THE GAMMA FUNCTION WITHOUT INFINITE PRODUCTS

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(Received 17th March 1964)

The aim of this note is described by the title; defining the Gamma function by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad (x>0), \qquad \text{.......................(1)}$$

to deduce its standard properties without using the infinite product for sin x, or its equivalent, the partial fractions expansion of cot x.

Although Euler ((**2**), p. 184) originally defined the Gamma function in terms of an infinite product and the modern convexity definition ((**2**), pp. 161-163) leads directly to Weierstrass's product, most text-books use such products only twice; firstly in the proof of

$$\Gamma(x)\Gamma(1-x) = \pi \operatorname{cosec} \pi x \qquad \text{..............................(2)}$$

and secondly in the proof of

$$-\frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} + \gamma + \sum_{r=1}^{\infty}\left(\frac{1}{x+r} - \frac{1}{r}\right). \qquad \text{.......................(3)}$$

There are at least four product-free proofs of (2), one ((**6**), p. 411) depends on Fourier series, another ((**3**), pp. 139-141) on contour integration, that of ((**4**), p. 400) uses partial fractions and the nth roots of unity, while the duplication formula for $\Gamma(x)$ is used in ((**1**), pp. 23-25).

Less attention seems to have been paid to proofs of (3). This note offers a proof using a formula due to Hermite ((**5**), p. 427) together with the facts that $\frac{d}{dx}f(x+t) = \frac{d}{dt}f(x+t)$ and $\frac{d^2}{dx^2}\log\Gamma(x)$ is positive for positive x. This last fact is equivalent to the convexity of $\log\Gamma(x)$.

Lemma 1. Raabe's Integral

For $x>0$, $\displaystyle\int_x^{x+1} \log\Gamma(t)dt = \int_0^1 \log\Gamma(x+t)dt = x\log x - x + \tfrac{1}{2}\log 2\pi.$

Proof. Let

$$G(x) = \int_x^{x+1}\log\Gamma(t)dt, \text{ then } \frac{dG(x)}{dx} = \log\Gamma(x+1) - \log\Gamma(x) = \log x.$$

https://doi.org/10.1017/S0013091500008919 Published online by Cambridge University Press

So $G(x) = x\log x - x + C$. Let $x\to 0$ and $C = \int_0^1 \log\Gamma(t)dt$.

By a change of variable in the Beta integral it can be shown that

$$\pi^{\frac{1}{2}}\Gamma(t) = 2^{t-1}\Gamma\left(\frac{t}{2}\right)\Gamma\left(\frac{t+1}{2}\right).$$

Taking logarithms we have

$$\int_0^1 \tfrac{1}{2}\log\pi\, dt + \int_0^1 \log\Gamma(t)dt$$
$$= \log 2\int_0^1 (t-1)dt + \int_0^1 \log\Gamma\left(\frac{t}{2}\right)dt + \int_0^1 \log\Gamma\left(\frac{t+1}{2}\right)dt.$$

That is

$$\tfrac{1}{2}\log\pi + C = -\tfrac{1}{2}\log 2 + 2\int_0^{\frac{1}{2}} \log\Gamma(u)du + 2\int_{\frac{1}{2}}^1 \log\Gamma(u)du$$
$$= -\tfrac{1}{2}\log 2 + 2\int_0^1 \log\Gamma(u)du.$$

So $C = \frac{1}{2}\log 2\pi$.

Lemma 2. Hermite's Formula

For $x>0$, *let* $A(x) = (x-\frac{1}{2})\log x - x + \frac{1}{2}\log 2\pi$ *and* $R(x) = \log\Gamma(x) - A(x)$, *then*

$$R(x) = \int_0^1 (t-\tfrac{1}{2})\frac{d}{dt}\log\Gamma(x+t)dt. \quad \text{.......................(4)}$$

Proof. On integrating by parts, the right hand side of (4) is

$$[(t-\tfrac{1}{2})\log\Gamma(x+t)]_0^1 - \int_0^1 \log\Gamma(x+t)dt,$$

which is the required result, by Lemma 1.

Lemma 3. *For* $x>0$,

$$\frac{d^2}{dx^2}\log\Gamma(x) \geqq 0.$$

Proof. Since

$$\frac{d^2}{dx^2}\log\Gamma(x) = \frac{\Gamma''(x)\Gamma(x) - \{\Gamma'(x)\}^2}{\{\Gamma(x)\}^2},$$

the result will follow if $H(x) = \Gamma''(x)\Gamma(x) - \{\Gamma'(x)\}^2$ is positive.

Now in (1), differentiate under the integral sign and we obtain

$$H(x) = \int_0^\infty\int_0^\infty (tu)^{x-1}e^{-t-u}\{(\log t)^2 - \log t\log u\}dtdu,$$

so interchanging t and u gives

$$2H(x) = \int_0^\infty\int_0^\infty (tu)^{x-1}e^{-t-u}(\log t - \log u)^2 dtdu \geqq 0.$$

Theorem 1. *For* $x>0$,

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma + \sum_{r=1}^{\infty}\left(\frac{1}{r} - \frac{1}{x+r}\right).$$

Proof. By Lemma 2,

$$\log \Gamma(x) = (x-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + \int_0^1 (t-\tfrac{1}{2})\frac{d}{dt}\log\Gamma(x+t)dt,$$

so

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{2x} + \frac{d}{dx}\int_0^1 (t-\tfrac{1}{2})\frac{d}{dt}\log\Gamma(x+t)dt$$

$$= \log x - \frac{1}{2x} + \int_0^1 (t-\tfrac{1}{2})\frac{d^2}{dt^2}\log\Gamma(x+t)dt. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Taking logarithms of $\Gamma(x+n+1) = (x+n)(x+n-1)\ldots(x+1)x\Gamma(x)$ and differentiating, we have

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \sum_{r=1}^{n}\frac{1}{x+r} + \frac{\Gamma'(x+n+1)}{\Gamma(x+n+1)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Now in (5) replace x by $x+n+1$ and substitute in the right hand side of (6) giving

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma_n + \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{x+r}\right) + \log\left(1+\frac{x+1}{n}\right) - \frac{1}{2(x+n+1)} + R_n(x), \quad \ldots\ldots(7)$$

where $\gamma_n = \sum_{r=1}^{n}\frac{1}{r} - \log n$ and $R_n(x) = \int_0^1 (t-\tfrac{1}{2})\frac{d^2}{dt^2}\log\Gamma(x+n+1+t)dt.$

Since $|t-\tfrac{1}{2}| \leqq \tfrac{1}{2}$ for $0 \leqq t \leqq 1$, and $\frac{d^2}{dt^2}\log\Gamma(x+n+1+t) \geqq 0$,

$$|R_n(x)| \leqq \tfrac{1}{2}\int_0^1 \frac{d^2}{dt^2}\log\Gamma(x+n+1+t)dt = \tfrac{1}{2}\frac{d}{dx}\int_0^1 \frac{d}{dt}\log\Gamma(x+n+1+t)dt$$

$$= \frac{1}{2(x+n+1)}.$$

The theorem follows by letting $n\to\infty$ in (7).

By differentiating we have

Theorem 2. *For* $x>0$ *and* $n>1$,

$$\frac{d^n}{dx^n}\log\Gamma(x) = \sum_{r=0}^{\infty}\frac{(-1)^n(n-1)!}{(x+r)^n}.$$

Theorem 3. *With the notation of Lemma* 2,

$$-\frac{1}{192x^3} < R(x) - \frac{1}{12x} < 0.$$

https://doi.org/10.1017/S0013091500008919 Published online by Cambridge University Press

Proof. Integrating the right hand side of (4) by parts gives

$$R(x) = \tfrac{1}{2}\int_0^1 (t-t^2)\,\frac{d^2}{dt^2}\log\Gamma(x+t)dt$$

$$= \frac{1}{12}\int_0^1 \frac{d^2}{dt^2}\log\Gamma(x+t)dt + \int_0^1 \left(-\frac{1}{12}+\frac{t}{2}-\frac{t^2}{2}\right)\frac{d^2}{dt^2}\log\Gamma(x+t)dt. \quad \ldots(8)$$

The first integral on the right hand side of (8) is

$$\frac{1}{12}\frac{d}{dx}\int_0^1 \frac{d}{dt}\log\Gamma(x+t)dt = \frac{1}{12}\frac{d}{dx}\log x = \frac{1}{12x},$$

and the second gives, on integrating twice by parts,

$$R(x) = \frac{1}{12x} - \frac{1}{24}\int_0^1 t^2(1-t)^2\,\frac{d^4}{dt^4}\log\Gamma(x+t)dt.$$

Now by Theorem 2,

$$\frac{d^4}{dt^4}\log\Gamma(x+t)\geqq 0, \text{ so } R(x)< \frac{1}{12x}.$$

Also, since

$$t^2(1-t)^2 \leqq \frac{1}{16} \text{ for } 0\leqq t\leqq 1,$$

$$\int_0^1 t^2(1-t)^2\,\frac{d^4}{dt^4}\log\Gamma(x+t)dt < \frac{1}{16}\frac{d^3}{dx^3}\int_0^1 \frac{d}{dt}\log\Gamma(x+t)dt = \frac{1}{8x^3}.$$

The extension of Theorem 3 to the case of complex z, $|\arg z|\leqq\pi-\delta$, $\delta>0$ can be effected by using the inequality $|z+p|\geqq(|z|+p)\sin\delta/2$, for $p>0$, when for example

$$\left|\frac{d^4}{dt^4}\log\Gamma(z+t)\right| = \left|\sum_{n=0}^{\infty}\frac{6}{(z+t+n)^4}\right| \leqq \frac{1}{\sin^4\delta/2}\sum_{n=0}^{\infty}\frac{6}{(|z|+t+n)^4}$$

$$= \frac{1}{\sin^4\delta/2}\frac{d^4}{dt^4}\log\Gamma(|z|+t),$$

which reduces the theory to the real case.

Theorem 4. *For $x>0$ and integers $n>0$*

$$(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-x}\Gamma(x) = \Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\ldots\Gamma\left(\frac{x+n-1}{n}\right).$$

Proof. Let $f(x)=\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\ldots\Gamma\left(\frac{x+n-1}{n}\right)$, then by Theorem 2,

$$\frac{d^2}{dx^2}\log f(x) = \sum_{r=0}^{\infty}\left\{\frac{1}{(x+nr)^2}+\frac{1}{(x+1+nr)^2}+\ldots+\frac{1}{(x+n-1+nr)^2}\right\}$$

$$= \sum_{s=0}^{\infty}\frac{1}{(x+s)^2} = \frac{d^2}{dx^2}\log\Gamma(x).$$

So $f(x) = be^{cx}\Gamma(x)$. But

$$f(x+1) = \frac{x}{n} f(x), \text{ so } e^c = \frac{1}{n}.$$

Now b is a function of n only, and

$$\log b = x \log n + \sum_{r=0}^{n-1} \log \Gamma\left(\frac{x+r}{n}\right) - \log \Gamma(x).$$

So by Theorem 3, and the result

$$\log\left(1+\frac{k}{x}\right) = \frac{k}{x} + O\left(\frac{1}{x^2}\right),$$

$$\log b = \tfrac{1}{2}(n-1)\log 2\pi + x \log n + \sum_{r=0}^{n-1}\left(\frac{x+r}{n} - \tfrac{1}{2}\right)\log\left(\frac{x+r}{n}\right)$$
$$- \sum_{r=0}^{n-1}\left(\frac{x+r}{n}\right) - (x-\tfrac{1}{2})\log x + x + O\left(\frac{1}{x}\right)$$
$$= \tfrac{1}{2}(n-1)\log 2\pi + \tfrac{1}{2}\log n + \sum_{r=0}^{n-1}\left(\frac{x+r}{n} - \tfrac{1}{2}\right)\log(x+r) - \tfrac{1}{2}(n-1)$$
$$- (x-\tfrac{1}{2})\log x + O\left(\frac{1}{x}\right)$$
$$= \tfrac{1}{2}(n-1)\log 2\pi + \tfrac{1}{2}\log n + \sum_{r=0}^{n-1}\left(\frac{x+r}{n} - \tfrac{1}{2}\right)\log\left(1+\frac{r}{x}\right)$$
$$- \tfrac{1}{2}(n-1) + O\left(\frac{1}{x}\right)$$
$$= \tfrac{1}{2}(n-1)\log 2\pi + \tfrac{1}{2}\log n + \sum_{r=0}^{n-1}\left(\frac{x+r}{n} - \tfrac{1}{2}\right)\left(\frac{r}{x}\right) - \tfrac{1}{2}(n-1) + O\left(\frac{1}{x}\right)$$
$$= \tfrac{1}{2}(n-1)\log 2\pi + \tfrac{1}{2}\log n + O\left(\frac{1}{x}\right).$$

Now let $x \to \infty$, and $b = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}}$.

REFERENCES

(1) E. Artin, *Einführung in die Theorie der Gammafunktion* (Leipzig, 1931).

(2) N. Bourbaki, *Éléments de Mathématique*, Livre IV, *Fonctions d'une Variable Réelle*, Chapter VII (Paris, 1951).

(3) E. T. Copson, *Functions of a complex Variable* (Oxford, 1935).

(4) G. A. Gibson, *Advanced Calculus* (London, 1931).

(5) C. Hermite, *Oeuvres*, Tome IV (Paris, 1917) = *Amer. J. Math.*, **17** (1895), 111-116.

(6) D. V. Widder, *Advanced Calculus* (2nd ed., London, 1961).

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