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SCHUR PROPERTY AND ℓ_P ISOMORPHIC COPIES IN MUSIELAK–ORLICZ SEQUENCE SPACES

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The author shows that if the dual of a Musielak–Orlicz sequence space ℓ_Φ is a stabilized asymptotic ℓ_∞ space with respect to the unit vector basis, then ℓ_Φ is saturated with complemented copies of ℓ_1 and has the Schur property. A sufficient condition is found for the isomorphic embedding of ℓ_p spaces into Musielak–Orlicz sequence spaces.

1. INTRODUCTION

The notion of asymptotic ℓ_p spaces first appeared in [**14**], where the collection of spaces that are now known as stabilised asymptotic ℓ_p spaces were introduced. Later in [**13**] more general collection of spaces, known as asymptotic ℓ_p spaces, were introduced. Characterisation of the stabilised asymptotic ℓ_∞ Musielak–Orlicz sequence space was given in [**4**].

A Banach space X is said to have the Schur property if every weakly null sequence is norm null. It is well known that ℓ_1 has the Schur property and it's dual ℓ_∞ is obviously a stabilised asymptotic ℓ_∞ space with respect to the unit vector basis. A characterisation of the Musielak–Orlicz sequence spaces ℓ_Φ possessing the Schur property, as well as sufficient conditions for ℓ_Φ and weighted Orlicz sequence spaces $\ell_M(w)$ to have the Schur property were found in [**8**]. Using an idea from [**1**] we find that if the dual of a Musielak–Orlicz sequence space is a stabilised asymptotic ℓ_∞ space then it is saturated with complemented copies of ℓ_1 and has the Schur property. While simple necessary conditions for embedding of ℓ_p spaces into Musielak–Orlicz spaces ℓ_Φ were found in [**16**], the problem of finding analogous sufficient conditions, as it is done in [**11**] for Orlicz ℓ_M, appeared more difficult. We find a sufficient condition for the existence of an ℓ_p copy in ℓ_Φ in Paragraph 4.

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2. Preliminaries

We use the standard Banach space terminology from **[11]**. Let us recall that an Orlicz function M is even, continuous, non-decreasing convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We say that M is a non-degenerate Orlicz function if $M(t) > 0$ for every $t > 0$. A sequence $\Phi = \{\Phi_i\}_{i=1}^{\infty}$ of Orlicz functions is called a Musielak-Orlicz function.

The Musielak-Orlicz sequence space ℓ_Φ, generated by a Musielak-Orlicz function Φ is the set of all real sequences $\{x_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty}\Phi_i(\lambda x_i) < \infty$ for some $\lambda > 0$. The Luxemburg's norm is defined by

$$\|x\|_\Phi = \inf\left\{r > 0 : \sum_{i=1}^{\infty} \Phi_i(x_i/r) \leqslant 1\right\}.$$

We denote by h_Φ the closed linear subspace of ℓ_Φ, generated by all $x \in \ell_\Phi$, such that $\sum_{i=1}^{\infty} \Phi_i(\lambda x_i) < \infty$ for every $\lambda > 0$.

If the Musielak-Orlicz function Φ consists of one and the same function M one obtains the Orlicz sequence spaces ℓ_M and h_M.

Let $1 \leqslant p_i$, $i \in \mathbb{N}$ be a sequence of reals. The Musielak-Orlicz sequence space ℓ_Φ, where $\Phi = \{t^{p_i}\}_{i=1}^{\infty}$ is called a Nakano sequence space and is denoted by $\ell_{\{p_i\}}$. In **[3]** it was proved that two Nakano sequence spaces $\ell_{\{p_i\}}$, $\ell_{\{q_i\}}$ are isomorphic if and only if there exists $0 < C < 1$ such that

$$\sum_{i=1}^{\infty} C^{1/|p_i - q_i|} < \infty.$$

An extensive study of Orlicz and Musielak-Orlicz spaces can be found in **[11, 15, 6, 9]**.

Definition 2.1: We say that the Musielak-Orlicz function Φ satisfies the δ_2 condition at zero if there exist constants $K, \beta > 0$ and a non-negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$ such that for every $n \in \mathbb{N}$

$$\Phi_n(2t) \leqslant K\Phi_n(t) + c_n$$

provided $t \in \left[0, \Phi_n^{-1}(\beta)\right]$.

The spaces ℓ_Φ and h_Φ coincide if and only if Φ has the δ_2 condition at zero.

Recall that given Musielak-Orlicz functions Φ and Ψ the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms if and only if Φ is equivalent to Ψ, that is there exist constants $K, \beta > 0$ and a non-negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$, such that for every $n \in \mathbb{N}$ the inequalities

$$\Phi_n(Kt) \leqslant \Psi_n(t) + c_n \quad \text{and} \quad \Psi_n(Kt) \leqslant \Phi_n(t) + c_n$$

hold for every $t \in \left[0, \min\left(\Phi_n^{-1}(\beta), \Psi_n^{-1}(\beta)\right)\right]$.

Throughout this paper M will always denote an Orlicz function while Φ is an Musielak–Orlicz function. As the properties we are dealing with are preserved by isomorphisms without loss of generality we may assume that Φ consists entirely of non-degenerate Orlicz functions, such that for every $i \in \mathbb{N}$ the Orlicz function Φ_i is differentiable, $\Phi_i'(0) = 0$ and $\Phi_i(1) = 1$. Indeed, we can always choose a sequence $\{\alpha_i\}$, such that $\alpha_i \leqslant 1/2$, $i \in \mathbb{N}$, $\sum_{i=1}^{\infty} \Phi_i(\alpha_i) < \infty$ and consider the sequence of functions $\varphi_i(t) = \int_0^t (\psi_i(s)/s)\, ds$, where

$$\psi_i(t) = \begin{cases} \dfrac{\Phi_i(\alpha_i)}{\alpha_i^2} t^2, & 0 \leqslant t \leqslant \alpha_i \\ \Phi_i(t), & t \geqslant \alpha_i \end{cases}.$$

Obviously the Musielak–Orlicz function $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ consists of differentiable functions and $\varphi_i'(0) = 0$ for every $i \in \mathbb{N}$.

For every $t \in [0, \alpha_i]$ we have $\varphi_i(\alpha_i) = (\Phi_i(\alpha_i))/2$ and

$$\varphi_i(t) = \int_0^t \frac{\psi_i(s)}{s}\, ds = \int_0^t \frac{\Phi_i(\alpha_i)}{\alpha_i^2}\, s ds = \frac{\Phi_i(\alpha_i)}{2\alpha_i^2} t^2 .$$

For every $t \geqslant \alpha_i$ we have

$$\varphi_i(t) = \int_0^{\alpha_i} \frac{\Phi_i(\alpha_i)}{\alpha_i^2}\, s ds + \int_{\alpha_i}^t \frac{\Phi_i(s)}{s}\, ds = \frac{\Phi_i(\alpha_i)}{2} + \int_{\alpha_i}^t \frac{\Phi_i(s)}{s}\, ds .$$

By the convexity of Φ_i follows that

$$\varphi_i(t) \leqslant \frac{\Phi_i(\alpha_i)}{2} + \Phi_i(t) \tag{1}$$

for every $t \geqslant 0$.

In order to get the opposite inequality we consider separately three cases:

(I) Let $\alpha_i \leqslant t/2$ then

$$\begin{aligned} \varphi_i(t) &= \int_0^{\alpha_i} \frac{\psi_i(s)}{s} ds + \int_{\alpha_i}^{t/2} \frac{\psi_i(s)}{s} ds + \int_{t/2}^t \frac{\psi_i(s)}{s} ds \\ &\geqslant \frac{\Phi_i(\alpha_i)}{2} + \int_{t/2}^t \frac{\Phi_i(s)}{s} ds \geqslant \frac{\Phi_i(\alpha_i)}{2} + \Phi_i(t/2) . \end{aligned}$$

(II) Let $t/2 \leqslant \alpha_i \leqslant t$ then

$$\begin{aligned} \varphi_i(t) &= \frac{\Phi_i(\alpha_i)}{2} + \int_{t/2}^t \frac{\Phi_i(s)}{s} ds - \int_{t/2}^{\alpha_i} \frac{\Phi_i(s)}{s} ds \\ &\geqslant \frac{\Phi_i(\alpha_i)}{2} + \Phi_i(t/2) - \Phi_i(\alpha_i) = \Phi_i(t/2) - \frac{\Phi_i(\alpha_i)}{2} . \end{aligned}$$

(III) Let $t \leqslant \alpha_i$ then

$$\frac{\Phi_i(t)}{2} \leqslant \varphi_i(t) + \frac{\Phi_i(\alpha_i)}{2}.$$

Thus

$$\frac{\Phi_i(t/2)}{2} \leqslant \varphi_i(t) + \frac{\Phi_i(\alpha_i)}{2} \tag{2}$$

for every $t \geqslant 0$. By (1) and (2) it follows that $\varphi \sim \Phi$ and thus $\ell_\varphi \cong \ell_\Phi$. To complete the proof, it is enough to normalise the functions φ_i by considering $\widetilde{\varphi} = \{\varphi_i/\varphi_i(1)\}_{i=0}^{\infty}$.

DEFINITION 2.2: For an Orlicz function M, such that $\lim\limits_{t\to 0} M(t)/t = 0$ the function

$$N(x) = \sup\{t|x| - M(t) : t \geqslant 0\},$$

is called the function complementary to M.

DEFINITION 2.3: The Musielak–Orlicz function $\Psi = \{\Psi_j\}_{j=1}^{\infty}$, defined by

$$\Psi_j(x) = \sup\{t|x| - \Phi_j(t) : t \geqslant 0\}, j = 1, 2, \ldots, n, \ldots$$

is called complementary to Φ.

Let us note that the condition $\lim\limits_{t\to 0} M(t)/t = 0$ ensures that the complementary function N is always non-degenerate. Observe that if N is function complementary to M, then M is complementary to N and if the Musielak–Orlicz function Ψ is complementary to the Musielak–Orlicz function Φ, then Φ is function complementary to Ψ. Throughout this paper the function complementary to the Musielak–Orlicz function Φ is denoted by Ψ.

It is well known that $h_M^* \cong \ell_N$ and $h_\Phi^* \cong \ell_\Psi$. The equivalent norm in ℓ_Φ is the Orlicz norm

$$\|x\|_\Phi^O = \sup\left\{\sum_{j=1}^{\infty} x_j y_j : \sum_{j=1}^{\infty} \Psi_j(y_j) \leqslant 1\right\},$$

which satisfies the inequalities (see for example,[7])

$$\|\cdot\|_\Phi \leqslant \|\cdot\|_\Phi^O \leqslant 2\|\cdot\|_\Phi.$$

We shall use the Hölder's inequality: $\sum\limits_{j=1}^{\infty} |x_j y_j| \leqslant \|x\|_\Phi^O \|y\|_\Psi$, which holds for every $x = \{x_j\}_{j=1}^{\infty} \in \ell_\Phi$ and $y = \{y_j\}_{j=1}^{\infty} \in \ell_\Psi$, where Φ and Ψ are complementary Musielak–Orlicz functions.

By $\{e_j\}_{j=1}^{\infty}$ and $\{e_j^*\}_{j=1}^{\infty}$ we denote the unit vector basis in h_Φ and h_Ψ respectively. For a Banach space X with a basis $\{v_i\}_{i=1}^{\infty}$ and element $x \in X$, $x = \sum\limits_{i=1}^{\infty} x_i v_i$ we define $\operatorname{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$. We write $n \leqslant x$ if $n \leqslant \min\{\operatorname{supp} x\}$ and $x < y$ if $\max\{\operatorname{supp} x\} < \min\{\operatorname{supp} y\}$. We say that x is a block vector with respect to the basis $\{v_i\}_{i=1}^{\infty}$ if $x = \sum\limits_{i=p}^{q} x_i v_i$ for some finite p and q and we say that x is a normalised block vector if it is a block vector and $\|x\| = 1$.

DEFINITION 2.4: A Banach space X is said to be stabilised asymptotic ℓ_∞ with respect to a basis $\{v_i\}_{i=1}^\infty$, if there exists a constant $C \geqslant 1$, such that for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$, so that whenever $N \leqslant x_1 < \cdots < x_n$ are successive normalised block vectors, then $\{x_i\}_{i=1}^n$ are C–equivalent to the unit vector basis of ℓ_∞^n; that is,

$$\frac{1}{C}\max_{1\leqslant i\leqslant n}|a_i| \leqslant \left\|\sum_{i=1}^n a_i x_i\right\| \leqslant C\max_{1\leqslant i\leqslant n}|a_i|.$$

The following characterisation of the stabilised asymptotic ℓ_∞ Musielak–Orlicz sequence spaces is due to Dew:

PROPOSITION 2.1. ([4, Proposition 4.5.1]) *Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a Musielak–Orlicz function. Then the following are equivalent:*

(i) *h_Φ is stabilised asymptotic ℓ_∞ (with respect to its natural basis $\{e_j\}_{j=1}^\infty$);*

(ii) *there exists $\lambda > 1$ such that for all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $N \leqslant p \leqslant q$ and $\sum_{j=p}^{q}\Phi_j(a_j) \leqslant 1$, then*

$$\sum_{j=p}^{q}\Phi_j(a_j/\lambda) \leqslant \frac{1}{n}.$$

Let X be a Banach space. By $Y \hookrightarrow X$ we denote that Y is isomorphic to a subspace of X.

3. MUSIELAK–ORLICZ SPACES WITH STABILISED ASYMPTOTIC ℓ_∞ DUAL WITH RESPECT TO THE UNIT VECTOR BASIS

We start with the following

LEMMA 3.1. *Let Φ have the δ_2 condition at zero and h_Ψ, generated by the Musielak–Orlicz function Ψ, complementary to Φ, be stabilised asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then every normalised block basis $\{x^{(n)}\}_{n=1}^\infty$ of the unit vector basis in ℓ_Φ contains a subsequence $\{x^{(n_i)}\}_{i=1}^\infty$ such that:*

(a) *$\{x^{(n_i)}\}_{i=1}^\infty$ is equivalent to the unit vector basis of ℓ_1;*

(b) *The closed subspace $[x^{(n_i)}]_{i=1}^\infty$ generated by $\{x^{(n_i)}\}_{i=1}^\infty$ is complemented in ℓ_Φ by means of a projection of norm less then or equal to 4λ, where λ is the constant from Proposition 2.1.*

PROOF: (a) Let $\{x^{(n)}\}_{n=1}^\infty$ be a normalised block basis of ℓ_Φ, where $x^{(n)} = \sum_{j=m_n+1}^{m_{n+1}} x_j^{(n)} e_j$, and $\{m_n\}$ is a strictly increasing sequence of naturals. For every $n \in \mathbb{N}$

there exists $y^{(n)} = \sum\limits_{j=1}^{\infty} y_j^{(n)} e_j^* \in h_\Psi$ such that

$$\sum_{j=1}^{\infty} \Psi_j(y_j^{(n)}) \leqslant 1 \quad \text{and} \quad \sum_{j=1}^{\infty} y_j^{(n)} x_j^{(n)} \geqslant 1/2.$$

Without loss of generality we may assume that supp $y^{(n)} \equiv$ supp $x^{(n)}$. We claim that

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \Psi_j\Bigl(\frac{y_j^{(n)}}{\lambda}\Bigr) = \lim_{n\to\infty} \sum_{j=m_n+1}^{m_{n+1}} \Psi_j\Bigl(\frac{y_j^{(n)}}{\lambda}\Bigr) = 0,$$

where $\lambda > 1$ is the constant from Proposition 2.1.

Indeed, by assumption h_Ψ is stabilised asymptotic ℓ_∞ space and according to Proposition 2.1 there exists $\lambda > 1$ such that for every $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ so, that whenever $m_n \geqslant N$ the inequality holds $\sum\limits_{j=m_n+1}^{m_{n+1}} \Psi_j\bigl(y_j^{(n)}/\lambda\bigr) \leqslant 1/m$. Thus $\lim\limits_{n\to\infty} \sum\limits_{j=m_n+1}^{m_{n+1}} \Psi_j\bigl(y_j^{(n)}/\lambda\bigr) = 0$.

Now passing to a subsequence we get a sequence $\{y^{(n_k)}\}_{k\in\mathbb{N}}$, $y^{(n_k)} = \sum\limits_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} e_j^*$ such that

$$\sum_{k=1}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) \leqslant 1.$$

Denote $y = \sum\limits_{k=1}^{\infty} y^{(n_k)} = \sum\limits_{k=1}^{\infty}\Bigl(\sum\limits_{j=p_k}^{q_k} y_j^{(n_k)} e_j^*\Bigr)$. Obviously $y \in \ell_\Psi$ and $\|y\|_\Psi \leqslant \lambda$. Thus $\|y\|_\Psi^O \leqslant 2\|y\|_\Psi \leqslant 2\lambda$.

Let now $a = \{a_k\}_{k=1}^{\infty} \in \ell_1$. Then

$$\begin{aligned}\Bigl\|\sum_{k=1}^{\infty} a_k x^{(n_k)}\Bigr\|_\Phi &\geqslant \frac{1}{\|y\|_\Psi^O} \sum_{k=1}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \bigl|a_k y_j^{(n_k)} x_j^{(n_k)}\bigr| \geqslant \frac{1}{2\lambda} \sum_{k=1}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \Bigl|a_k y_j^{(n_k)} x_j^{(n_k)}\Bigr| \\ &\geqslant \frac{1}{2\lambda} \sum_{k=1}^{\infty} |a_k| \sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j^{(n_k)} \geqslant \frac{1}{4\lambda} \sum_{k=1}^{\infty} |a_k| = \frac{1}{4\lambda}\|a\|_1.\end{aligned}$$

Obviously $\Bigl\|\sum\limits_{k=1}^{\infty} a_k x^{(n_k)}\Bigr\|_\Phi \leqslant \|a\|_1$ and thus $\{x^{(n_k)}\}_{k=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_1.

(b) Define now for each $k \in \mathbb{N}$ the functional $F_k : \ell_\Phi \to \mathbb{R}$ by

$$F_k(x) = \frac{1}{\sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j^{(n_k)}} \sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j$$

and the map $P : \ell_\Phi \to \ell_\Phi$ by $P(x) = \sum\limits_{k=1}^{\infty} F_k(x) x^{(n_k)}$. Then for every $k \in \mathbb{N}$, $\|F_k\| \leqslant 2\|y^{(n_k)}\|_\Psi \leqslant 2\Bigl(1 + \sum\limits_{j=p_{n_k}}^{q_{n_k}} \Psi_j(y_j^{(n_k)})\Bigr) \leqslant 4$. Furthermore P is a projection of ℓ_Φ onto

$[x_{n_k}]_{k=1}^{\infty}$ with

$$\begin{aligned} \|P\| &= \sup_{\|x\|_\Phi \leqslant 1} \left\| \sum_{k=1}^{\infty} \frac{\sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j}{\sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j^{(n_k)}} x^{(n_k)} \right\| \leqslant 2 \sup_{\|x\|_\Phi \leqslant 1} \sum_{k=1}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \left| y_j^{(n_k)} x_j \right| \\ &\leqslant 2 \sup_{\|x\|_\Phi \leqslant 1} \sum_{j=1}^{\infty} |y_j x_j| \leqslant 2 \sup_{\|x\|_\Phi \leqslant 1} \|y_j\|_\Psi^O \|x\|_\Phi \leqslant 4\lambda . \end{aligned}$$

∎

The following two theorems are simple corollaries of Lemma 3.1.

THEOREM 1. *Let Φ have the δ_2 condition at zero and h_Ψ, generated by the Musielak–Orlicz function Ψ, complementary to Φ, be stabilised asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$. Then ℓ_Φ has the Schur property.*

PROOF: The proof is an easy consequence of the Kaminska, Mastylo characterisation of Musielak–Orlicz spaces possessing Schur property ([8, Theorem 4.4]). Consider a Φ–convex block of Φ, that is, a sequence of convex functions $\left\{ M_i(t) = \sum_{j=n_i+1}^{n_{i+1}} \Phi_j(t\alpha_j) \right\}_{i=1}^{\infty}$, where n_i is a strongly increasing sequence in $\mathbb{N}$ and $\{\alpha_j\}_{j=1}^{\infty}$ is a sequence of positive numbers such that $\sum_{j=n_i+1}^{n_{i+1}} \Phi_j(\alpha_j) = 1$ for each $i \in \mathbb{N}$. It is easy to observe that the sequence $\left\{ u_i = \sum_{j=n_i+1}^{n_{i+1}} \alpha_j e_j \right\}_{i=1}^{\infty}$ is a normalised block-basis of the unit vector basis of ℓ_Φ. Lemma 3.1 now implies that the closed linear span $[u_{i_k}]_{k=1}^{\infty}$ for appropriate subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ is isomorphic to ℓ_1. On the other hand $[u_{i_k}]_{k=1}^{\infty}$ is obviously isometrically isomorphic to the Musielak–Orlicz space $\ell_{\{M_{i_k}\}}$, generated by the subsequence $\{M_{i_k}\}$ of the given Φ–convex block. Thus every Φ–convex block contains a subsequence equivalent to a linear function and therefore ℓ_Φ has the Schur property. ∎

THEOREM 2. *Let Φ have the δ_2 condition at zero and h_Ψ, generated by the Musielak–Orlicz function Ψ, complementary to Φ, be stabilised asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$. Then every subspace Y of ℓ_Φ contains an isomorphic copy of ℓ_1 which is complemented in ℓ_Φ.*

PROOF: According to a well known result of Bessaga and Pelczinski [2] every infinite dimensional closed subspace Y of ℓ_Φ has a subspace Z isomorphic to a subspace of ℓ_Φ, generated by a normalised block basis of the unit vector basis of ℓ_Φ. Now to finish the proof it is enough to observe that by Lemma 3.1 the space Z contains a complemented subspace of ℓ_Φ, which is isomorphic to ℓ_1. ∎

REMARK. It is well known ([18]) that every subspace of Musielak–Orlicz sequence space ℓ_Φ with Φ satisfying the δ_2 condition, contains ℓ_p for some $p \in [1, \infty]$. If ℓ_Φ has in addition the Schur property, as no $\ell_p, p \neq 1$ has the Schur property, it follows that ℓ_Φ is ℓ_1 saturated.

4. ℓ_p copies in Musielak–Orlicz sequence spaces

Let Φ be a Musielak–Orlicz function consisting of differentiable Orlicz functions. Denote:

$$a(\Phi_n) = \sup\Big\{p > 0 : p \leqslant \frac{x\Phi_n'(x)}{\Phi_n(x)}, x \in (0,1]\Big\};$$

$$b(\Phi_n) = \inf\Big\{q > 0 : q \geqslant \frac{x\Phi_n'(x)}{\Phi_n(x)}, x \in (0,1]\Big\}.$$

The following indexes, introduced by Yamamuro ([**17**])

$$a(\Phi) = \liminf_{n\to\infty} a(\Phi_n) \ , \ b(\Phi) = \limsup_{n\to\infty} b(\Phi_n)$$

appear to be useful in the study of Musielak–Orlicz sequence spaces (see for example [**11, 16, 8, 12**]). Obviously $1 \leqslant a(\Phi) \leqslant b(\Phi) \leqslant \infty$. By the results of Woo ([**18**]) and Katirtzoglou ([**9**]) it follows that an Musielak–Orlicz function Φ satisfies the δ_2 condition at zero if and only if $b(\Phi) < \infty$. Analogously to the case of the classical Orlicz sequence spaces if ℓ_p, ,$p \geqslant 1$ or c_0 for $p = \infty$ is isomorphic to a subspace of h_Φ, then $p \in \big[a(\Phi), b(\Phi)\big]$ (see [**16, 18**]). However, the converse fails to be true in general (see [**16**]) for Musielak–Orlicz sequence spaces, which confirms their more complex structure. Sufficient conditions for the isomorphical embedding of ℓ_p, ,$p \geqslant 1$ in h_Φ are given by the following.

Theorem 3. *Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a Musielak–Orlicz function and $p \in \big[a(\Phi), b(\Phi)\big]$. If there exist sequences $\{\tau_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$, $\{\varepsilon_j\}_{j=1}^\infty$ and constants $0 < k < 1 < K$ such that:*

(1) $\varepsilon_j \geqslant 0\,,\ 0 < y_j \leqslant 1 \quad 0 < \tau_j < 1$ *for every* $j \in \mathbb{N}$;

(2) $\lim\limits_{j\longrightarrow\infty} \tau_j = 0\,;$

(3) $\sum\limits_{j=1}^\infty \Phi_j(y_j) = \infty\,;$

(4) $kt^{\varepsilon_j} \leqslant \big(\Phi_j(ty_j)\big)/\big(t^p\Phi_j(y_j)\big) \leqslant K(1/t)^{\varepsilon_j}$ *for every* $t \in [\tau_j, 1]\,;$

(5) $\sum\limits_{j=1}^\infty C^{1/\varepsilon_j} < \infty$ *for some* $0 < C < 1\,,$

then $\ell_p \hookrightarrow h_\Phi$.

Proof: The condition (5) obviously implies $\lim\limits_{j\longrightarrow\infty} \varepsilon_j = 0$.

We may assume that $\tau_j < 1/2$ for every j. Indeed, by (2) we easily get $\tau_j < 1/2$, $j < j_0$ for some j_0 and can consider the Musielak–Orlicz sequence space $h_{\{\Phi_j\}_{j=j_0}^\infty} \cong h_\Phi$.

Consider first the case: $\sharp\big\{j \in \mathbb{N} : \Phi(y_j) \geqslant 1/2\big\} < \infty$. For the same reason as above we may assume that $\Phi(y_j) \leqslant 1/2$ for every $j \in \mathbb{N}$.

Find sequence of naturals $\{k_n\}_{n=1}^\infty$, $k_1 = 0$, such that for every $n \in \mathbb{N}$:

$$\frac{1}{2} \leqslant \sum_{j=k_n+1}^{k_{n+1}-1} \Phi_j(y_j) < 1\ ,\ \Phi_{k_{n+1}}(y_{k_{n+1}}) \geqslant 1 - \sum_{j=k_n+1}^{k_{n+1}-1} \Phi_j(y_j)\,.$$

Put

$$\varphi_n(t) = \sum_{j=k_n+1}^{k_{n+1}-1} \Phi_j(y_j t) + \Phi_{k_{n+1}}(\overline{y}_{k_{n+1}} t),$$

where

$$(3) \qquad \sum_{j=k_n+1}^{k_{n+1}-1} \Phi_j(y_j) + \Phi_{k_{n+1}}(\overline{y}_{k_{n+1}}) = 1.$$

Obviously

$$(4) \qquad \sum_{j=k_n+1}^{k_{n+1}} \Phi(y_j) < \frac{3}{2}$$

and $0 < \overline{y}_{k_{n+1}} \leqslant y_{k_{n+1}}$. Let us note that $u_n = \sum_{j=k_n+1}^{k_{n+1}-1} y_j e_j + \overline{y}_{k_{n+1}} e_{k_{n+1}}, n = 1, 2, \ldots$ represents a normalised block basis of the unit vector basis of h_Φ. Obviously the Musielak–Orlicz sequence space $h_{\overline{\Phi}}$, generated by the sequence $\{\varphi_n\}$ is isometrically isomorphic to $[u_n]_{n=1}^\infty$, which in turn is isomorphic to a subspace of h_Φ. Further we find a sequence of $\{n_m\}_{m=1}^\infty$, such that $\tau_j \leqslant 1/m^2$ for $j > k_{n_m}$. Following [**11, 10**] we easily check that the functions φ_{n_m}, $m = 1, 2, \ldots$ are equi–continuous in $[0, 1/2]$. Indeed, from

$$\Phi_j(t) = \int_0^t \Phi_j'(t)dt \geqslant \int_{t/2}^t \Phi_j'(t)dt \geqslant \frac{1}{2} t\Phi_j(t/2)$$

it follows immediately

$$\left|\frac{\Phi_j(\mu t_1)}{\Phi_j(\mu)} - \frac{\Phi_j(\mu t_2)}{\Phi_j(\mu)}\right| \leqslant |t_1 - t_2| \frac{\mu\Phi_j'(\mu/2)}{\Phi_j(\mu)} \leqslant 2|t_1 - t_2|$$

for every $0 \leqslant t_1, t_2 \leqslant 1/2$ and any $\mu > 0$. Now it is enough to apply the last inequality to the functions φ_{n_m}, taking into account (3). The functions φ_{n_m}, $m = 1, 2, \ldots$ are also uniformly bounded in $[0, 1/2]$. Using the Arzela-Ascoli theorem by passing to a subsequence if necessary, which in order to simplify the notations we denote $\{\varphi_{n_m}\}_{m=1}^\infty$ too, we have that $\{\varphi_{n_m}\}_{m=1}^\infty$ converges uniformly to a function φ on $[0, 1/2]$, satisfying the inequalities $\|\varphi_{n_m} - \varphi\|_\infty \leqslant 1/2^m$ for every $m \in \mathbb{N}$. Obviously φ is an Orlicz function on [0,1/2] as uniform limit of Orlicz functions and the Musielak–Orlicz sequence space $h_{\{\varphi_{n_m}\}}$ is isomorphic to the Orlicz space h_φ, when φ is non-degenerated. If we take into account that $h_{\{\varphi_{n_m}\}}$ is isometrically isomorphic to $[u_{n_m}]_{m=1}^\infty$ to finish the proof it is enough to show that h_φ and ℓ_p consist of the same sequences. Before starting the last part of the proof we mention that according to the result from [**3**], mentioned in the preliminaries, the condition (5) implies that the Nakano spaces $\ell_{\{p+\nu_j\epsilon_j\}_{j=1}^\infty}$ are isomorphic to ℓ_p for every choice of the sequence of signs $\{\nu_j = \pm 1\}_{j=1}^\infty$.

Define the sets:

$$A_m = \{j \in \mathbb{N} : k_{n_m} + 1 \leqslant j \leqslant k_{n_{m+1}}, \tau_j \geqslant \alpha_m\}$$

and

$$B_m = \{j \in \mathbb{N} : k_{n_m} + 1 \leqslant j \leqslant k_{n_{m+1}}, \tau_j < \alpha_m\}.$$

It is obvious that $A_m \cap B_m = \emptyset$ and $A_m \cup B_m = \{k_{n_m} + 1, \dots, k_{n_{m+1}}\}$. Let $\delta_m = \max\{\varepsilon_j : k_{n_m} + 1 \leqslant j \leqslant k_{n_{m+1}}\}$. Then $\{\delta_m\}_{m=1}^{\infty}$ is a subsequence of $\{\varepsilon_j\}_{j=1}^{\infty}$ and thus by (5) we obtain $\sum\limits_{m=1}^{\infty} C^{1/\delta_m} < \infty$. So the Nakano spaces $\ell_{\{p+\nu_m\delta_m\}}$ consist of the same sequences as ℓ_p for every choice of the signs $\{\nu_m = \pm 1\}$.

Let now $\{\alpha_j\}_{j=1}^{\infty} \in \ell_p$ that is, $\sum\limits_{j=1}^{\infty} \alpha_j^p < \infty$. We may assume that $\alpha_j \leqslant 1/2$ for every $j \in \mathbb{N}$. Now we can write the chain of inequalities.

$$\begin{aligned}
\sum_{m=1}^{\infty} \varphi_{n_m}(\alpha_m) &= \sum_{m=1}^{\infty} \left(\sum_{j=k_{n_m}+1}^{k_{n_{m+1}}-1} \Phi_j(\alpha_m y_j) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \right) \\
&\leqslant \sum_{m=1}^{\infty} \sum_{j=k_{n_m}+1}^{k_{n_{m+1}}} \Phi_j(\alpha_m y_j) \leqslant \sum_{m=1}^{\infty} \sum_{j \in A_m} \Phi_j(\alpha_m y_j) + \sum_{m=1}^{\infty} \sum_{j \in B_m} \Phi_j(\alpha_m y_j) \\
&\leqslant \sum_{m=1}^{\infty} \sum_{j \in A_m} \Phi_j(\tau_m y_j) + \sum_{m=1}^{\infty} \sum_{j \in B_m} K \alpha_m^{p-\delta_m} \Phi_j(y_j) \\
&\leqslant \sum_{m=1}^{\infty} \sum_{j=k_{n_m}+1}^{k_{n_{m+1}-1}} \tau_j \Phi_j(y_j) + K \sum_{m=1}^{\infty} \alpha_m^{p-\delta_m} \sum_{j=k_{n_m}+1}^{k_{n_{m+1}-1}} \Phi_j(y_j) \\
&\leqslant \frac{3K}{2} \left\{ \sum_{m=1}^{\infty} \frac{1}{m^2} + \sum_{m=1}^{\infty} \alpha_m^{p-\delta_m} \right\} < \infty,
\end{aligned}$$

where we used that $0 < \overline{y}_{k_{n+1}} \leqslant y_{k_{n+1}}$ for the second and (4) for the last inequality. Let now $\alpha = \{\alpha_m\}_{m=1}^{\infty} \in \ell_{\{\varphi_{n_m}\}}$, that is,

$$\sum_{m=1}^{\infty} \varphi_{n_m}(\alpha_m) = \sum_{m=1}^{\infty} \left(\sum_{j=k_{n_m}+1}^{k_{n_{m+1}}-1} \Phi_j(\alpha_m y_j) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \right) < \infty.$$

It is not difficult to check that for every $m \in \mathbb{N}$ the estimate holds:

$$(5) \qquad |\alpha_m|^{p+\delta_m} \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) \leqslant \frac{1}{m^2} \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}).$$

Denote $A'_m = A_m \setminus \{n_{m+1}\}$ and $B'_m = B_m \setminus \{n_{m+1}\}$ Now taking into account (3), (4) and (5) we can write the chain of inequalities:

$$\begin{aligned}
\sum_{m=1}^{\infty} |\alpha_m|^{p+\delta_m} &= \sum_{m=1}^{\infty} |\alpha_m|^{p+\delta_m} \Big(\sum_{j=k_{n_m}+1}^{k_{n_{m+1}}-1} \Phi_j(y_j) + \Phi_{n_{m+1}}(\overline{y}_{k_{n_{m+1}}}) \Big) \\
&\leqslant \sum_{m=1}^{\infty} \Big(|\alpha_m|^{p+\delta_m} \Big(\sum_{j\in A'_m} \Phi_j(y_j) + \sum_{j\in B'_m} \Phi_j(y_j) \Big) \\
&\qquad + \frac{1}{m^2} \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) + \Phi_{n_{m+1}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \Big) \\
&\leqslant \sum_{m=1}^{\infty} \Big(\sum_{j\in A'_m} (\tau_j)^{p+\delta_m} \Phi_j(y_j) + \sum_{j\in B'_m} |\alpha_m|^{p+\delta_m} \Phi_j(y_j) \\
&\qquad + \frac{1}{m^2} \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \Big) \\
&\leqslant \sum_{m=1}^{\infty} \frac{1}{m^2} \Big(\sum_{j\in A'_m} \Phi_j(y_j) + \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) \Big) \\
&\qquad + \frac{1}{k} \sum_{m=1}^{\infty} \Big(\sum_{j\in B'_m} \Phi_j(\alpha_m y_j) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \Big) \\
&\leqslant \frac{1}{k} \Big(\sum_{m=1}^{\infty} \frac{1}{m^2} + \sum_{m=1}^{\infty} \varphi_{n_m}(\alpha_m) \Big) < \infty ,
\end{aligned}$$

which concludes the proof.

Let now $1/2 \leqslant \Phi(y_{j_k}) \leqslant 1$ for some increasing sequence of naturals $\{j_k\}_{k=1}^{\infty}$. Passing to a subsequence if necessary we may assume that $\sum_{k=1}^{\infty} \tau_{j_k} < \infty$. Then

$$\Phi_{j_k}(t) \geqslant \Phi_{j_k}(t y_{j_k}) \geqslant k t^{p+\varepsilon_{j_k}} \Phi_{j_k}(y_{j_k}) \geqslant \frac{k}{2} t^{p+\varepsilon_{j_k}}$$

for every $t \in [\tau_{j_k}, 1]$. Consequently

$$u^{p+\varepsilon_{j_k}} \leqslant \frac{2}{k} \Phi_{j_k}(u) + \tau_{j_k}. \tag{6}$$

holds for every $u \in [0,1]$. Similarly

$$\Phi_{j_k}(t/2) \leqslant \Phi_{j_k}(t y_{j_k}) \leqslant 2^{p-\varepsilon_{j_k}} K \Big(\frac{t}{2}\Big)^{p-\varepsilon_{j_k}} \Phi_{j_k}(y_{j_k})$$

for every $t \in [\tau_{j_k}, 1]$. Thus

$$\Phi_{j_k}(u) \leqslant K_1 u^{p-\varepsilon_{j_k}}$$

holds for every $u \in [\tau_{j_k}/2, 1/2]$, where $K_1 = 2^p K$. So

$$\Phi_{j_k}(u) \leqslant K_1 u^{p-\varepsilon_{j_k}} + \tau_{j_k} \tag{7}$$

holds for every $u \in [0,1/2]$. Consequently by (6) and (7) it follows that $\ell_p \cong \ell_{\{\Phi_{j_k}\}} \hookrightarrow \ell_\Phi$. ∎

REMARK. If the conditions in Theorem 3 hold for a subsequence $\{\Phi_{n_k}\}_{k=1}^\infty$ then $\ell_p \hookrightarrow \ell_{\{\Phi_{n_k}\}} \hookrightarrow \ell_\Phi$.

COROLLARY 4.1. *Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a Musielak–Orlicz function and $\big(\Phi_j(ty_j)\big)/\big(\Phi_j(y_j)\big)$ converge uniformly to t^p on $[0,1]$ for some sequence $\{y_j\}_{j=1}^\infty$ such that, $0 < y_j \leqslant 1$, $\sum_{j=1}^\infty \Phi_j(y_j) = \infty$ and $p \in \big[a(\Phi), b(\Phi)\big]$. Then $\ell_p \hookrightarrow h_\Phi$.*

PROOF: Pick a decreasing sequence $\{\delta_k\}_{k=1}^\infty$, such that $\lim_{k\to\infty} \delta_k = 0$. There exists $j(k)$ such that for every $j \geqslant j(k)$ the inequalities hold.

$$t^p - \delta_k < \frac{\Phi_j(ty_j)}{\Phi_j(y_j)} < t^p + \delta_k \tag{8}$$

for every $t \in [0,1]$. Thus (8) implies

$$(1/2)t^0 \leqslant 1 - \delta_k/t^p < \frac{\Phi_j(ty_j)}{t^p\Phi_j(y_j)} < 1 + \delta_k/t^p \leqslant 2(1/t)^0$$

for every $t \in \big[(2\delta_k)^{1/p}, 1\big]$ and for every $j \geqslant j(k)$. We define inductively sequences $\big\{r(k)\big\}$ and $\big\{s(k)\big\}$ in the following way. We put $r(1) = j(1)$ and choose $s(1)$ with $\sum_{j=r(1)}^{r(1)+s(1)} \Phi_j(y_j) > 1/2$. If $r(k)$, $s(k)$ are already chosen we put $r(k+1) = \max\big(r(k) + s(k), j(k+1)\big)$ and choose $s(k+1)$ such that $\sum_{j=r(k+1)}^{r(k+1)+s(k+1)} \Phi_j(y_j) > 1/2$. Now we can apply Theorem 3 for the subsequence $\{\Phi_{j_m}\}_{m=1}^\infty$ and the sequences $\{\varepsilon_m = 0\}$, $\big\{\tau_m = (2\delta_m)^{1/p}\big\}$, $m \in \mathbb{N}$, where for every m the index j_m is of the form $j_m = \sum_{i=1}^{k-1} s(i) + p$ for some $k \in \mathbb{N}$ and p with $1 \leqslant p \leqslant s(k)$, while $\varepsilon_m = 0$, $\delta_m = \delta_k$. ∎

REMARK. In particular if the sequence of Orlicz functions $\Phi = \{\Phi_j\}_{j=1}^\infty$ converges uniformly on $[0,1]$ to t^p for some $p \in \big[a(\Phi), b(\Phi)\big]$ then $\ell_p \hookrightarrow h_\Phi$.

An easy to apply form of Theorem 3 is given by the following

COROLLARY 4.2. *Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a Musielak–Orlicz function and $p \in \big[a(\Phi), b(\Phi)\big]$. If there exist sequences $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$, $\{\varepsilon_j\}_{j=1}^\infty$ such that:*

(1) $\varepsilon_j \geqslant 0$, $0 < x_j \leqslant y_j \leqslant 1$ *for every* $j \in \mathbb{N}$;

(2) $\lim_{j\to\infty} x_j/y_j = 0$;

(3) $\sum_{j=1}^\infty \Phi_j(y_j) = \infty$;

(4) $p - \varepsilon_j \leqslant \big(u\Phi_j'(u)\big)/\big(\Phi_j(u)\big) \leqslant p + \varepsilon_j$ *for every* $u \in [x_j, y_j]$;

(5) $\sum_{j=1}^\infty C^{1/\varepsilon_j} < \infty$ *for some* $0 < C < 1$,

then $\ell_p \hookrightarrow h_\Phi$.

For the proof it is enough to rewrite the inequalities from (4) in the form:

$$(9)\qquad p-\varepsilon_j \leqslant \frac{ty_j\Phi_j'(ty_j)}{\Phi_j(ty_j)} \leqslant p+\varepsilon_j \quad \text{for every} \quad t\in[x_j/y_j,1].$$

After integration in (9) we easily get for every $n\in\mathbb{N}$:

$$(10)\qquad t^{p+\varepsilon_j}\Phi_j(y_j) \leqslant \Phi_j(ty_j) \leqslant t^{p-\varepsilon_j}\Phi_j(y_j)$$

for every $t\in[x_j/y_j,1]$. Now we can apply Theorem 3 with $\tau_j = x_j/y_j$. □

We shall illustrate some applications of Theorem 3 and the necessity of some of the conditions in it by the following four examples. By examples (1) and (2) we show that conditions (2) and (3) in Theorem 3 could not be omitted.

The next example represents a convex analog to an example from [**16**]

EXAMPLE 1. Let

$$f_n(x) = \begin{cases} x & if \quad x \geqslant 1/n^2 \\ n^2x^2 & if \quad x\in[0,1/n^2]. \end{cases}$$

Obviously

$$\frac{f_n(x)}{x} = \begin{cases} 1 & if \quad x \geqslant 1/n^2 \\ n^2x & if \quad x\in[0,1/n^2] \end{cases}$$

is an increasing function and therefore

$$\Phi_n(x) = \int_0^x \frac{f_n(t)}{t}dt = \begin{cases} x-\dfrac{1}{2n^2} & if \quad x\geqslant 1/n^2 \\ \dfrac{n^2}{2}x^2 & if \quad x\in[0,1/n^2]. \end{cases}$$

is an Orlicz function.

It is easy to check that

$$\frac{\Phi_n(t/n^2)}{t^2\Phi_n(1/n^2)} = 1$$

for every $n\in\mathbb{N}$ and every $t\in[0,1]$. Therefore for the sequences $\{y_n = 1/n^2\}_{n=1}^\infty$, $\{\varepsilon_n = 0\}_{n=1}^\infty$ and any arbitrary sequence $\{\tau_n\}_{n=1}^\infty$ such that $\tau_n \searrow 0$ all the conditions of Theorem 3 hold except for the condition (3) $\left(\sum_{n=1}^\infty y_n = \sum_{n=1}^\infty 1/n^2 < \infty\right)$. Nonetheless $\ell_2 \not\hookrightarrow \ell_{\Phi_n}$ because the inequalities

$$\Phi_n(x)\leqslant x \quad\text{and}\quad x\leqslant \Phi_n(x)+\frac{1}{2n^2}, \quad \text{for every} \quad x\in[0,+\infty).$$

imply $\ell_1 \cong \ell_\Phi$.

Then for the next two examples $k_n = 2n\big(1-\sqrt{1-(1/n)}\big)$, $b_n = 1-k_n$, $\alpha_n = 1-\sqrt{1-(1/n)}$, $n\in\mathbb{N}$. It is easy to see that $1/2n \leqslant \alpha_n \leqslant 1/n$.

EXAMPLE 2. Consider the functions

$$\Phi_n(x) = \begin{cases} k_n x + b_n & \text{if} \quad x \geqslant \alpha_n \\ n x^2 & \text{if} \quad x \in \big[(\alpha_n/2), \alpha_n\big] \\ \dfrac{n\alpha_n}{2} x & \text{if} \quad x \in \big[0, (\alpha_n/2)\big]. \end{cases}$$

Obviously by the choice of the sequences k_n, b_n and α_n it follows that Φ_n are Orlicz functions.

It is easily to check that

$$\frac{\Phi_n(t\alpha_n)}{t^2\Phi_n(\alpha_n)} = 1$$

for every $n \in \mathbb{N}$ and for every $t \in [1/2, 1]$. Obviously $\sum_{n=1}^{\infty} \Phi_n(\alpha_n) = \sum_{n=1}^{\infty} n.\alpha_n^2 = \infty$. Therefore for the sequences $\{y_n = \alpha_n\}_{n=1}^{\infty}$, $\{\varepsilon_n = 0\}_{n=1}^{\infty}$ and $\{\tau_n = 1/2\}_{n=1}^{\infty}$ all the conditions of Theorem3 hold except for the condition (2) $\big(\lim_{n\to\infty} \tau_n = 0\big)$. Nonetheless $\ell_2 \not\hookrightarrow \ell_{\Phi_n}$ because $\ell_1 \cong \ell_\Phi$.

Indeed consider now the Nakano sequence space $\ell_{\{p_n\}}$, where $p_n = 1 + (1/\ln n^2)$. According to [**3**] $\ell_1 \cong \ell_{\{p_n\}}$. It is easy to check that $x^{p_n} \leqslant \Phi_n(x) \leqslant x$, for every $x \in [0, 1]$, because the solutions of the equation: $nx^2 = x^{p_n}$ are $x_1 = 0$ and $x_2 = (1/n)^{1/(2-p_n)}$ and $x_2 < 1/(4n) < \alpha_n/2$. Thus $\ell_1 \cong \ell_\Phi$ which in turn implies $\ell_2 \not\hookrightarrow \ell_{\Phi_n}$.

Similar calculations can be done in Examples (1) and (2) to show that conditions (2) and (3) in Corollary 4.2 do not hold.

The next example shows that the indexes

$$\alpha_\Phi = \liminf_{n\to\infty} \alpha_{\Phi_n}, \quad \beta_\Phi = \limsup_{n\to\infty} \beta_{\Phi_n},$$

where α_{Φ_n} and β_{Φ_n} are the Boyd indexes of Φ_n (see for example, [**11**, p. 143]) are irrelevant when embedding of ℓ_p – spaces into ℓ_Φ is investigated. This fact is not surprising taking into account that among the Musielak–Orlicz functions Ψ equivalent to a given Musielak–Orlicz function Φ there exist such with $\alpha_\Psi = \beta_\Psi = 1$ ([**18**]).

EXAMPLE 3. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} t_n = 0$ and $t_n < 1/2$ for every $n \in \mathbb{N}$. Define the functions

$$\Phi_n(x) = \begin{cases} k_n x + b_n & \text{if} \quad x \geqslant \alpha_n \\ n x^2 & \text{if} \quad x \in \Big[(t_n/n), \alpha_n\Big] \\ t_n x & \text{if} \quad x \in \Big[0, (t_n/n)\Big], \end{cases}$$

Obviously by the choice of the sequences k_n, b_n and α_n follows that Φ_n are Orlicz functions which are differentiable for every $x \in [0, 1]$ except for $x = t_n/n$ and $x = \alpha_n$.

It easy to see that $\ell_1 \cong \ell_{\{\Phi_{2^n}\}} \hookrightarrow \ell_\Phi$ because $\Phi_{2^n}(x) \leqslant x \leqslant \Phi_{2^n}(x) + \alpha_{2^n}$ and $\sum_{n=1}^{\infty} \alpha_{2^n} < \infty$.

The conditions $\big(u\Phi_n'(u)\big)/\big(\Phi_n(u)\big) = 2$ for every $u \in \big[(t_n/n), \alpha_n\big]$, $\sum_{n=1}^{\infty} \Phi_n(\alpha_n) = \infty$ and $\lim_{n\to\infty} (t_n)/(n\alpha_n) = 0$ ensure that by Corollary 4.2 $\ell_2 \hookrightarrow \ell_\Phi$.

To calculate the Boyd indexes we have to observe that the functions Φ_n are linear for $t \in [0, t_n/n]$ and thus $1 = \alpha_\Phi = \beta_\Phi$.

We have that $\big(u\Phi_n'(u)\big)/\big(\Phi_n(u)\big) = 1$ for every $u \in [0, t_n/n]$. So we obtain that $1 = a(\Phi) < b(\Phi) = 2$. Thus there exists a Musielak–Orlicz sequence space ℓ_Φ such that $\ell_2 \hookrightarrow \ell_\Phi$ and $2 \notin [\alpha_\Phi, \beta_\Phi]$.

Following [5] we shall construct an example of a weighted Orlicz sequence space which contains an isomorphic copy of ℓ_1.

EXAMPLE 4. Let the sequences $\{d_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ be such that $d_n \leqslant d_{n+1}$, $a_n \leqslant a_{n+1}$, $\lim_{n\to\infty} d_n/d_{n+1} = 0$, $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} a_n(d_n/d_{n+1}) = 0$ and $\sum_{n=1}^{\infty} C^{a_n} < \infty$ for some $0 < C < 1$. Define the Orlicz function

$$M(x) = \begin{cases} x^2 & if \quad 0 \leqslant x \leqslant 1 \\ A_n x + B_n & if \quad d_n \leqslant x \leqslant d_{n+1}, \end{cases}$$

where $A_n = d_{n+1} + d_n$, $B_n = -d_{n+1}d_n$.

Let the sequence $w = \{w_n\}_{n=1}^{\infty}$ be defined by $w_n = 1/\big(\Phi(d_{n+1})\big) = 1/(d_{n+1}^2)$. Then $\ell_\Phi(w) \cong \ell_{\{\Phi_n\}}$, where $\Phi_n(x) = \big(\Phi(d_{n+1}x)\big)/\big(\Phi(d_{n+1})\big)$.

Thus

$$\frac{x\Phi_n'(x)}{\Phi_n(x)} = \frac{xd_{n+1}(\Phi_n'(d_{n+1}x))/(\Phi_n(d_{n+1}))}{(\Phi_n(d_{n+1}x))/(\Phi_n(d_{n+1}))} = \frac{xd_{n+1}A_n}{xd_{n+1}A_n + B_n}$$

for $d_n/d_{n+1} \leqslant s \leqslant 1$.

After easy calculations we obtain the inequalities:

$$1 - \frac{1}{a_n - 1} < 1 + \frac{d_n}{d_{n+1}} \leqslant \frac{x\Phi_n'(x)}{\Phi_n(x)} \leqslant 1 + \frac{1}{a_n - 1}$$

for every $a_n(d_n/d_{n+1}) \leqslant x \leqslant 1$.

Thus $\sum_{n=1}^{\infty} C^{a_n - 1} = 1/C \sum_{n=1}^{\infty} C^{a_n} < \infty$ and we can apply Corollary 4.2 with $y_n = 1$, $x_n = a_n(d_n/d_{n+1})$, $\varepsilon_n = 1/(a_n - 1)$ to show that $\ell_1 \hookrightarrow \ell_\Phi(w) \cong \ell_{\{\Phi_n\}}$.

REMARK. If

$$\sum_{n=1}^{\infty} (d_n)/(d_{n+1}) < 1/2 \tag{11}$$

it is proved in [5] that $\ell_1 \cong \ell_M(w)$.

REMARK. By choosing the sequences $\{d_n = n!\}_{n=1}^{\infty}$ and $\{a_n = \log n^2\}_{n=1}^{\infty}$ in Example 4 we get a weighted Orlicz sequence space $\ell_M(w)$ generated by an Orlicz function M which does not satisfy the Δ_2–condition at infinity and a weight sequence

$$w = \left\{w_n = \frac{1}{\big((n+1)!\big)^2}\right\}_{n=1}^{\infty},$$

but containing an isomorphic copy of ℓ_1. Indeed $\big(M(2n!)/M(n!)\big) = 3 + n$ and thus M does not satisfy the Δ_2–condition at ∞. The sequences $\{d_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ satisfy the conditions imposed on them in Example 4 and thus $\ell_1 \hookrightarrow \ell_M(w)$.

Following [4] we define a sequence of real numbers $\{\psi_\lambda(j)\}_{j=1}^{\infty}$ by

$$\psi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : t > 0\}.$$

PROPOSITION 4.1. ([4, Proposition 4.5.3]) *Let $\Phi = \{\Phi_j\}_{j=1}^{\infty}$ be a Musielak–Orlicz function. Suppose that for some $\lambda > 1$, $\lim\limits_{j\to\infty} \psi_\lambda(j) = \infty$, then h_Φ is stabilised asymptotic ℓ_∞.*

Let us mention that in the proof of Proposition 4.1, a_j were chosen such that $\sum\limits_{j=p}^{q} \Phi(a_j) \leqslant 1$. Thus the function $\psi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : t > 0\}$ can be replaced by
$\psi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : 0 < t \leqslant 1\}$.

COROLLARY 4.3. *Let Φ has δ_2 condition at zero and h_Ψ, generated by the Musielak–Orlicz function Ψ, complementary to Φ If there exist sequences: $\{x_j\}_{j=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$ and $\{\varepsilon_j\}_{j=1}^{\infty}$ satisfying:*

(1′) $\varepsilon_j > 0$, $0 < x_j \leqslant y_j \leqslant 1$ *for every* $j \in \mathbb{N}$;

(2′) $\lim\limits_{j\to\infty} (x_j/y_j) = 0$;

(3′) $\sum\limits_{j=1}^{\infty} \Phi_j(y_j) = \infty$;

(4′) $b(\Phi) - \varepsilon_j \leqslant \big(u\Phi_j'(u)\big)/\big(\Phi_j(u)\big) \leqslant b(\Phi) + \varepsilon_j$ *for any* $u \in [x_j, y_j]$;

(5′) $\sum\limits_{j=1}^{\infty} C^{1/\varepsilon_j} < \infty$ *for some* $0 < C < 1$. *and* ℓ_Φ *is* ℓ_1 *saturated, then holds:*

(a) $a(\Phi) = b(\Phi) = 1$;

(b) *h_Ψ is stabilised asymptotic ℓ_∞ respect to the basis $\{e_j^*\}_{j=1}^{\infty}$.*

PROOF: (a) By [**16**] it follows that if $\ell_1 \hookrightarrow \ell_\Phi$ then $1 \in [a(\Phi), b(\Phi)]$ and thus $a(\Phi) = 1$. Let $a(\Phi) \neq b(\Phi)$. By Corollary 4.2 follows that $\ell_{b(\Phi)} \hookrightarrow \ell_\Phi$, which is a contradiction. Thus $1 = a(\Phi) = b(\Phi)$.

(b) By (a) we have $a(\Phi) = b(\Phi) = 1$. So we have $\lim\limits_{j\to\infty} a(\Phi_j) = \lim\limits_{j\to\infty} b(\Phi_j) = 1$. Then using the well known connections $1/a(\Phi_j) + 1/b(\Psi_j) = 1$ and $1/a(\Psi_j) + 1/b(\Phi_j) = 1$ (see

[8]) it follows that $\lim_{j\to\infty} a(\Psi_j) = \lim_{j\to\infty} b(\Psi_j) = \infty$. Then by the definition of the indices $a(\Psi_j)$ and $b(\Psi_j)$ there is $\varepsilon > 0$, such that for every p_j, q_j: $0 < a(\Psi_j) - \varepsilon < p_j < a(\Psi_j)$ and $b(\Psi_j) < q_j < b(\Psi_j) + \varepsilon$

$$2^{a(\Psi_j)-\varepsilon} < 2^{p_j} < \frac{\Psi_j(2t)}{\Psi_j(t)} < 2^{q_j} < 2^{b(\Psi_j)+\varepsilon}.$$

Thus

$$\lim_{j\to\infty}\left(\inf\left\{\frac{\Psi_j(2t)}{\Psi_j(t)} : t > 0\right\}\right) \geqslant \lim_{j\to\infty} 2^{p_j} = \infty,$$

and by Propositon 4.1 it follows that h_Ψ is stabilised asymptotic ℓ_∞ with respect to the basis $\{e_j^*\}_{j=1}^{\infty}$. ∎

REMARK. Kaminska and Mastylo have given some sufficient and some necessary conditions for the Schur property in terms of the generating Musielak–Orlicz function Φ [8]. Sometimes we know only the complementary function Ψ. For example let the Musielak–Orlicz function $\Psi = \{\Psi_j\}_{j=1}^{\infty}$ be defined by $\Psi_j = e^{\alpha_j} e^{-(\alpha_j)/(|x|^{c_j})}$, where $\lim_{j\to\infty} \alpha_j = \infty$ and $0 < c_j$. Then ℓ_Ψ is stabilised asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$ because

$$\lim_{j\to\infty} \inf\left\{\frac{\Psi_j(2x)}{\Psi_j(x)} : 0 \leqslant x \leqslant 1\right\} = \lim_{j\to\infty} \inf\left\{e^{\alpha_j(2^{c_j}-1)/(2^{c_j}|x|^{c_j})} : 0 \leqslant x \leqslant 1\right\}$$
$$= \lim_{j\to\infty} e^{\alpha_j(2^{c_j}-1)/(2^{c_j})} = \infty.$$

Thus we conclude that ℓ_Φ has the Schur property without considering the functions Φ_n, $n \in \mathbb{N}$.

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