{rk}(\mathscr{E}) = \mathrm{rk}(\mathscr{E}') + \mathrm{rk}(\mathscr{E}'') \quad \text{and} \quad \mathrm{ZCar}^\dagger(\mathscr{E}) = \mathrm{ZCar}^\dagger(\mathscr{E}') + \mathrm{ZCar}^\dagger(\mathscr{E}''). \qquad \square$$

Lemma 2.4.5. *Consider the cartesian diagram*

$$\begin{array}{ccc} V & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle \overline{f}} \\ U & \longrightarrow & X \end{array}$$

where all the schemes are proper smooth schemes over k, X and Y are schemes with d-quadruples, $\overline{f}$ is a proper morphism, and f is a finite étale morphism. Suppose that $Z := X\backslash U$ and $W := Y\backslash V$ are divisors. Let $\mathscr{E}$ be an overholonomic $\mathscr{D}^\dagger_{V,\mathbb{Q}}$-module which is a locally projective $\mathcal{O}_{V,\mathbb{Q}}$-module of finite rank. Then we have

$$\mathrm{Sw}^{\mathscr{D}}_X(f_+\mathscr{E}) = \overline{f}_*\mathrm{Sw}^{\mathscr{D}}_Y(\mathscr{E}) + \mathrm{rk}(\mathscr{E}) \cdot \mathrm{Sw}^{\mathscr{D}}_X(f_+\mathcal{O}_{V,\mathbb{Q}})$$

in $\mathrm{CH}_0(X)_{\mathbb{Q}}$.

Proof. For simplicity, let $\alpha_X := (\mathrm{ZCar}^\dagger(\mathcal{O}_{X,\mathbb{Q}}(^\dagger Z)) \cdot [X]) \in \mathrm{CH}_0(X)$. By definition, we have:

- $\mathrm{Sw}^{\mathscr{D}}_X(f_+\mathscr{E}) = (-1)^d\{\mathrm{rk}(\mathscr{E}) \cdot \deg(f) \cdot \alpha_X - \mathrm{ZCar}^\dagger(f_+\mathscr{E}) \cdot [X]\}$;
- $\overline{f}_*\mathrm{Sw}^{\mathscr{D}}_Y(\mathscr{E}) = (-1)^d\overline{f}_*\{\mathrm{rk}\mathscr{E} \cdot \alpha_Y - \mathrm{ZCar}^\dagger(\mathscr{E}) \cdot [Y]\}$;
- $\mathrm{Sw}^{\mathscr{D}}_X(f_+\mathcal{O}_{Y,\mathbb{Q}}(^\dagger W)) = (-1)^d\{\deg(f) \cdot \alpha_X - \mathrm{ZCar}^\dagger(f_+\mathcal{O}_{Y,\mathbb{Q}}(^\dagger W)) \cdot [X]\}$.

By using the relative Kashiwara–Dubson formula, Corollary 2.3.17, we get the lemma. $\square$

2.4.6 Let U be a smooth scheme with d-quadruples, and let $\mathscr{E}$ be an overholonomic $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-module. Let $(U, X, Z, \mathscr{X})$ be a proper d-quadruple, and denote the realization also by $\mathscr{E}$. Let $p : \mathscr{X} \to \mathscr{S}$ be the structure morphism. We define $\chi_{\mathrm{DR}}(\mathscr{E})$ by $\chi_{\mathrm{DR}}(p_+\mathscr{E})$. It can easily be seen by using Theorem 1.1.6 that the definition does not depend on the choice of d-quadruple.

Corollary 2.4.7 (Grothendieck–Ogg–Shafarevich formula). *Let X be a proper smooth scheme with d-quadruples over k, and let U be an open subscheme whose complement is a divisor. Let $\mathscr{E}$ be an overholonomic $\mathscr{D}^\dagger_{U,\mathbb{Q}}$-module and a locally projective $\mathcal{O}_{U,\mathbb{Q}}$-module of rank n. Then we have*

$$\chi_{\mathrm{DR}}(\mathscr{E}) = n \cdot \chi_{\mathrm{DR}}(U) - \deg \mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E}).$$

Proof. This follows directly from the definition of $\mathrm{Sw}^{\mathscr{D}}_X$ and Corollary 2.2.9. $\square$

Remark 2.4.8. $\mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E})$ should be a positive cycle (here positive means that

$$(-1)^d\{n \cdot \mathrm{ZCar}^\dagger(\mathcal{O}_{X,\mathbb{Q}}(^\dagger Z)) - \mathrm{ZCar}^\dagger(\mathscr{E})\} \in \mathrm{CH}_{\dim X}(T^*X)$$

is a positive cycle), but we do not know how to prove this. For certain cases, we know that this is indeed the case by using the main theorem of this paper.

We conclude this section by calculating the Swan conductor for the geometric case.

Theorem 2.4.9. *Consider the same situation as in Lemma 2.4.5. In addition, assume that W and Z are simple normal crossing divisors, and write $Z = \bigcup_{i\in I} Z_i$ and $W = \bigcup_{i\in I'} W_i$ where Z_i and W_i are irreducible components. For $I \supset J \neq \emptyset$, we define $Z_J := \bigcap_{i\in J} Z_i$, and similarly*

https://doi.org/10.1112/S0010437X09004485 Published online by Cambridge University Press

for $W_{J'}$, where $I' \supset J' \neq \emptyset$. Then

$$\begin{aligned}&\mathrm{Sw}^{\mathscr{D}}_X(f_+(\mathcal{O}_{V,\mathbb{Q}}))\\ &\quad = (-1)^d\biggl\{\deg(V/U)\cdot\sigma_X^*\biggl([X]+\sum_{I\supset J\neq\emptyset}[N^*_{Z_J/X}]\biggr)-\overline{f}_*\sigma_Y^*\biggl([Y]+\sum_{I'\supset J'\neq\emptyset}[N^*_{W_{J'}/Y}]\biggr)\biggr\}\end{aligned}$$

in $\mathrm{CH}_0(X)_{\mathbb{Q}}$. Here $d=\dim X$, σ_X and σ_Y are zero-sections of T^*X and T^*Y, and $[N^*_{Z_J/X}]$ ($I\supset J\neq\emptyset$) denotes the d-dimensional cycle in T^*X defined by the conormal bundle of Z_J in X and similarly for $[N^*_{W_{J'}/Y}]$ with $J'\subset I'$.

Proof. Since f is finite étale, $f_+\mathcal{O}_{V,\mathbb{Q}}$ is an overholonomic module, and we may use Corollary 2.3.17 to reduce the proof of the theorem to showing the following lemma.

Lemma 2.4.10. *Let X be a proper smooth scheme with d-quadruples over k, and let U be an open subscheme with $Z:=X-U$ being a simple normal crossing divisor. Let $Z=\bigcup_{i\in I}Z_i$ where the Z_i are irreducible components. Then we have*

$$\mathrm{ZCar}^\dagger(\mathcal{O}_{X,\mathbb{Q}}({}^\dagger Z)) = [X]+\sum_{I\supset J\neq\emptyset}[N^*_{Z_J/X}]$$

in $\mathrm{CH}_d(T^*X)$ (or even in $Z_d(T^*X)$).

Proof. Since $\mathrm{ZCar}^\dagger(\mathscr{E})$ is defined in $Z_d(T^*X)$, we may calculate locally. Thus we are reduced to showing the following claim.

Claim. Let $\mathscr{X}$ be a smooth formal scheme over $\mathscr{S}$, and let $Z\subset X$ be a simple normal crossing divisor where X denotes the special fiber of $\mathscr{X}$. Let $Z=\bigcup_{i\in I}Z_i$ where the Z_i are irreducible components of Z. For $J\subset I$, let $Z_J:=\bigcap_{i\in J}Z_i$. Assume that Z_J can be lifted to a smooth formal closed subscheme $\mathscr{Z}_J$ of $\mathscr{X}$ over $\mathscr{S}$. Then we have

$$\mathrm{ZCar}^\dagger(\mathcal{O}_{\mathscr{X},\mathbb{Q}}({}^\dagger Z)) = [X]+\sum_{I\supset J\neq\emptyset}[N^*_{Z_J/X}]$$

in $Z_d(T^*X)$.

By definition, there exists the distinguished triangle

$$\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathscr{X},\mathbb{Q}})\to\mathcal{O}_{\mathscr{X},\mathbb{Q}}\to\mathcal{O}_{\mathscr{X},\mathbb{Q}}({}^\dagger Z)\to .$$

This reduces the problem to calculating $\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathscr{X},\mathbb{Q}})$. Since there are Mayer–Vietoris exact sequences for $\mathbb{R}\underline{\Gamma}^\dagger_Z$ (i.e. the distinguished triangle in [Car05a, Théorème 2.2.16]), all we have to calculate is $\mathbb{R}\underline{\Gamma}^\dagger_{Z_J}(\mathcal{O}_{\mathscr{X},\mathbb{Q}})$. Let $i_J:\mathscr{Z}_J\hookrightarrow\mathscr{X}$ be the closed immersion, and let d_J be the codimension of Z_J in X. Then we have $i^!_J(\mathcal{O}_{\mathscr{X},\mathbb{Q}})=\mathcal{O}_{\mathscr{Z}_J}[d_J]$. Indeed, $i^!_J(\mathcal{O}_{\mathscr{X},\mathbb{Q}})$ equals $\mathbb{L}i^*_J(\mathcal{O}_{\mathscr{X},\mathbb{Q}})[d_J]$ as a $\mathcal{O}_{\mathscr{Z}_J,\mathbb{Q}}$-module, and we have $\mathscr{H}_j i^!_J(\mathcal{O}_{\mathscr{X},\mathbb{Q}})=0$ for $j\neq d_J$ and $\mathscr{H}_{d_J}i^!_J(\mathcal{O}_{\mathscr{X},\mathbb{Q}})=\mathcal{O}_{\mathscr{Z}_J}$. Let

$$g:T^*X\times_X Z_J\to T^*Z_J\quad\text{and}\quad \bar{i}_J:T^*X\times_X Z_J\to T^*X$$

be the canonical morphisms. Then we have

$$\begin{aligned}\mathrm{ZCar}^\dagger(\mathbb{R}\underline{\Gamma}^\dagger_{Z_J}(\mathcal{O}_{\mathscr{X},\mathbb{Q}})) &= \mathrm{ZCar}^\dagger(i_{J+}i^!_J(\mathcal{O}_{\mathscr{X},\mathbb{Q}}))\\ &= \mathrm{ZCar}^\dagger(i_{J+}(\mathcal{O}_{\mathscr{Z}_J,\mathbb{Q}}[d_J]))\\ &= \bar{i}_{J*}g^*\,\mathrm{ZCar}^\dagger(\mathcal{O}_{\mathscr{Z}_J,\mathbb{Q}}[d_J])\\ &= (-1)^{d_J}[N^*_{Z_J/X}],\end{aligned}$$

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where the third equality follows from Proposition 2.2.10 and the last equality from the short exact sequence

$$0 \to N^*_{Z_J/X} \to T^*X \times_X Z_J \to T^*Z_J \to 0$$

together with the claim below.

Claim. Let $\mathscr{X}$ be a smooth formal scheme of dimension d over $\mathscr{S}$. Then we have $\mathrm{ZCar}^\dagger(\mathcal{O}_{\mathscr{X},\mathbb{Q}}) = [X] \in Z_d(T^*X)$, where X denotes the special fiber of $\mathscr{X}$ and $[X]$ denotes the zero-section.

We know that $F^*\mathcal{O}_{\mathscr{X},\mathbb{Q}} \cong \widehat{\mathscr{D}}^{(m+1)}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \mathcal{O}_{\mathscr{X},\mathbb{Q}}$. Thus, for $p^m > e/(p-1)$, the Frobenius descent of $\mathcal{O}_{\mathscr{X},\mathbb{Q}}$ is $\mathcal{O}_{\mathscr{X},\mathbb{Q}}$ as a $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module. Take the coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module $\mathcal{O}_{\mathscr{X}}$ as its lifting. Thus we are reduced to calculating the characteristic variety for the $\mathscr{D}^{(m)}_X$-module $\mathcal{O}_X$. Since we have the canonical Frobenius structure on $\mathcal{O}_X$, we need only calculate the characteristic variety for the $\mathscr{D}^{(0)}_X$-module $\mathcal{O}_X$, and this is easy.

Therefore, Lemma 2.4.10 is proved. □

With this lemma in hand, Theorem 2.4.9 is also established. □

3. Calculation of the Swan conductor defined by Kato and Saito

In this section, we calculate the Swan conductor in the sense of Kato and Saito in the case where the boundary is a simple normal crossing divisor. We use the theory of intersection due to Fulton.

Definition 3.0.1 [KS08, Definition 1.1.1]. Let X be a smooth scheme over a perfect field k, and let $X \supset Z$ be a simple normal crossing divisor. Let $Z = \bigcup_{i\in I} Z_i$ $(I = \{1, 2, \dots, n\})$ where the Z_i are irreducible components. For $i \in I$, let $(X \times X)'_i$ be the blow-up of $X \times X$ along $Z_i \times Z_i$, and take $(X \times X)^\sim$ to be the complement of the strict transforms of $Z_i \times X$ and $X \times Z_i$ in $(X \times X)'_i$. We define the log blow-up $(X \times X)'$ to be the fiber product of $(X \times X)'_i$ over $X \times X$ for $i \in I$. We define the log product $(X \times X)^\sim$ to be the fiber product of $(X \times X)^\sim_i$ over $X \times X$ for $i \in I$. We denote the diagonals by $\Delta : X \to X \times X$, $\Delta^{\log} : X \to (X \times X)^\sim$.

Remark 3.0.2. Using the notation in Definition 3.0.1, for $J \subset I$ we denote by $(X \times X)'_J$ the log-blow up with respect to the divisor $\bigcup_{i\in J} Z_i$. Let $J \subsetneq I$, take $i \in I$ such that $i \notin J$, and let $J' := J \cup \{i\}$. Then we see easily from the universal property of blowing-up that the canonical homomorphism

$$(X \times X)'_{J'} \to (X \times X)'_J$$

is the blow-up of $(X \times X)'_J$ along the strict transformation of $Z_i \times Z_i$.

3.0.3 Consider the following cartesian diagram.

$$\begin{array}{ccc} V & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle \bar{f}} \\ U & \longrightarrow & X \end{array}$$

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Here, all the schemes are smooth over k, $U \hookrightarrow X$ is an open immersion, and the complements $W := Y - V$ and $Z := X - U$ are assumed to be simple normal crossing divisors. Furthermore, we assume f to be a finite étale morphism and $\overline{f}$ to be a proper morphism. Let $\overline{f}^{\log} : (Y \times Y)^{\sim} \to (X \times X)^{\sim}$ be the canonical morphism. Then we define

$$D^{\log}_{V/U,Y} := \Delta_Y^{\log *}(\overline{f}^{\log *}([X]) - [Y]) \in \mathrm{CH}_0(Y)$$

and call it the wild different (cf. [KS08, Proposition 3.4.10]). For a more thorough treatment of this concept, see [KS08, Definition 3.4.1.1]. We define the discriminant $d^{\log}_{V/U,X} \in \mathrm{CH}_0(X)$ as $\overline{f}_*(D^{\log}_{V/U,Y})$.

Now let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_\ell$-lisse sheaf over U. In [KS08, Definition 4.2.8], the Swan conductor $\mathrm{Sw}^{\mathrm{KS}}_X(\mathcal{F}) \in \mathrm{CH}_0(X)_{\mathbb{Q}}$ is defined. We do not review the definition here and content ourselves with listing a few properties of this Swan conductor.

PROPERTIES 3.0.4.

(i) $\mathrm{Sw}^{\mathrm{KS}}_X(f_*\overline{\mathbb{Q}_\ell}) = d^{\log}_{V/U,X}$ (cf. [KS08, Corollary 4.3.4]).

(ii) $\mathrm{Sw}^{\mathrm{KS}}_X$ is additive with respect to exact sequences (cf. [KS08, Lemma 4.2.4]).

(iii) Let $\chi : \pi_1(U) \to \overline{\mathbb{Q}_\ell}^{\times}$ be a character factoring through a finite group and let $\mathcal{F}(\chi)$ be a lisse sheaf corresponding to the character. For i prime to p, we have $\mathrm{Sw}^{\mathrm{KS}}_X(\mathcal{F}(\chi)) = \mathrm{Sw}^{\mathrm{KS}}_X(\mathcal{F}(\chi^i))$ (by definition).

The main result of this section is the next theorem, whose proof will be completed at the end of this section.

THEOREM 3.0.5. *Consider the same situation as above. Let $Z = \bigcup_{i\in I} Z_i$ and $W = \bigcup_{i\in I'} W_i$, where Z_i and W_i are irreducible components. For $I \supset J \neq \emptyset$, we define $Z_J := \bigcap_{i\in J} Z_i$, and similarly for $W_{J'}$, where $I' \supset J' \neq \emptyset$. Then we have*

$$D^{\log}_{V/U,Y} = (-1)^d \bigg\{ \overline{f}^* \sigma_X^* \bigg([X] + \sum_{I \supset J \neq \emptyset} [N^*_{Z_J/X}] \bigg) - \sigma_Y^* \bigg([Y] + \sum_{I' \supset J' \neq \emptyset} [N^*_{W_{J'}/Y}] \bigg) \bigg\}.$$

Here $d = \dim X$, σ_X and σ_Y are zero-sections of T^*X and T^*Y, and $[N^*_{Z_J/X}]$ $(I \supset J \neq \emptyset)$ denotes the d-dimensional cycle in T^*X defined by the conormal bundle of Z_J in X and similarly for $[N^*_{W_{J'}/Y}]$.

To establish this theorem, we first prove the following proposition.

PROPOSITION 3.0.6. *Let X be a smooth scheme, and let $X \supset Z$ be a simple normal crossing divisor such that $Z = \bigcup_{i\in I} Z_i$ where Z_i are the irreducible components of Z. Let $Z_J := \bigcap_{i\in J} Z_i$. Then*

$$\Delta^*([X]) = \Delta^{\log *}([X]) + (-1)^{d+1} \sigma^* \bigg(\sum_{I \supset J \neq \emptyset} [N^*_{Z_J/X}] \bigg)$$

in $\mathrm{CH}_0(X)$, where σ denotes the zero-section $X \to T^*X$.

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Proof. Let $d = \dim X$. For $J \subset I$, we denote by $(X \times X)^{\sim}_J$ and $(X \times X)'_J$ the log product and log blow-up, respectively, with respect to the divisor $\bigcup_{i \in J} Z_i$. Consider the following diagram.

$$\begin{array}{ccc} & & (X \times X)' \\ & & \downarrow{\scriptstyle q_J} \\ (Z_J \times Z_J)'_J & \xrightarrow{j_J} & (X \times X)'_J \\ {\scriptstyle g_J}\downarrow & \square & \downarrow{\scriptstyle p_J} \\ Z_J \times Z_J & \longrightarrow & X \times X \end{array} \qquad p = p_I = q = q_\emptyset : (X \times X)' \to X \times X$$

Here, $(Z_J \times Z_J)'_J$ is the pull-back of $Z_J \times Z_J$ in $(X \times X)'_J$. By definition, $(Z_J \times Z_J)'_J \to Z_J \times Z_J$ is a composition of projective bundles. We define $Z_\emptyset = X$ and $(X \times X)'_\emptyset = X \times X$.

Claim.

$$q_J^*([g_J^{-1}(Z_J)]) = \sum_{I \supset K \supset J} [(g_K^{-1}(Z_K))^{\sim}].$$

Here, $(g_K^{-1}(Z_K))^{\sim}$ denotes the strict transform of $g_K^{-1}(Z_K) \subset (X \times X)'_K$ in $(X \times X)'$.

To verify this claim, we shall prove a more general statement. For $I \supset J' \supset J$, we will show that

$$q_{J'}^*([(g_J^{-1}(Z_J))^{\sim J'}]) = \sum_{J' \cap K = J} [(g_K^{-1}(Z_K))^{\sim}],$$

where $(g_J^{-1}(Z_J))^{\sim J'}$ denotes the strict transform of $g_J^{-1}(Z_J) \subset (X \times X)'_J$ in $(X \times X)'_{J'}$. The claim corresponds to the special case where $J' = J$. We prove the above statement using descending induction on $\#J$. The statement is trivial for $J = I$. Now, fixing J, we again use descending induction on $\#J'$. The statement is trivial for $J' = I$. Choose an $i \in I$ such that $i \notin J'$. Let

$$J'' = J' \cup \{i\} \quad \text{and} \quad \alpha : (X \times X)'_{J''} \to (X \times X)'_{J'},$$

the canonical morphism. We will calculate $\alpha^*([(g_J^{-1}(Z_J))^{\sim J'}])$.

We define $(Z_i \times Z_i)'$ by the following cartesian diagram.

$$\begin{array}{ccc} (Z_i \times Z_i)' & \xrightarrow{j} & (X \times X)'_{J''} \\ {\scriptstyle \alpha'}\downarrow & \square & \downarrow{\scriptstyle \alpha} \\ p_{J'}^{-1}(Z_i \times Z_i) & \longrightarrow & (X \times X)'_{J'} \end{array}$$

Note that α is the blow-up of $(X \times X)'_{J'}$ along $p_{J'}^{-1}(Z_i \times Z_i)$ by definition. Let V be a scheme and W a closed subscheme of V over k. We denote the Segre class of W in V by $s(W, V)$ (see [Ful98, ch. 4]). If $W \hookrightarrow V$ is a regular closed immersion, then $s(W, V) = c(N_{W/V})^{-1} \cap [W]$.

Since $g_J^{-1}(Z_J) \cap p_J^{-1}(Z_{j'} \times Z_{j'})$ are divisors in $g_J^{-1}(Z_J)$ for $j' \notin J$, the morphism

$$g_J^{-1}(Z_J)^{\sim J'} \to g_J^{-1}(Z_J)$$

induced by the canonical projection $(X \times X)'_{J'} \to (X \times X)'_J$ is an isomorphism. Thus the morphism

$$g_J^{-1}(Z_J)^{\sim J'} \to Z_J \tag{3.0.6.1}$$

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is smooth of relative dimension $\#J$, and $\dim(g_J^{-1}(Z_J)^{\sim J'}) = d$. By [Ful98, Theorem 6.7], one can write

$$\begin{aligned}\alpha^*([(g_J^{-1}(Z_J))^{\sim J'}]) &= [((g_J^{-1}(Z_J))^{\sim J'})^{\sim J''}] \\ &\quad + j_*\{c(E) \cap \alpha'^* s((g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i),\ (g_J^{-1}(Z_J))^{\sim J'})\}_d\end{aligned} \quad ①$$

using some vector bundle E on $(Z_i \times Z_i)'$. By definition,

$$((g_J^{-1}(Z_J))^{\sim J'})^{\sim J''} = (g_J^{-1}(Z_J))^{\sim J''}. \quad ②$$

On the other hand, consider the following cartesian diagram.

$$\begin{array}{ccc} (g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i) & \longrightarrow & g_J^{-1}(Z_J)^{\sim J'} \\ \downarrow \beta & \square & \downarrow \\ Z_{J\cup\{i\}} & \longrightarrow & Z_J \end{array}$$

Since (3.0.6.1) is smooth of relative dimension $\#J$, we get that the upper horizontal morphism is a regular immersion, and that

$$\begin{aligned} &s((g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i),\ (g_J^{-1}(Z_J))^{\sim J'}) \\ &\quad = c(\beta^* N_{Z_{J\cup\{i\}}/Z_J})^{-1} \cap [(g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i)]. \end{aligned}$$

Since the dimension of $Z_{J\cup\{i\}}$ is $d-1-\#J$, the dimension of $(g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i)$ is $d-1-\#J+\#J = d-1$, and considering that α' is flat of relative dimension one, we conclude that

$$\dim \alpha'^{-1}((g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i)) = d. \quad (*)$$

On the other hand, we have

$$\alpha'^{-1}((g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i)) = (g_{J\cup\{i\}}^{-1}(Z_{J\cup\{i\}}))^{\sim J''}.$$

Consider the following cartesian diagram.

$$\begin{array}{ccc} (g_{J\cup\{i\}}^{-1}(Z_{J\cup\{i\}}))^{\sim J''} & \xrightarrow{i'} & (Z_i \times Z_i)' \\ \downarrow \alpha'' & \square & \downarrow \alpha' \\ (g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i) & \xrightarrow[i]{} & p_{J'}^{-1}(Z_i \times Z_i) \end{array}$$

We have

$$\begin{aligned} &\{c(E) \cap \alpha'^* s((g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i),\ (g_J^{-1}(Z_J))^{\sim J'})\}_d \\ &\quad = \{c(E) \cap i'_* c(\alpha''^* \beta^* N_{Z_{J\cup\{i\}}/Z_J})^{-1} \cap [\alpha'^{-1}\{(g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i)\}]\}_d \\ &\quad = [\alpha'^{-1}((g_J^{-1}(Z_J))^{\sim J'} \cap p_{J'}^{-1}(Z_i \times Z_i))] = [(g_{J\cup\{i\}}^{-1}(Z_{J\cup\{i\}}))^{\sim J''}], \end{aligned} \quad ③$$

where the first equality holds because α' is flat and the second holds by $(*)$ above. Combining ①, ② and ③, we have

$$\alpha^*([(g_J^{-1}(Z_J))^{\sim J'}]) = [(g_J^{-1}(Z_J))^{\sim J''}] + [(g_{J\cup\{i\}}^{-1}(Z_{J\cup\{i\}}))^{\sim J''}].$$

Thus,

$$\begin{aligned} q^*_{J'}([(g_J^{-1}(Z_J))^{\sim J'}]) &= q^*_{J''}\alpha^*([(g_J^{-1}(Z_J))^{\sim J'}]) \\ &= q^*_{J''}([(g_J^{-1}(Z_J))^{\sim J''}] + [(g_{J\cup\{i\}}^{-1}(Z_{J\cup\{i\}}))^{\sim J''}]) \\ &= \sum_{J''\cap K=J} [(g_K^{-1}(Z_K))^{\sim}] + \sum_{J''\cap K=J\cup\{i\}} [(g_K^{-1}(Z_K))^{\sim}] \\ &= \sum_{J'\cap K=J} [(g_K^{-1}(Z_K))^{\sim}], \end{aligned}$$

where we have used the induction hypothesis in the third equality. The claim then follows.

Now let $\Delta_J^{\log} : X \to (X \times X)'_J$. Then we make the following assertion.

Claim.

$$\Delta_J^{\log\,*}([g_J^{-1}(Z_J)]) = (-1)^{d+\#J}\sigma^*([N^*_{Z_J/X}]).$$

Consider the following cartesian diagram.

$$\begin{array}{ccc} Z_J & \xrightarrow{\ k_J\ } & X \\ {\scriptstyle \Delta^{\log}_{Z_J}}\downarrow & \square & \downarrow{\scriptstyle \Delta^{\log}_J} \\ (Z_J \times Z_J)'_J & \xrightarrow{\ j_J\ } & (X \times X)'_J \\ {\scriptstyle g_J}\downarrow & \square & \downarrow{\scriptstyle p_J} \\ Z_J \times Z_J & \longrightarrow & X \times X \end{array}$$

We know that $Z_J \to (Z_J \times Z_J)'_J$ and $X \to (X \times X)'_J$ are regular closed immersions with the same codimension. Thus, by [Ful98, Theorem 6.2(a) and (c)], we have $\Delta_J^{\log\,*} j_{J*} = k_{J*}\Delta_{Z_J}^{\log\,*}$. We therefore have

$$\begin{aligned} \Delta_J^{\log\,*}([g_J^{-1}(Z_J)]) &= \Delta_J^{\log\,*} j_{J*} g_J^*([Z_J]) = k_{J*}\Delta_{Z_J}^{\log\,*} g_J^*([Z_J]) \\ &= k_{J*}\Delta^*_{Z_J}([Z_J]) = k_{J*}\{c(TZ_J) \cap [Z_J]\}_0, \end{aligned}$$

where Δ_{Z_J} denotes the diagonal morphism $Z_J \to Z_J \times Z_J$ and the last equality holds by [Ful98, Proposition 6.1(a)]. Now, since the dimension of Z_J is $d - \#J$, we get that

$$\{c(TZ_J) \cap [Z_J]\}_0 = (-1)^{d-\#J}\{c(T^*Z_J) \cap [Z_J]\}_0$$

by [Ful98, Remark 3.2.3(a)]. Since there is an exact sequence

$$0 \to N^*_{Z_J/X} \to T^*X \times_X Z_J \to T^*Z_J \to 0,$$

we obtain

$$\begin{aligned} k_{J*}\{c(T^*Z_J) \cap [Z_J]\}_0 &= k_{J*}\{c(T^*X \times_X Z_J) \cap c(N^*_{Z_J/X})^{-1} \cap [Z_J]\}_0 \\ &= \sigma^*([N^*_{Z_J/X}]), \end{aligned}$$

where the second equality holds by [Ful98, Proposition 6.1(a)]. Combining these facts, we conclude the proof of the claim.

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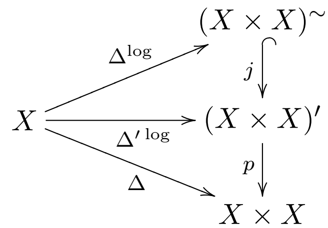
Now we are ready to complete the proof of the proposition. Consider the following diagram.



Using the first claim with $J=\emptyset$, we have

$$\begin{aligned}\Delta^*([X]) &= \Delta^{\log *} j^* p^*([X]) \\ &= \Delta^{\log *}([X]) + \Delta'^{\log *}\biggl(\sum_{I\supset J\neq\emptyset}[(g_J^{-1}(Z_J))^\sim]\biggr).\end{aligned}$$

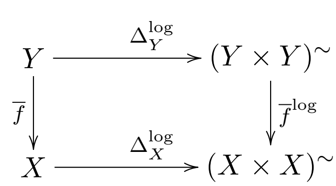
Now, by the fact that $\Delta'^{\log *} q_J^* = \Delta_J^{\log *}$ and using the first claim, we have

$$\begin{aligned}\Delta'^{\log *}\biggl(\sum_{I\supset J\neq\emptyset}[(g_J^{-1}(Z_J))^\sim]\biggr) &= \Delta'^{\log *}\biggl(\sum_{I\supset J\neq\emptyset}(-1)^{\#J+1}\sum_{I\supset K\supset J}[(g_K^{-1}(Z_K))^\sim]\biggr) \\ &= \Delta'^{\log *}\biggl(\sum_{I\supset J\neq\emptyset}(-1)^{\#J+1}\ q_J^*([g_J^{-1}(Z_J)])\biggr) \\ &= \sum_{I\supset J\neq\emptyset}(-1)^{\#J+1}\ \Delta_J^{\log *}([g_J^{-1}(Z_J)]).\end{aligned}$$

Then, using the second claim, we conclude the proof. □

This proposition gives us the following calculation.

Proof of Theorem 3.0.5. By considering the commutative diagram



we get

$$D^{\log}_{V/U,Y} = \Delta_Y^{\log *}(\overline{f}^{\log *}([X]) - [Y]) = \overline{f}^*\Delta_X^{\log *}([X]) - \Delta_Y^{\log *}([Y]).$$

By Proposition 3.0.6, we have

$$\begin{aligned}\Delta_X^{\log *}([X]) &= \Delta_X^*([X]) + (-1)^d\sum_{I\supset J\neq\emptyset}\sigma_X^*([N^*_{Z_J/X}]) \\ &= (-1)^d\sigma_X^*\biggl([X] + \sum_{I\supset J\neq\emptyset}[N^*_{Z_J/X}]\biggr).\end{aligned}$$

The same calculation for $\Delta_Y^{\log *}([Y])$ then gives us the theorem. □

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4. Comparison of Swan conductors

4.0.1 We fix isomorphisms $\overline{\mathbb{Q}}_\ell \cong \mathbb{C} \cong \overline{\mathbb{Q}}_p$. We assume that $\sigma(\pi) = \pi$ and let Λ be the subfield of K fixed by σ unless otherwise stated. We suppose that $\zeta_n \in \Lambda$, where ζ_n denotes a primitive root of unity for some integer n.

Before going to the main theorem, we prove the theorem for the geometric case, which can also be seen as a corollary of the main theorem.

Theorem 4.0.2. *Consider the following cartesian diagram.*

$$\begin{array}{ccc} V & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle \overline{f}} \\ U & \longrightarrow & X \end{array}$$

Here, X and Y are proper smooth schemes over k with d-quadruples, $U \hookrightarrow X$ is an open immersion, and the complements $W := Y - V$ and $Z := X - U$ are assumed to be simple normal crossing divisors. Furthermore, we assume f to be a finite étale morphism and $\overline{f}$ to be a proper morphism. Then we have

$$\mathrm{Sw}^{\mathrm{KS}}_X(f_*\overline{\mathbb{Q}}_\ell) = d^{\log}_{V/U,X} = \mathrm{Sw}^{\mathscr{D}}_X(f_+\mathcal{O}_{V,\mathbb{Q}})$$

in $\mathrm{CH}_0(X)_{\mathbb{Q}}$.

Proof. This follows immediately from Theorems 2.4.9 and 3.0.5, together with the fact that $\overline{f}_*\overline{f}^* = \deg(V/U)$ for Chow groups. □

We introduce some more notation.

Definition 4.0.3. Let X be a smooth projective scheme over k, and let U be an open subscheme whose complement in X is a divisor. Let $\chi : \pi_1(U) \to \mathbb{C}^\times$ be a character of finite order m such that $m|n$.

(i) Let $\chi_\ell : \pi_1(U) \to \overline{\mathbb{Q}}_\ell^\times$ be a character arising from χ via the fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$. We denote the corresponding $\overline{\mathbb{Q}}_\ell$-lisse sheaf over U by $\mathcal{F}(\chi)$.

(ii) Let $\chi_p : \pi_1(U) \to \Lambda^\times$ be a character arising from χ via the fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. We denote $\mathrm{sp}_+G^\dagger(\chi_p)$ (see § 1.3) by $\mathscr{E}(\chi)$.

Recall the following hypothesis.

Resolution of singularities. Let X be a scheme of finite type over k, and let U be a dense open subscheme that is smooth over k. Then there exists a *projective* morphism $f : X' \to X$ such that X' is smooth, $f^{-1}(U) \to U$ is an isomorphism, and the complement of $f^{-1}(U)$ in X' is a simple normal crossing divisor.

Lemma 4.0.4. *Let X be a smooth scheme over k, and let U be an open subscheme. Assuming resolution of singularities, (U, X) is resolvable (cf. Definition 0.0.1).*

Proof. Let V be a scheme that is finite étale over U. Since V is finite over U, there exists an immersion $i : V \hookrightarrow \mathbb{P}^n \times X$. Take Y' to be the closure of V in $\mathbb{P}^n \times X$. Then Y' is a projective

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scheme over X and there exists a cartesian diagram as follows.

$$\begin{CD} V @>>> Y' \\ @VVV \square @VVV \\ U @>>> X \end{CD}$$

By using the resolution of singularities, we may find a projective morphism $Y \to Y'$ such that Y is smooth, $V \to Y'$ factors through $Y \to Y'$, and $Y \backslash V$ is a simple normal crossing divisor, which is what we wanted to prove. $\square$

THEOREM 4.0.5. *Let X be a projective smooth scheme over k, and let $U \hookrightarrow X$ be an open immersion whose complement Z is a simple normal crossing divisor. Suppose that (U, X) is resolvable. Let $\chi : \pi_1(U) \to \mathbb{C}^\times$ be a character of finite order p^i for some $i \geqslant 0$ such that $p^i|n$. Then we have*

$$\mathrm{Sw}_X^{\mathrm{KS}}(\mathcal{F}(\chi)) = \mathrm{Sw}_X^{\mathscr{D}}(\mathscr{E}(\chi))$$

in $\mathrm{CH}_0(X)_{\mathbb{Q}}$.

Proof. The idea of the proof is the same as that of [KS08, Proposition 5.1.4]. We may suppose that χ is not the trivial character; for the trivial character, both sides of the equality are zero, and we have nothing to prove. We make the following two claims.

CLAIM. Consider a map Sw from the set of characters $\chi : \pi_1(U) \to \mathbb{C}^\times$ factoring through a finite cyclic group $\mathbb{Z}/p^i\mathbb{Z}$ to $\mathrm{CH}_0(X)_{\mathbb{Q}}$ that satisfies the following two conditions.

(i) $\mathrm{Sw}(\chi) = \mathrm{Sw}(\chi^j)$ for j prime to p.

(ii) Let χ be a character of order p^j (with $j \leqslant i$). Let V be a finite étale covering of U corresponding to the kernel of χ. Then

$$\mathrm{Sw}(\mathbf{1}) + \mathrm{Sw}(\chi) + \cdots + \mathrm{Sw}(\chi^{p^j-1}) = d^{\log}_{V/U,X},$$

where $\mathbf{1}$ denotes the trivial character.

Then Sw is uniquely determined.

CLAIM. $\mathrm{Sw}_X^{\mathscr{D}}(\mathscr{E}(\chi)) = \mathrm{Sw}_X^{\mathscr{D}}(\mathscr{E}(\chi^j))$ for j prime to p.

The first claim is easy to verify. We shall prove only the second claim. To do this, we have to show that $\mathrm{ZCar}^\dagger(\mathscr{E}(\chi)) = \mathrm{ZCar}^\dagger(\mathscr{E}(\chi^j))$. Since $\mathrm{ZCar}^\dagger$ is defined on the level of cycles (not only in Chow groups), the problem is local. Since the problem is local, we may assume that X can be lifted to some smooth formal scheme $\mathscr{X}'$ over $W(k)$. There exists an automorphism ι of R (i.e. a valuation ring of K) over $W(k)$ sending ζ_{p^i} to $\zeta_{p^i}^j$ which is compatible with Frobenius structure. (Indeed, let $\Lambda' := R^\sigma$; then $R = \Lambda' \otimes_{\mathbb{Z}_p} W(k)$. Take ι' to be an isomorphism of Λ' over $\mathbb{Z}_p$ sending ζ_{p^i} to $\zeta_{p^i}^j$. Now we may define ι to be $\iota' \otimes \mathrm{id}$. Since $\sigma = \mathrm{id} \otimes F$ where $F : W(k) \to W(k)$ is the canonical Frobenius, the compatibility of σ and ι is obvious.) We define $\mathscr{X}$ to be $\mathscr{X}' \otimes_{W(k)} R$. Then we have the cartesian diagram

$$\begin{CD} \mathscr{X} @>{f}>> \mathscr{X} \\ @VVV @VVV \\ \mathrm{Spf}(R) @>{\iota}>> \mathrm{Spf}(R) \end{CD}$$

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where $f = \mathrm{id}_{\mathscr{X}'} \otimes \iota$. Since ι induces the identity on the residue field of R, f induces the identity on the special fiber of $\mathscr{X}$. Let $V := K$ be the one-dimensional vector space and $\widetilde{\chi} : \pi_1(U) \to \mathrm{GL}(V)$ the representation induced by χ_p. Then we get that the following diagram is commutative.

$$\begin{array}{ccc} \pi_1(U) & \xrightarrow{1 \otimes \widetilde{\chi}} & \mathrm{GL}(K \otimes_{\iota, K} V) \\ & \searrow^{\widetilde{\chi^j}} & \downarrow \sim \\ & & \mathrm{GL}(V) \end{array}$$

Thus, by (1.4.3.1), we get $(\mathscr{E}(\chi))^{\iota} = \mathscr{E}(\chi^j)$ (using the notation of Lemma 2.2.12). Using Lemma 2.2.12, we then get the second claim.

Now it is easy to finish the proof of the main theorem. For a character χ as in the first claim, we have two Swan conductors $\mathrm{Sw}^{\mathrm{KS}}_X(\mathcal{F}(\chi))$ and $\mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E}(\chi))$. By the uniqueness given by the first claim, all we have to do is check the conditions of the first claim for both Swan conductors.

For $\mathrm{Sw}^{\mathrm{KS}}_X$, see Properties 3.0.4. For $\mathrm{Sw}^{\mathscr{D}}_X$, the first condition is just the second claim. Let us look at the second condition. By the assumption that (U, X) is resolvable, there exists a cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle g} \\ U & \longrightarrow & X \end{array}$$

such that Y is smooth and g is projective. Since X is projective, Y is projective, and Y is a scheme with d-quadruples. Now, we get

$$\sum_{k=0}^{p^j-1} \mathrm{Sw}^{\mathscr{D}}_X(\mathscr{E}(\chi^k)) = \mathrm{Sw}^{\mathscr{D}}_X(f_+ \mathcal{O}_{V,\mathbb{Q}}) = d^{\log}_{V/U,X},$$

where the first equality comes from combining Lemmas 1.4.6 and 2.4.4, and the second from Theorem 4.0.2 for the above U, X, V and Y. Therefore, the second condition holds, and the theorem is proved. □

Remark 4.0.6. It seems that Theorem 4.0.5 is true not only for p-groups but also for modules coming from representations factoring through finite groups. To prove it in the general case, we need a more detailed study of characteristic cycles. More precisely, we need to prove that if $\mathscr{E}$ is a holonomic $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module which is also a coherent $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}({}^{\dagger}Z)$-module whose restriction to $\mathscr{U} := \mathscr{X} \setminus Z$ is a locally free $\mathcal{O}_{\mathscr{U},\mathbb{Q}}$-module of rank one, then the characteristic variety of $\mathscr{E}$ contains $[X] + \sum [N_{Z_J/X}]$. This is equivalent to saying that Swan conductor is a positive cycle (cf. Remark 2.4.8). It is easy to show that the characteristic variety is contained in the above variety, but it is difficult to show that they are equal.

Corollary 4.0.7. *Assume the resolution of singularities. Let X be a smooth projective scheme over k and let U be an open subscheme whose complement is a simple normal crossing divisor. Then, for any $\mathbb{Q}_\ell$-lisse sheaf $\mathcal{F}$ on U, $\mathrm{Sw}^{\mathrm{KS}}_X(\mathcal{F})$ is in the image of the canonical homomorphism* $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X)_{\mathbb{Q}}$.

Proof. By [KS08, Lemma 4.3.8], we may assume $\mathcal{F}$ to be a $\mathbb{F}_\ell$-lisse sheaf of rank one. Thus we are reduced to proving the assertion for a sheaf $\mathcal{F}$ coming from a character $\pi_1(U) \to \overline{\mathbb{Q}}_\ell^{\times}$ factoring through a finite group. Hence there is a finite étale covering V such that $G := \mathrm{Gal}(V/U)$ is a cyclic group and $\mathcal{F}$ is trivialized by V. For an abelian group H, we denote by $H_{(p)}$ the subgroup of elements of order p. We may write $G = G_{(p)} \times G'$ such that the order of G' is prime to p. Let $\mathcal{F}'$ be a sheaf on U corresponding to the character $G \twoheadrightarrow G_{(p)} \hookrightarrow G \to \overline{\mathbb{Q}}_\ell^\times$. For sheaves $\mathcal{G}, \mathcal{G}'$ corresponding to two characters $\phi, \psi : G \to \overline{\mathbb{Q}}_\ell^\times$ such that $\phi|_{G_{(p)}} = \psi|_{G_{(p)}}$, we have $\mathrm{Sw}_X^{\mathrm{KS}}(\mathcal{G}) = \mathrm{Sw}_X^{\mathrm{KS}}(\mathcal{G}')$ by the definition of $\mathrm{Sw}_X^{\mathrm{KS}}$. Thus we have $\mathrm{Sw}_X^{\mathrm{KS}}(\mathcal{F}) = \mathrm{Sw}_X^{\mathrm{KS}}(\mathcal{F}')$, and we are reduced to proving the assertion in the case where G is a commutative p-group. In this case, let q be the number of elements of G. Let $\Lambda := \mathbb{Q}_p(\zeta_q)$. Note that this is totally ramified over $\mathbb{Q}_p$. Then let $K := \Lambda \otimes_{\mathbb{Q}_p} W(k)$ and take a Frobenius lift $\sigma : K \to K$ to be $\mathrm{id} \otimes F$ where $F : W(k) \to W(k)$ is the canonical Frobenius. Since $\mathrm{Sw}_X^{\mathscr{D}}$ is defined in $\mathrm{CH}_0(X)$, the corollary follows upon using Theorem 4.0.5 with the K and σ just defined. □

By [KS08, Lemma 4.3.9], the Serre conjecture for Artin characters is true under the hypothesis of resolution of singularities.

Remark 4.0.8. The assumption that X be projective should not be essential. To eliminate this hypothesis, we need to define the category of overholonomic modules for schemes that may not be embedded into smooth proper formal schemes by gluing. The construction of this category has been done in the papers by Caro. However, push-forwards between these categories have yet to be defined.

Acknowledgements

The author wishes to thank Professor Shuji Saito for discussions that began long before this work took shape. Prof. Saito provided numerous suggestions and much encouragement; he also read the chaotic manuscript many times and helped to improve this paper. Without Prof. Saito's help, the author could not have finished this work. The author is also grateful to Prof. Pierre Berthelot and Prof. Ahmed Abbes for reading the manuscript and giving helpful comments. Finally, he thanks the referee for reading the manuscript very carefully.

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