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ON MAJOR AND MINOR BRANCHES OF ROOTED TREES

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1. Introduction. Let $\mathcal{T}_n$ denote a rooted tree with n nodes. (For definitions not given here, see, e.g. [**4**]). For any node v of $\mathcal{T}_n$, let $B(v)$ denote the subtree of $\mathcal{T}_n$ determined by v and all nodes u such that v is between u and the root of $\mathcal{T}_n$; node v serves as the root of $B(v)$. The *branches* of $\mathcal{T}_n$ are the subtrees $B(v)$ such that node v is joined to the root of $\mathcal{T}_n$. A branch B with i nodes is a *primary branch* of $\mathcal{T}_n$ if $n/2 \leqq i \leqq n - 1$; if $\mathcal{T}_n$ has a primary branch B with i nodes, then a branch C with j nodes is a *secondary branch* if $(n - i)/2 \leqq j \leqq n - 1 - i$; if $\mathcal{T}_n$ has a primary branch B with i nodes and a secondary branch C with j nodes, then a branch D with h nodes is a *tertiary branch* if

$$(n - i - j)/2 \leqq h \leqq n - 1 - i - j.$$

A branch is a *major branch* if it is a primary, secondary or tertiary branch; otherwise it is a *minor branch*. Our main object here is to establish some results on the existence, sizes, and heights of the major and minor branches of trees in certain families $\mathcal{F}$ of rooted trees.

In Section 2 we define the families $\mathcal{F}$ of rooted trees we shall consider and we give some lemmas on the asymptotic behaviour of the coefficients in certain generating functions. Our main results are in the remaining sections. In Section 3 we show that the probability that a tree $\mathcal{T}_n$ in $\mathcal{F}$ does not have a primary branch is $O(n^{-1})$ and in Section 4 we show, among other things, that the expected number of nodes of $\mathcal{T}_n$ that do not belong to a primary branch is equal to $O(n^{1/2})$. In the later sections we show that the probabilities that a tree $\mathcal{T}_n$ has a secondary branch, a tertiary branch, or some minor branches all tend to a limit as $n \to \infty$. We find that the expected number of nodes in a secondary branch of $\mathcal{T}_n$ is equal to $O(n^{1/2})$ and the expected number in a tertiary branch is equal to $O(\log n)$. These major branches are, in a sense, the only significant branches, for the expected number of nodes in all the minor branches tends to a constant as $n \to \infty$. We also appeal to a result of Flajolet and Odlyzko [**3**] to show that the expected heights of the primary and secondary branches of $\mathcal{T}_n$ are $O(n^{1/2})$ and $O(\log n)$, respectively, and that the expected value of the sum of the heights of all the remaining branches is $O(1)$.

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2. Preliminaries. Let y_n denote the number of trees $\mathscr{T}_n$ in a given family $\mathscr{F}$ of rooted trees; if there are weights associated with trees in $\mathscr{F}$, then each tree is counted according to its weight in these definitions. The family $\mathscr{F}$ is a *simply generated family* if the generating function $Y = \sum y_n x^n$ satisfies a relation of the type

$$(2.1) \qquad Y = x\phi(Y)$$

where

$$\phi(Y) = 1 + c_1 Y + c_2 Y^2 + \dots$$

is a power series in Y with non-negative coefficients. This implies that the trivial tree $\mathscr{T}_1$ is in $\mathscr{F}$ and that any non-trivial tree $\mathscr{T}_n$ in $\mathscr{F}$ can be constructed from an ordered collection of smaller trees in $\mathscr{F}$ by joining their roots to a new node that serves as the root of $\mathscr{T}_n$. The factor x in (2.1) takes the new root-node into account and the coefficients c_i determine weights associated with trees in $\mathscr{F}$. If the tree $\mathscr{T}_n$ has $D_i(\mathscr{T}_n)$ nodes incident with i edges leading away from the root, then the *weight* $\omega(\mathscr{T}_n)$ of $\mathscr{T}_n$ is defined by the formula

$$\omega(\mathscr{T}_n) = \prod_i c_i^{D_i(\mathscr{T}_n)}$$

where we adopt the convention that $c_0 = 1$. Relation (2.1) implies that

$$y_n = \sum \omega(\mathscr{T}_n)$$

where the sum is over all trees $\mathscr{T}_n$ in $\mathscr{F}$. Two familiar examples of simply generated families are the plane trees and the labelled trees, for which $c_i = 1$ and $c_i = 1/i!$, respectively, for $i \geqq 1$.

In what follows we let $\mathscr{C}_n\{f(x)\}$ denote the coefficient of x^n in the power series $f(x)$; and, in all statements involving asymptotic or limit relations, it is to be understood that the relation holds as the appropriate parameter (usually n) tends to infinity. We assume henceforth that $\mathscr{F}$ is some given simply generated family of rooted trees and that the function ϕ in (2.1) satisfies the hypothesis of the following result.

LEMMA 1. *Suppose*

$$\phi(t) = 1 + c_1 t + c_2 t^2 + \dots$$

is a regular function of t *for* $|t| < R \leqq \infty$, *and let*

$$Y = Y(x) = x + y_2 x^2 + \dots$$

denote the unique solution of $Y(x) = x\phi(Y(x))$ *in the neighbourhood of* $x = 0$. *If*

(i) $c_i \geqq 0$ *for* $i \geqq 1$,

(ii) $c_i c_j > 0$ *for some distinct* i *and* j *such that* $\gcd(i, j) = 1$, *and*

(iii) $\tau\phi'(\tau) = \phi(\tau)$ *for some* τ, *where* $0 < \tau < R$, *then*

(2.2) $y_n \sim c\rho^{-n} n^{-3/2}$

where

$$\rho = \tau/\phi(\tau) \quad \textit{and} \quad c = \tau(2\pi\rho\tau\phi''(\tau))^{-1/2}.$$

Furthermore, if K is any fixed positive integer, then

(2.3) $\mathscr{C}_n\{x\phi^{(k)}(Y)\} \sim \rho\phi^{(k+1)}(\tau)y_n$

and

(2.4) $\mathscr{C}_n\{Y^k(x)\} \sim k\tau^{k-1}y_n.$

Relation (2.2) was proved in [**7**] and a closely related result was proved earlier in [**12**]. Relation (2.3) may be proved in a similar way, as was pointed out in [**8**] when $k = 1$ (we shall use (2.3) only when $k \leqq 3$ here). Relation (2.4) was proved in [**10**]. We remark that (2.3) and (2.4) are special cases of a more general result proved in [**11**; Lemma 4].

In the following lemmas, which we shall need later, a_n and b_n denote non-negative functions of n and a, b, and ρ denote positive constants; the function f_n is defined by the relation

$$f_n = \Sigma' \, b_{n-i}a_i$$

where the sum is over i such that $1 \leqq i \leqq n/2$.

LEMMA 2. *If*

(i) $b_n \sim b\rho^{-n}n^{-\beta}$

for some constant β, *and if*

(ii) $0 < A \doteq \sum_1^{\infty} a_n\rho^n < \infty$

then

(2.5) $f_n \sim Ab_n.$

Proof. For any given $\epsilon > 0$, let $N = N_\epsilon$ denote the least integer such that

$$\delta_N \doteq A - \sum_1^N a_i\rho^i < \epsilon.$$

It follows readily from (i) that if $1 \leqq i \leqq n/2$ then

$$b_{n-i} = \rho^i O(b_n)$$

as $n \to \infty$ and that if $1 \leqq i \leqq N$ then

$$b_{n-i} = b_n\rho^i(1 + o(1))$$

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as $n \to \infty$, where the error terms depend only on n (and ϵ in the second case). Hence,

$$f_n = \sum_{i=1}^{N} b_{n-i}a_i + \sum_{i=N+1}^{n/2} b_{n-i}a_i$$
$$= (1 + o(1))b_n(A - \delta_N) + O(\delta_N b_n)$$
$$= (1 + o(1))Ab_n + O(\delta_N b_n)$$

as $n \to \infty$. This suffices to prove the required result since $\delta_N < \epsilon$.

(We remark that the conclusion of Lemma 2 still holds if condition (i) is replaced by the conditions that

(iii) $b_{n-1}/b_n \to \rho$ as $n \to \infty$ and

there exists a constant K such that

(iv) $b_{n-i}/b_n \leqq K\rho^i$ for $1 \leqq i \leqq n/2$.

If only two of the conditions (ii), (iii), and (iv) are assumed to hold, then the conclusion does not necessarily follow).

LEMMA 3. *Suppose that* $b_n \sim b\rho^{-n}n^{-3/2}$.
(i) *If* $a_n \sim a\rho^{-n}n^{-1}$, *then* $f_n \sim a \log n \cdot b_n$.
(ii) *If* $a_n \sim a\rho^{-n}n^{-1/2}$, *then* $f_n \sim 2an^{1/2}b_n$.

These results follow readily upon approximating the appropriate sums by an integral and then passing to the limit; we shall omit the details.

3. Trees with a primary branch. Let N_n denote the number of trees $\mathcal{T}_n$ in $\mathcal{F}$ that do not have a primary branch. We begin by deriving an expression for the number P_n of trees $\mathcal{T}_n$ in $\mathcal{F}$ that have a primary branch. (We adopt the convention that an empty sum equals zero.)

THEOREM 1. $P_n = \sum_{i \leqq n/2} y_{n-i}\mathcal{C}_i\{x\phi'(Y)\}$.

Proof. The generating function $c_k x Y^k$ enumerates those trees in $\mathcal{F}$ that have k branches, for $k = 0, 1, \ldots$. It follows, therefore, that the number of trees $\mathcal{T}_n$ with k branches one of which is a primary branch with $n - i$ nodes is equal to

$$y_{n-i}\mathcal{C}_i\{xkc_kY^{k-1}\},$$

for $i \leqq i \leqq n/2$. When we sum this over the admissible values of k and i, we obtain the required expression for P_n.

Sometimes it is possible to deduce from Theorem 1 an explicit formula for P_n or, equivalently, for $N_n = y_n - P_n$. The following are two examples of such results; we shall omit the details of their derivation other than to point out that the result for labelled trees is an immediate consequence of Jordan's result [**2**] on centroids in trees. (We adopt the convention that y_m equals zero if m is not a positive integer.)

COROLLARY 1.1. *If $\mathcal{F}$ is the family of rooted labelled trees, then*

$$(3.1) \quad N_n = \frac{1}{n} y_n - \frac{1}{2} y_{n/2}^2$$

where

$$y_n = \frac{n^{n-1}}{n!}.$$

If $\mathcal{F}$ is the family of plane trees, then

$$(3.2) \quad N_n = \frac{3}{n+1} y_n - y_{n/2} y_{n/2+1} - \frac{1}{2} y_{(n+1)/2}^2$$

where

$$y_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

When we apply relation (2.2) and Lemma 2 to the expression for P_n in Theorem 1, we find that

$$P_n/y_n \to \rho\phi'(\tau) = 1$$

or, equivalently, that

$$N_n/y_n \to 0.$$

One might suspect, in view of (3.1) and (3.2), that $N_n/y_n \sim kn^{-1}$, k constant, for all simply generated families. We now show that this is indeed the case.

THEOREM 2. $\lim_{n\to\infty} (nN_n/y_n) = \tau\phi'''(\tau)/\phi''(\tau)$.

Proof. For each integer $n \geqq 2$, let

$$S_n = S_n(x) = \sum_{i \leqq n/2} y_i x^i.$$

For each pair of integers n and k where $2 \leqq k \leqq n - 1$, let

$$L_{nk} = \mathscr{C}_{n-1}\{ (S_{n-1}(x))^k \}.$$

Since N_n is the number of trees $\mathcal{T}_n$ in which each branch has at most $(n - 1)/2$ nodes, it follows that

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$$(3.3) \qquad N_n = \sum_{k=2}^{n-1} c_k L_{nk}$$

for $n \geqq 3$. To prove the theorem we shall first show that

$$(3.4) \qquad \lim_{n\to\infty} nL_{nk}/y_n = k(k-1)(k-2)\tau^{k-2}/\phi''(\tau)$$

for each fixed integer $k \geqq 2$.

It is not difficult to see that

$$L_{n2} = y^2_{(n-1)/2} = O(n^{-3/2}y_n),$$

so (3.4) certainly holds if $k = 2$. Consider now any fixed integer $k \geqq 3$ and suppose that $n \geqq k$. We observe that

$$(3.5) \qquad L_{n+1,k} = \sum y_{i_1} \dots y_{i_k}$$

where the sum is over all integers $i_1, \dots, i_k$ such that

$$i_1 + \dots + i_k = n$$

and $1 \leqq i_j \leqq n/2$ for $1 \leqq j \leqq k$. Let us temporarily assume that n is odd. For each term in the sum in (3.5), let p denote the largest integer such that

$$i_1 + \dots + i_p < n/2;$$

let $r = i_1 + \dots + i_p$, $s = i_{p+1}$, and $t = i_{p+2} + \dots + i_k$. Since $i_j \leqq n/2$ for all j and n is odd, it follows that $1 \leqq p \leqq k - 2$ and $1 \leqq r, s, t < n/2$. Relation (3.5) may therefore be rewritten as

$$L_{n+1,k} = \sum_{p=1}^{k-2} \Sigma^* \mathscr{C}_r\{S_n^p\} \cdot \mathscr{C}_s\{S_n\} \cdot \mathscr{C}_t\{S_n^{k-1-p}\}$$

where the sum Σ^* is over all integers r, s, and t such that $r + s + t = n$ and $1 \leqq r, s, t < n/2$. But

$$\mathscr{C}_h\{S_n^m\} = \mathscr{C}_h\{Y^m\} \quad \text{if } h \leqq n/2;$$

hence,

$$(3.6) \qquad L_{n+1,k} = \sum_{p=1}^{k-2} \Sigma^* \mathscr{C}_r\{Y^p\} \cdot Y_s \cdot \mathscr{C}_t\{Y_n^{k-1-p}\}.$$

Now consider the inner sum Σ^* in the right hand side for any fixed value of p. We recall that relation (2.4) and (2.2) imply that

$$(3.7) \qquad \mathscr{C}_h\{Y^m\} \sim m\tau^{m-1}y_h \sim cm\tau^{m-1}\rho^{-h}h^{-3/2}$$

for any fixed integer m, as $h \to \infty$. If we apply relation (3.7) to the terms in Σ^*, multiply by n/y_n and then approximate this sum by an integral, we find that for every fixed p

(3.8) $$\lim_{n\to\infty} (n/y_n) \sum{}^* = c^2\tau^{k-3}p(k - 1 - p)I,$$

where

$$I = \int_0^{1/2} \int_{1/2-x}^{1/2} (xz(1 - x - z))^{-3/2}dzdx.$$

One way to evaluate I is to let $x + z = u$ and $x - z = v$; then integrate with respect to v and let

$$u = (1 + \cos^2\theta)^{-1}.$$

This gives

$$I = 8\int_{1/2}^{1} u^{-2}((1 - u)(2u - 1))^{-1/2}du$$
$$= 16\int_0^{\pi/2} (1 + \cos^2\theta)d\theta = 12\pi.$$

Thus it follows from (3.6)-(3.8) that

$$\lim_{n\to\infty} (nL_{n+1,k}/y_n) = 12\pi c^2\tau^{k-3} \sum_{p=1}^{k-2} p(k - 1 - p)$$
$$= 2\pi c^2 k(k - 1)(k - 2)\tau^{k-3}$$

for each fixed integer k. This establishes (3.4) since

$$2\pi c^2 = \phi(\tau)/\phi''(\tau) \quad \text{and} \quad y_n/y_{n+1} \to \rho = \tau/\phi(\tau).$$

(See Lemma 1.)

It was shown in [**10**, Lemma 3] that there exists an absolute constant K_1 such that

(3.9) $$\mathscr{C}_h\{Y^m\} \leqq K_1\tau^{m-1}\rho^{-h}(m/h)^{3/2}$$

for all positive integers m and h. If we apply this inequality to the terms in the inner sum $\sum^*$ in (3.6) for any fixed value of p, we find that there exists a constant K_2 such that

$$(n/y_n) \sum{}^* \leqq K_2\tau^k(p(k - 1 - p))^{3/2} \leqq K_2k^3\tau^k$$

for all n, k, and p and hence

$$nL_{n+1,k}/y_n \leqq K_2k^4\tau^k$$

for all n and k. Since y_n/y_{n+1} is bounded, there exists a constant K_3 such that

$$nL_{n,k}/y_n = K_3k^4\tau^k$$

for all n and k. Since $\phi(t)$ is analytic in $|t| < R$ and $\tau < R$, the series $\sum c_kk^4\tau^k$ converges; so by appealing to Tannery's theorem [**1**; p. 136] we may interchange the order of limit and summation in the second expression below and conclude that

$$\lim_{n\to\infty} (nN_n/y_n) = \lim_{n\to\infty} \sum_{k=2}^{\infty} c_k nL_{nk}/y_n$$

$$= \sum_{k=2}^{\infty} c_k \lim_{n\to\infty} (nL_{nk}/y_n)$$

$$= \sum_{k=2}^{\infty} c_k k(k - 1)(k - 2)\tau^{k-2}/\phi''(\tau)$$

on using (3.4). This proves Theorem 2, since the last sum equals $\tau\phi'''(\tau)/\phi''(\tau)$.

In the foregoing argument we assumed that n was odd. When n is even we find that some additional terms should be included in formula (3.7): namely, (i) terms in which $s = n/2$ in the inner sum, and (ii) terms of the form

$$\mathscr{C}_{n/2}\{Y^p\} \cdot \mathscr{C}_{n/2}\{Y^{k-p}\} \quad \text{for } 1 \leqq p \leqq k - 1.$$

It is not difficult to see that the contribution of these terms is negligible with respect to the contribution of the terms we have already considered. Thus the required result also holds when n is even.

A slight modification of the preceding argument shows that if

$$(3.10) \quad H_n = \sum_{k=3}^{n} kc_k L_{n,k-1},$$

then

$$(3.11) \quad nH_n/y_n \to \tau\phi^{(iv)}(\tau)/\phi''(\tau).$$

We shall use this result later.

We conclude this section by pointing out that if $d(n)$ denotes the expected number of nodes v in a random tree $\mathscr{T}_n$ such that the subtree $B(v)$ has a primary branch, then

$$d(n)/n \to P(\rho)/\tau \quad \text{where } P(x) = \sum p_i x^i;$$

the argument is similar to an argument applied to a different problem in [**9**; Corollary 2.2]. It follows from (3.1) and (3.2) that

$$P(\rho)/\tau = \frac{1}{2} + \frac{1}{2} \sum_{1}^{\infty} \left(\frac{n^{n-1}}{n!} e^{-n}\right)^2 = .58 \ldots$$

for the family of rooted labelled trees, and that

$$P(\rho)/\tau = 12 \sum_{1}^{\infty} \frac{1}{n(n+1)} \left(\binom{2n-2}{n-1} 4^{-n} \right)^2 = .42\ldots$$

for the family of plane trees. Much of the difference between these values arises from the fact that the expected fraction of nodes that are end-nodes in a random tree $\mathcal{T}_n$ tends to e^{-1} for labelled trees and to $1/2$ for plane trees, and these end nodes contribute nothing to $d(n)$.

4. The size of the principal branch. In this section we obtain some results on the distribution of the number of nodes in a primary branch of a tree. For convenience we shall formulate these results in terms of the number $q(\mathcal{T}_n)$ of nodes of the tree $\mathcal{T}_n$ that do not belong to a principal branch of $\mathcal{T}_n$.

It follows readily from the definition of a principal branch and the argument used to prove Theorem 1, that

$$(4.1) \quad \Pr\{q(\mathcal{T}_n) = i\} \cdot y_n = \begin{cases} y_{n-i} \cdot \mathscr{C}_i\{x\phi'(Y)\}, & \text{if } 1 \leqq i \leqq n/2; \\ N_n, & \text{if } i = n. \end{cases}$$

If we apply relations (2.2) and (2.3) to the first formula, we find that

$$\lim_{n\to\infty} \Pr\{q(\mathcal{T}_n) = i\} = \rho^i \mathscr{C}_i\{x\phi'(Y)\}$$

for each fixed integer i, and that

$$(4.2) \quad \Pr\{q(\mathcal{T}_n) = i\} \sim c\rho\phi''(\tau)(i(1 - i/n))^{-3/2}$$

if $i \leqq n/2$ as $i, n \to \infty$. The following result implies that almost all trees $\mathcal{T}_n$ have a primary branch that contains at least $(1 - \epsilon)n$ nodes for any fixed $\epsilon > 0$.

THEOREM 3. *Let γ denote a constant such that $0 \leqq \gamma < 1/2$. If $i/n \to \gamma$ as $i, n \to \infty$, then*

$$\Pr\{q(\mathcal{T}_n) > i\} \sim 2c\rho\phi''(\tau)(1 - 2\gamma)((1 - \gamma)i)^{-1/2}.$$

Proof. We first observe that relations (4.1) and (4.2) and Theorem 2 imply that

$$(4.3) \quad \Pr\{q(\mathcal{T}_n) > i\} \sim c\rho\phi''(\tau) \sum{}' (k(1 - k/n))^{-1/2} + O(n^{-1})$$

as $i, n \to \infty$, where the sum is over k such that $i < k \leqq n/2$. If $i/n \to 0$, then

$$\begin{aligned} \Pr\{q(\mathcal{T}_n) > i\} &\sim c\rho\phi''(\tau) \sum{}' k^{-3/2}(1 + O(k/n)) + O(n)^{-1} \\ &\sim c\rho\phi''(\tau)(2i^{-1/2} + o(i^{-1/2}) + O(n^{-1/2})) \\ &\sim 2c\rho\phi''(\tau)i^{-1/2}, \end{aligned}$$

as required in this case.

Now suppose that $i/n \to \gamma$ where $0 < \gamma < 1/2$. If we multiply both sides of relation (4.3) by $i^{1/2} \sim (\gamma n)^{1/2}$ and approximate the sum Σ' by an integral, we find that

$$i^{1/2}\Pr\{q(\mathscr{T}_n) > i\} \to c\rho\phi''(\tau)\gamma^{1/2} \int_{\gamma}^{1/2} (x(1 - x))^{-3/2}dx$$
$$= 2c\rho\phi''(\tau)(1 - 2\gamma)(1 - \gamma)^{-1/2},$$

and this completes the proof of the theorem.

If $g(\mathscr{T}_n)$ denotes some parameter defined on trees $\mathscr{T}_n$, we shall let

$$g_n = \Sigma\, \omega(\mathscr{T}_n) \cdot g(\mathscr{T}_n)$$

where the sum is over all trees $\mathscr{T}_n$ in $\mathscr{F}$ and where $\omega(\mathscr{T}_n)$ denotes the weight function defined in Section 2; we shall refer to g_n simply as the sum of the numbers $g(\mathscr{T}_n)$ without explicitly mentioning the weights each time. Notice that g_n/y_n is the expected value of $g(\mathscr{T}_n)$ over all trees $\mathscr{T}_n$; in particular, q_n/y_n is the expected value of $q(\mathscr{T}_n)$ over all trees $\mathscr{T}_n$.

THEOREM 4. $q_n/y_n \sim 2c\rho\phi''(\tau)n^{1/2}$.

Proof. It follows from (4.1) that

$$(4.4) \qquad q_n = \sum_{i \leqq n/2} y_{n-i} \cdot i\mathscr{C}_i\{x\phi'(Y)\} + nL_n.$$

Now $nL_n = O(y_n)$ by Theorem 2, and

$$i\mathscr{C}_i\{x\phi'(Y)\} \sim \rho\phi''(\tau)iy_i$$

by (2.3). Hence we may apply Lemma 3 (ii) to the sum and conclude that

$$q_n \sim 2c\rho\phi''(\tau)n^{1/2}y_n$$

as required.

It can be shown by a similar argument that the variance of $q(\mathscr{T}_n)$ over all trees $\mathscr{T}_n$ is asymptotic to $(2 - \pi/2)\rho\phi''(\tau)n^{3/2}$.

We remark that a weaker version of Theorem 2 can be proved by making use of a second expression for q_n, namely,

$$(4.5) \qquad q_n = y_n + \sum_{i<n/2} iy_i \cdot \mathscr{C}_{n-i}\{x\phi'(Y)\}.$$

The isolated term y_n records the contribution of the roots of all trees $\mathscr{T}_n$ to q_n and the sum records the contribution of all nodes belonging to non-primary branches. Hence,

$$nL_n = y_n + \Sigma_1 - \Sigma_2$$

where Σ_1 and Σ_2 denote the sums in expressions (4.5) and (4.4), respectively. But both Σ_1 and Σ_2 are asymptotically equal to

$$2c\rho\phi''(\tau)n^{1/2}y_n,$$

by Lemma 3 (ii); consequently,

$$nL_n = o(n^{1/2}y_n) \quad \text{or} \quad N_n/y_n = o(n^{-1/2}).$$

5. Heights and the primary path length. The *length* of a path is the number of edges it contains and the *height* $h(\mathcal{T}_n)$ of a rooted tree $\mathcal{T}_n$ is the length of any longest path starting at the root of $\mathcal{T}_n$. Flajolet and Odlyzko [**3**] have shown that h_n, the sum of the heights $h(\mathcal{T}_n)$, satisfies the relation

$$(5.1) \qquad h_n \sim (c\rho\phi''(\tau))^{-1}n^{1/2}y_n \sim (\rho\phi''(\tau))^{-1}\rho^{-n}n^{-1}.$$

Let $\pi(\mathcal{T}_n)$ denote the height of the primary branch of $\mathcal{T}_n$ and let π_n denote the sum of the heights $\pi(\mathcal{T}_n)$. (If $\mathcal{T}_n$ does not have a primary branch then $\pi(\mathcal{T}_n)$ is understood to equal zero, and we adopt a similar convention elsewhere.) Since, as we have shown, almost all trees $\mathcal{T}_n$ have a primary branch that contains a very large majority of all the nodes of $\mathcal{T}_n$, the following result is not surprising.

THEOREM 5. $\pi_n \sim h_n$.

Proof. It follows readily from the definition of π_n and the same type of argument as was used to prove Theorem 1, that

$$\pi_n = \sum_{i \leqq n/2} h_{n-i}\mathscr{C}_i\{x\phi'(Y)\}.$$

This sum satisfies the hypothesis of Lemma 2 in view of (5.1). Hence we may conclude that

$$\pi_n \sim \rho\phi'(\tau)h_n = h_n,$$

as required.

We define the *primary path* of $\mathcal{T}_n$ to be the unique maximal path in $\mathcal{T}_n$ of the form $(v_0, v_1, \ldots, v_m)$ where v_0 is the root of $\mathcal{T}_n$ and $B(v_j)$ is a primary branch of the subtree $B(v_{j-1})$ for $1 \leqq j \leqq m$. Let $l(\mathcal{T}_n)$ denote the length of the primary path of $\mathcal{T}_n$. If $\mathcal{T}_n$ has a primary branch B, then $l(\mathcal{T}_n) = 1 + l(B)$; if not, then $l(\mathcal{T}_n) = 0$. Hence, if l_n denotes the sum of the primary path lengths $l(\mathcal{T}_n)$, then

$$(5.2) \qquad l_n = P_n + \sum_{i \leqq n/2} l_{n-i}\mathscr{C}_i\{x\phi'(Y)\}.$$

For, the term P_n records the contribution of the 1 from all trees $\mathcal{T}_n$ with a primary branch and the sum records the contribution from the primary branches of these trees. Now $l(\mathcal{T}_n) \leqq h(\mathcal{T}_n)$ for any tree $\mathcal{T}_n$, so

$$l_n/y_n \leqq h_n/y_n \sim (c\rho\phi''(\tau))^{-1}n^{1/2}.$$

On the other hand, it can be deduced from (5.2) that there exists a positive constant α such that $l_n/y_n > \alpha n^{1/2}$ for all sufficiently large n. We suspect that $n^{-1/2}l_n/y_n$ tends to a limit as $n \to \infty$, but the following result is all we can prove in this direction.

THEOREM 6. *If there exist constants* λ *and* κ *such that*

$$(5.3) \qquad l_n/y_n = \lambda n^{1/2} + \kappa + o(1)$$

as $n \to \infty$, *then*

$$\lambda = A(c\rho\phi''(\tau))^{-1}$$

where

$$A = (2^{3/2} - \log(3 + 2^{3/2}))^{-1} = .93836\ldots.$$

Proof. If each number l_m in formula (5.2) is expressed in terms of y_m, in accordance with relation (5.3), then the resulting equation can be rewritten as

$$(\lambda n^{1/2} + \kappa + o(1))N_n$$
$$= P_n - \lambda \sum_{i \leqq n/2} i(n^{1/2} + (n-i)^{1/2})^{-1} \cdot y_{n-i}\mathscr{C}_i\{x\phi'(Y)\},$$

upon appealing to Theorem 1 and the identity

$$(n-i)^{1/2} = n^{1/2} - i(n^{1/2} + (n-i)^{1/2})^{-1}.$$

Now $P_n/y_n \to 1$ and $N_n/y_n = O(n^{-1})$, by Theorem 2, so

$$(5.4) \qquad \lambda \sum_{i \leqq n/2} i(n^{1/2} + (n-i)^{1/2})^{-1}(y_{n-i}/y_n)\mathscr{C}_i\{x\phi'(Y)\} = 1 + o(1)$$

as $n \to \infty$. It follows from (2.2) and (2.3) that

$$(y_{n-i}/y_n)\mathscr{C}_i\{x\phi'(Y)\} = (1 + o(1))c\rho\phi''(\tau)(n/(i(n-i)))^{3/2}$$

as $i, n \to \infty$. Thus if we approximate the sum in (5.4) by an integral and take the limit as $n \to \infty$, we find that

$$\lambda c\rho\phi''(\tau)J = 1,$$

where

$$J = \int_0^{1/2} \frac{dx}{x^{1/2}(1-x)^{3/2}(1+(1-x)^{1/2})}$$
$$= 2\int_0^{\pi/4} \frac{1-\cos\theta}{\cos^2\theta \sin^2\theta}d\theta = 2^{3/2} - \log(3 + 2^{3/2}).$$

This implies the required result.

Some values of the numbers $g_n = n^{-1/2} l_n / y_n$ for the family of plane trees and labelled trees are given in the following table (truncated after two digits). In the case of plane trees $\lambda = 1.66\ldots$ while for labelled trees $\lambda = 2.35\ldots$; so it would seem that if g_n does tend to a limit, the rate of convergence is rather slow.

n	2	3	4	5	10	50	100	500	1000	2000
g_n (plane)	.70	.57	.80	.67	.82	1.02	1.11	1.29	1.36	1.41
g_n (labelled)	.70	.76	1.03	1.00	1.28	1.65	1.77	2.00	2.07	2.13

TABLE. Values of g_n for plane trees and labelled trees.

6. The secondary branch. In this section we determine the number S_n of trees $\mathscr{T}_n$ that have a secondary branch and we determine the expected size and height of a secondary branch. The derivations of the results in this section and the next are straightforward extensions of the derivations of the analogous results for the primary branch, so we shall omit some of the details.

THEOREM 7. *Let*

$$R(x) = \sum_1^\infty R_i x^i,$$

where

$$R_i = \sum_{j \leqq i/2} y_{i-j} \mathscr{C}_j \{x\phi''(Y)\}.$$

Then

$$S_n \sim R(\rho) y_n.$$

Proof. The number of trees $\mathscr{T}_n$ with k branches one of which is a primary branch with $n - i$ nodes and one of which is a secondary branch with $i - j$ nodes is equal to

$$y_{n-i} y_{i-j} \mathscr{C}_j \{xk(k-1)c_k Y^{k-2}\},$$

for any integers i, j, and k such that $1 \leqq i \leqq n/2$, $1 \leqq j \leqq i/2$, and $k \geqq 0$; hence,

$$(6.1) \quad S_n = \sum_{i \leqq n/2} y_{n-i} \sum_{j \leqq n/2} y_{i-j} \mathscr{C}_j \{x\phi''(Y)\}$$
$$= \sum_{i \leqq i/2} y_{n-i} R_i.$$

Now $R_i \sim \rho\phi''(\tau) y_i$, by (2.3) and Lemma 2, so the series for $R(\rho)$ converges. Hence we may apply Lemma 2 to the last expression for S_n and conclude that $S_n \sim R(\rho) y_n$, as required.

If $\mathscr{F}$ is the family of labelled trees, then $Y = xe^Y$, $\rho = e^{-1}$, and $y_n = n^{n-1}/n!$. In this case $R_i = P_i$ and it can be shown that

$$R(\rho) = \frac{1}{2} + \frac{1}{2}\sum_1^\infty (y_n\rho^n)^2 = .58\ldots.$$

If $\mathscr{F}$ is the family of plane trees, then $Y = x(1 - Y)^{-1}$, $\rho = 1/4$, and

$$y_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

In this case,

$$\mathscr{C}_j\{x\phi''(Y)\} = \mathscr{C}_j(2x^{-2}Y^3\} = \mathscr{C}_j\{2x^{-2}(Y - xY - x)\}$$
$$= 2(y_{j+2} - y_{j+1}) = 12\frac{j(2j-1)}{(j+1)(j+2)}y_j$$

and

$$\sum_{h \geqq j} y_h\rho^h = 2jy_j\rho^j.$$

Hence,

$$R(\rho) = \sum_{i=1}^\infty \sum_{j \leqq i/2} y_{i-j}\mathscr{C}_j\{x\phi''(Y)\}\rho^i$$
$$= \sum_{j=1}^\infty \mathscr{C}_j\{x\phi''(Y)\}\rho^j \sum_{h \geqq j} y_h\rho^h$$
$$= 24\sum_{j=1}^\infty \frac{j^2(2j-1)}{(j+1)(j+2)}(y_j\rho^j)^2.$$

This series converges rather slowly; and it can be shown by a more complicated argument, the details of which we omit, that

$$R(\rho) = \frac{3}{14} + \frac{1}{14}\sum_{j=1}^\infty \frac{447j + 24}{(j+1)(j+2)}(y_j\rho^j)^2 = .59\ldots$$

for the family of plane trees.

Let $s(\mathscr{T}_n)$ denote the number of nodes of $\mathscr{T}_n$ that belong to a secondary branch and let s_n denote the sum of the secondary branch sizes $s(\mathscr{T}_n)$.

THEOREM 8. $s_n/y_n \sim 2c\rho\phi''(\tau)n^{1/2}$.

Proof. It is not difficult to see, in view of formula (6.1), that

$$s_n = \sum_{i \leqq n/2} y_{n-i} \sum_{j \leqq i/2} (i - j)y_{i-j}\mathscr{C}_j\{x\phi''(Y)\}.$$

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Now

$$\sum_{j \leqq i/2} (i - j) y_{i-j} \mathscr{C}_j \{ x\phi''(Y) \} \sim \rho\phi''(\tau) i y_i$$

as $i \to \infty$, by Lemma 2. Hence we may apply Lemma 3 (ii) to the formula for s_n and conclude that

$$s_n \sim 2c\rho\phi''(\tau) n^{1/2} y_n,$$

as required.

Notice that it follows from Theorem 4 and 8 that the expected number of nodes in a tree $\mathscr{T}_n$ that belong neither to a principal branch nor to a secondary branch is equal to $o(n^{1/2})$. We shall obtain stronger versions of this observation in the next two sections.

Let $\sigma(\mathscr{T}_n)$ denote the height of the secondary branch of $\mathscr{T}_n$ and let σ_n denote the sum of the heights $\sigma(\mathscr{T}_n)$.

THEOREM 9. $\sigma_n / y_n \sim \log n$.

Proof. It follows readily from the definition of σ_n that

$$\sigma_n = \sum_{i \leqq n/2} y_{n-i} \sum_{j \leqq i/2} h_{i-j} \mathscr{C}_j \{ x\phi''(Y) \}.$$

Now

$$\sum_{j \leqq i/2} h_{i-j} \mathscr{C}_j \{ x\phi''(Y) \} \sim \rho\phi''(\tau) h_i \sim \rho^{-i} i^{-1}$$

as $i \to \infty$, by Lemma 2 and (5.1). Hence we may apply Lemma 3 (i) to the formula for σ_n and conclude that $\sigma_n \sim \log n \cdot y_n$, as required.

We remark that s_n / y_n and σ_n / y_n are the expected values of $s(\mathscr{T}_n)$ and $\sigma(\mathscr{T}_n)$ over all trees $\mathscr{T}_n$; to obtain the expected values over those trees $\mathscr{T}_n$ that actually have a secondary branch one should multiply by

$$y_n / s_n \sim (R(\rho))^{-1}.$$

7. The tertiary branch. In this section we determine the number T_n of trees $\mathscr{T}_n$ that have a tertiary branch and we determine the average size and height of a tertiary branch; we assume there that $\phi'''(\tau) \neq 0$, for if $\phi'''(\tau) = 0$ then trees in $\mathscr{F}$ could not have a tertiary branch.

THEOREM 10. *Let*

$$U(x) = \sum_{i}^{\infty} U_i x^i$$

where

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$$U_i = \sum_{j \leqq i/2} y_{i-j} \sum_{h \leqq j/2} y_{j-h} \mathscr{C}_h \{x\phi'''(Y)\}.$$

Then

$$T_n \sim U(\rho) y_n.$$

Proof. The number of trees $\mathscr{T}_n$ with a primary branch with $n - i$ nodes, a secondary branch with $i - j$ nodes, a tertiary branch with $j - h$ nodes, and with k branches altogether, is equal to

$$y_{n-i} y_{i-j} y_{j-h} \mathscr{C}_h \{xk(k - 1)(k - 2)c_k Y^{k-3}\},$$

for any integers i, j, h and k such that $1 \leqq i \leqq n/2$, $1 \leqq j \leqq i/2$, $1 \leqq h \leqq j/2$, and $k \geqq 0$; hence,

$$(7.1) \quad T_n = \sum_{i \leqq n/2} y_{n-i} \sum_{j \leqq i/2} y_{i-j} \sum_{h \leqq j/2} y_{j-h} \mathscr{C}_h \{x\phi'''(Y)\}$$

$$= \sum_{i \leqq n/2} y_{n-i} U_i.$$

If we appeal to (2.3) and apply Lemma 2 twice to the expression for U_i, we find that $U_i = O(y_i)$ as $i \to \infty$ and so the series for $U(\rho)$ converges. Hence we may apply Lemma 2 to the last expression for T_n and conclude that

$$T_n \sim U(\rho) y_n,$$

as required.

We remark that it can be shown that $U(\rho) = .19\ldots$ for the family of labelled trees and that $.237 < U(\rho) < .25$ for the family of plane trees.

Let $t(\mathscr{T}_n)$ denote the number of nodes of $\mathscr{T}_n$ that belong to a tertiary branch and let t_n denote the sum of the tertiary branch sizes $t(\mathscr{T}_n)$.

THEOREM 11. $t_n/y_n \sim 2c^2 \rho\phi'''(\tau) \log n$.

Proof. It is not difficult to see, in view of formula (7.1), that

$$(7.2) \quad t_n = \sum_{i \leqq n/2} y_{n-i} \sum_{j \leqq i/2} y_{i-j} \sum_{h \leqq j/2} (j - h) y_{j-h} \mathscr{C}_h \{x\phi'''(Y)\}.$$

It follows from Lemma 2 that the innermost sum in this expression is asymptotic to $\rho\phi'''(\tau) j y_j$ as $j \to \infty$; and it then follows from Lemma 3 (ii) that the intermediate sum is asymptotic to

$$2c\rho\phi'''(\tau) i^{1/2} y_i \quad \text{as } i \to \infty.$$

Hence we may apply Lemma 3 (i) to the outer sum and conclude that

$$t_n \sim 2c^2 \rho\phi'''(\tau) \log n \cdot y_n,$$

as required.

Let $\tau(\mathcal{T}_n)$ denote the height of the tertiary branch of $\mathcal{T}_n$ and let τ_n denote the sum of the heights $\tau(\mathcal{T}_n)$.

THEOREM 12. *Let*

$$V(x) = \sum_{i}^{\infty} V_i x^i,$$

where

$$V_i = \sum_{j \leqq i/2} y_{i-j} \sum_{k \leqq j/2} h_{j-k} \mathscr{C}_k \{x\phi'''(Y)\}.$$

Then

$$\tau_n \sim V(\rho)v_n.$$

Proof. It follows readily from the definition of τ_n that

$$\tau_n = \sum_{i \leqq n/2} y_{n-i} \sum_{j \leqq i/2} y_{i-j} \sum_{k \leqq j/2} h_{j-k} \mathscr{C}_k \{x\phi'''(Y)\}$$
$$= \sum_{i \leqq n/2} y_{n-i} V_i.$$

If we appeal to (5.1) and apply Lemmas 2 and 3 (i) to the expression for V_i, we find that

$$V_i = O(\log i \cdot y_i) \quad \text{as } i \rightarrow \infty$$

and so the series for $V(\rho)$ converges. Hence we may apply Lemma 2 to the last expression for τ_n and conclude that

$$\tau_n \sim V(\rho)y_n,$$

as required.

8. The minor branches. In this section we determine the number M_n of trees $\mathcal{T}_n$ that have at least one minor branch and we show that the expected number of nodes belonging to minor branches of $\mathcal{T}_n$ approaches a constant as $n \rightarrow \infty$.

THEOREM 13. $M_n/y_n \rightarrow 1 - \rho(c_1 + 2c_2\rho + 3c_3\rho^2) + 3c_3\rho \sum_{1}^{\infty} (y_i\rho^i)^2.$

Proof. There are four types of trees with at least one minor branch: all trees with at least four branches; trees with three branches in which one branch is a principal branch and the other two branches each have the same size; trees with three branches in which no branch is a principal branch; and trees with two branches each of which has the same size. When we count the trees of each type, we find that

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$$M_n = y_n - \mathscr{C}_n\{x(1 + c_1Y + c_2Y^2 + c_3Y^3)\} + 3c_3\rho \sum_{i \leqq n/2} y_{n-i}y^2_{(i-1)/2} + c_3L_{n3} + y^2_{(n-1)/2}.$$

That M_n/y_n tends to the limit stated above now follows upon applying (2.4), Lemma 2, (3.3) and (2.2), respectively, to the contributions from these four types of trees.

We remark that it can be shown that

$$M_n/y_n \to 1 - 5(2e)^{-1} + (2e^{-1}) \sum_1^\infty (y_i\rho^i)^2 = .11 \ldots$$

for the family of labelled trees, and that

$$M_n/y_n \to \frac{5}{16} + \frac{3}{4} \sum_1^\infty (y_i\rho^i)^2 = .36 \ldots$$

for the family of plane trees. Notice that most of the difference between these values arises from the fact that the limiting fraction of trees with four or more branches is considerably larger for the plane trees than for the labelled trees.

Let $m(\mathscr{T}_n)$ denote the number of nodes of $\mathscr{T}_n$ that belong to a minor branch and let m_n denote the sum of the numbers $m(\mathscr{T}_n)$.

THEOREM 14. *There exists a constant C such that $m_n/y_n \to C$.*

Proof. Let Σ_i denote the contribution to $m(\mathscr{T}_n)$ from those trees $\mathscr{T}_n$ with exactly i major branches, for $0 \leqq i \leqq 3$. It follows from the definition of Σ_0 and Theorem 2 that

(8.1) $$\Sigma_0/y_n = (n - 1)N_n/y_n \to \tau\phi'''(\tau)/\phi''(\tau).$$

Next, recall that in (3.10) we defined the numbers H_n by the relation

$$H_n = \sum_{k=3}^{n} kc_kL_{n,k-1},$$

where $L_{n,k-1}$ is the number of ordered collections of $k - 1$ trees from $\mathscr{F}$ such that these trees have $n - 1$ nodes altogether and at most $(n - 1)/2$ nodes each. It is not difficult to see that

$$\Sigma_1 = \sum_{i \leqq n/2} y_{n-i}(i - 1)H_i.$$

Now $(i - 1)H_i = O(y_i)$, by (3.11), so the series

$$H(\rho) = \sum (i - 1)H_i\rho^i$$

converges.

Hence we may apply Lemma 2 to the formula for $\sum_1$ and conclude that

(8.2) $\quad \sum_1 / y_n \to H(\rho).$

If a tree $\mathcal{T}_n$ has a primary branch with $n - i$ nodes and a secondary branch with $i - j$ nodes, where $1 \leqq i \leqq n/2$ and $1 \leqq j \leqq i/2$, then any remaining branch with h nodes is a minor branch if $h < j/2$ irrespective of whether $\mathcal{T}_n$ has a tertiary branch. It follows, therefore, that

$$\sum_2 + \sum_3 = \sum_{i \leqq n/2} y_{n-i} \sum_{j \leqq i/2} y_{i-j} \sum_{h < j/2} \mathscr{C}_{j-h}\{x\phi'''(Y)\} \cdot h y_h,$$

since the contribution of each minor branch is counted separately in the inner sum. Let W_i denote the intermediate sum in this formula so that

$$\sum_2 + \sum_3 = \sum_{i \leqq n/2} y_{n-i} W_i.$$

If we appeal to (2.3) and apply Lemmas 3 (ii) and 3 (i) to the expression for W_i, we find that

$$W_i = O(\log i \cdot y_i) \quad \text{as } i \to \infty,$$

and so the series

$$W(\rho) = \sum W_i \rho^i$$

converges. Hence we may apply Lemma 2 to the expression for $\sum_2 + \sum_3$ and conclude that

(8.3) $\quad \left(\sum_2 + \sum_3\right) / y_n \to W(\rho).$

The required result now follows from (8.1), (8.2), and (8.3).

Let $\nu(\mathcal{T}_n)$ denote the sum of the heights of the minor branches of $\mathcal{T}_n$ and let ν_n denote the sum of the numbers $\nu(\mathcal{T}_n)$. Since $\nu_n \leqq m_n$ it follows from Theorem 14 that

$$\nu_n / y_n = O(1) \text{ as } \quad n \to \infty;$$

in fact, ν_n / y_n tends to a limit, as one could expect, but we shall omit the proof of this.

9. The product of the branch sizes. Beyond the results obtained in the earlier paragraphs, additional information can be provided concerning the distribution of branch sizes by considering the quantity

$$b_n = \sum_{\mathcal{T}_n \in \mathcal{F}} b(\mathcal{T}_n)$$

where $b(\mathcal{T}_n)$ is the product of the number of nodes in the different branches of a tree $\mathcal{T}_n$ in $\mathcal{F}$. It follows readily from the definition of b_n that

$$b_n = \mathscr{C}_n\{x\phi(xY')\}.$$

The asymptotic behaviour of b_n seems to depend very strongly on the particular nature of the function ϕ. We shall content ourselves here with stating the behaviour of b_n for three families $\mathscr{F}$: in the first case, $\phi(t)$ is a polynomial; in the second case, $\phi(t)$ is an infinite series with a finite radius of convergence; and in the third case, $\phi(t)$ is an infinite series that converges everywhere.

Example 1. If $\mathscr{F}$ is a family such that

$$\phi(t) = 1 + \sum_1^k c_i t^i$$

where $c_k \neq 0$, then

$$b_n/y_n \sim \alpha(\rho b^2/4)^{(k-1)/2} \cdot n^{(k+1)/2}$$

as $n \to \infty$, where

$$\alpha = \rho\pi^{1/2}c_k/\Gamma(k/2) \quad \text{and} \quad \rho b^2 = 2\phi(\tau)/\phi''(\tau).$$

Example 2. If $\mathscr{F}$ is the family of plane trees for which $\phi(t) = (1 - t)^{-1}$, then

$$b_n/y_n \sim \alpha\delta^n n^{3/2}$$

as $n \to \infty$, where

$$\alpha = 4\pi^{1/2}(9.5^{-1/2} - 4) \quad \text{and} \quad \delta = (2 + 5^{1/2})/4.$$

Example 3. If $\mathscr{F}$ is the family of labelled trees for which $\phi(t) = e^t$, then

$$b_n/y_n \sim \alpha \cdot \exp(n^{1/3}/2) \cdot n^{2/3}$$

for some constant α, as $n \to \infty$.

In the first example, it follows from a result in [**7**; Theorem 3.1] that $x\phi(xY')$ is regular when $|x| \leqq \rho$, $x \neq \rho$, and that in the neighbourhood of ρ it has an expansion of the form

$$x\phi(xY') = \rho c_k(b\rho/2)^k(\rho - x)^{-k/2} + b_2(\rho - x)^{-(k-1)/2} + \dots.$$

The conclusion then follows upon appealing to a result of Darboux [**2**].

In the second example, it is not difficult to show that

$$x\phi(xY') = x\{1 - 4x + x(1 - 4x)^{1/2}\}(1 - 4x - x^2)^{-1},$$

and the required conclusion follows readily upon expanding in partial fractions. More generally, if $\phi(t)$ has a finite radius of convergence R, then the equation $xY'(x) = R$ has a solution $x = \rho_1$ where $\rho_1 < \rho$; since the radius of convergence of $x(xY')$ is not larger than ρ_1, it follows that

$b_n > \rho_2^{-n}$ infinitely often for any $\rho_2 > \rho_1$.

In the third example, we readily find that

$$x\phi(xY') = x \exp(Y(1 - Y)^{-1}).$$

In this case the conclusion follows by a rather intricate argument that makes use of Hayman's result [5] on admissible entire functions.

Addendum. While preparing this manuscript we learned that recently J. Komlos and W. O. J. Moser (Almost all trees have tribe number at most three, submitted) have considered a related problem concerning the family of labelled trees. They have shown that for fixed, sufficiently small ϵ, large n and for all nodes v of almost all of the n^{n-2} labelled trees $\mathcal{T}_n$ the following assertion holds: If $\mathcal{T}_n$ is rooted at v, then the three largest branches of $\mathcal{T}_n$ collectively contain more than $(1 - \epsilon)n$ nodes.

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