# POLAR MEANS OF CONVEX BODIES AND A DUAL TO THE BRUNN-MINKOWSKI THEOREM

WILLIAM J. FIREY

**1. Introduction.** This paper deals with processes of combining convex bodies in Euclidean $n$-space which are, in a sense, dual to the process of Minkowski addition and some of its generalizations.

All the convex bodies considered will have a common interior point $Q$. Variables $x$ and $y$ denote vectors drawn from $Q$; we shall speak of their terminal points as the points $x$ and $y$. Unit vectors will be denoted by $u$; $||x||$ signifies the length of $x$. Convex bodies will be symbolized by $K$ with distinguishing marks. $\partial K$ means the boundary of $K$. $\lambda K$ will mean the image of $K$ under a homothetic transformation in the ratio $\lambda : 1$. The centre of the homothety will always be $Q$.

The distance function $F(x)$ of a convex body is defined as follows: let $y$ be the vector having the same direction as $x$ which terminates at $\partial K$, then $F(x) = ||x||/||y||$. If $x = 0$, we set $F(0) = 0$. The points $x$ of $K$ satisfy $F(x) \leqq 1$ with equality if and only if $x$ is a point of $\partial K$. Let $u = x/||x||$; then $\rho = 1/F(u) = f(u)$ is the polar co-ordinate equation of $\partial K$ with respect to a co-ordinate system with pole at $Q$. Since $Q$ is an interior point of $K$, $F(u)$ is continuous and bounded.

The distance function satisfies: (a) $F(x) > 0$ for $x \neq 0$, $F(0) = 0$; (b) $F(\mu x) = \mu F(x)$ for $\mu > 0$; (c) $F(x + y) \leqq F(x) + F(y)$ for any two vectors $x$ and $y$. Conversely, any function $F(x)$ satisfying (a) through (c) is the distance function of a unique convex body $K$ (cf. **(1)**, p. 22).

The following observations regarding distance functions should be borne in mind; they follow immediately from the definition. $F_0(x) \geqq F_1(x)$ if and only if $K_0 \subseteq K_1$. If the distance function of $K$ is $F(x)$, that of $\lambda K$ is $F(x)/\lambda$.

If $F_i(x)$, $(i = 0, 1)$, is the distance function of the body $K_i$ containing $Q$ as an interior point, then

$$F_\vartheta^{(1)}(x) = (1 - \vartheta)F_0(x) + \vartheta F_1(x), \qquad 0 \leqq \vartheta \leqq 1,$$

and, more generally,

$$F_\vartheta^{(p)}(x) = \sqrt[p]{[(1 - \vartheta)F_0^p(x) + \vartheta F_1^p(x)]}, \qquad 1 \leqq p \leqq \infty,$$

satisfy conditions (a) through (c). By $F_\vartheta^{(\infty)}(x)$ we mean

$$\lim_{p\to\infty} F_\vartheta^{(p)}(x) = \max(F_0(x), F_1(x))$$

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for $0 < \vartheta < 1$ with $F_i^{(\infty)}(x) = F_i(x)$. Conditions (a) and (b) are obviously satisfied. Condition (c) is a consequence of Minkowski's inequality. Let $a_i = b_i + c_i$; Minkowski's inequality is

$$\sqrt[p]{[(1-\vartheta)a_0^p + \vartheta a_1^p]} \leqq \sqrt[p]{[(1-\vartheta)b_0^p + \vartheta b_1^p]} + \sqrt[p]{[(1-\vartheta)c_0^p + \vartheta c_1^p]}.$$

If $a_i \leqq b_i + c_i$, the inequality is clearly still valid. Set $a_i = F_i(x+y)$, $b_i = F_i(x)$ and $c_i = F_i(y)$ and condition (c) is verified for $F_\vartheta^{(p)}$. A limit argument establishes (c) for $p = \infty$. Consequently we may speak of a unique convex body $\dot{K}_\vartheta^{(p)}$ having the distance function $F_\vartheta^{(p)}$. We will call this body the $p$th dot-mean of $K_0$ and $K_1$. It clearly contains $Q$ as an interior point. For $1 \leqq p < \infty$, the body

$$\sqrt[p]{2}\dot{K}_{1/2}^{(p)}$$

will be denoted by $\dot{S}^{(p)}(K_0, K_1)$ and called the $p$th dot-sum of $K_0$ and $K_1$. Its distance function is $\sqrt[p]{[F_0^p(x) + F_1^p(x)]}$. We set

$$\dot{S}^{(\infty)}(K_0, K_1) = \dot{K}_{1/2}^{(\infty)}.$$

We obtain a direct geometric meaning for $\dot{K}_\vartheta^{(p)}$ as follows. If the polar co-ordinate equation of $\partial K_i$ is $\rho = f_i(u)$, then the polar co-ordinate equation of $\partial\dot{K}_\vartheta^{(p)}$ is

$$\rho = 1 \Big/ \sqrt[p]{\left[\frac{(1-\vartheta)}{f_0^p(u)} + \frac{\vartheta}{f_1^p(u)}\right]} \qquad \text{for } 1 \leqq p < \infty,$$

$$\rho = \min\,(f_0(u), f_1(u)) \qquad \text{for } p = \infty.$$

In particular if $p = 1$, $\rho$ is the harmonic mean of the distances to $\partial K_0$ and $\partial K_1$ in the direction $u$.

$$\dot{K}_\vartheta^{(\infty)} = K_0 \cap K_1$$

for $0 < \vartheta < 1$.

In § 2, we first take up some elementary rules about such combinations of convex bodies. A deviation or metric in a space of convex bodies is introduced. The duality mentioned at the beginning of the paper is discussed and with its aid, we examine the topology induced by the deviation measure.

Section 3 is devoted to the dependence of the family $\{\dot{K}_\vartheta^{(p)}\}$ on $K_0$, $K_1$ and the parameters $p$ and $\vartheta$, for $1 \leqq p < \infty$. The dependence is continuous; the family is monotonic decreasing in $p$ and concave with respect to $\vartheta$. The special case $p = \infty$ is considered separately.

We establish a theorem of the Brunn–Minkowski type for the family $\{\dot{K}_\vartheta^{(p)}\}$ in the final section. This is

$$V^{1/n}(\dot{K}_\vartheta^{(p)}) \leqq 1/\sqrt[p]{[(1-\vartheta)V^{-p/n}(K_0) + \vartheta V^{-p/n}(K_1)]} \qquad \text{for } 1 \leqq p < \infty,$$

$$V(\dot{K}_\vartheta^{(\infty)}) \leqq \min(V(K_0), V(K_1)) \qquad \text{for } 0 < \vartheta < 1.$$

Here $V(K)$ signifies the volume of the convex body $K$.

A discussion of the cases of equality is included.

**2. Measures of deviation.** The following rules follow immediately from the properties of $S_p(a_0, a_1) = \sqrt[p]{[a_0^p + a_1^p]}$ for non-negative numbers $a_i$ applied to the appropriate distance functions.

(i) $\dot{S}^{(p)}(\lambda K_0, \lambda K_1) = \lambda \dot{S}^{(p)}(K_0, K_1)$.

(ii) $\dot{S}^{(p)}(K_0, K_1) = \dot{S}^{(p)}(K_1, K_0)$.

(iii) $\dot{S}^{(p)}(\dot{S}^{(p)}(K_0, K_1), K_2) = \dot{S}^{(p)}(K_0, \dot{S}^{(p)}(K_1, K_2))$.

This last rule allows us to write without misunderstanding $\dot{S}^{(p)}(K_0, K_1, \ldots, K_m)$ defined inductively as

$$\dot{S}^{(p)}(\dot{S}^{(p)}(K_0, K_1, \ldots, K_{m-1}), K_m).$$

In turn we set

$$\dot{S}^{(p)}(\sqrt[p]{w_0}K_0, \sqrt[p]{w_1}K_1, \ldots, \sqrt[p]{w_m}K_m) = \dot{M}^{(p)}(K_0, K_1, \ldots, K_m)$$

if

$$\sum_{i=1}^{m} w_i = 1, w_i \geqq 0, 1 \leqq p < \infty.$$

$\dot{M}^{(p)}(K_0, K_1) = \dot{K}_\vartheta^{(p)}$ with $\vartheta = w_1$. We define $\dot{M}^{(\infty)}(K_0, K_1, \ldots, K_m)$ and $\dot{S}^{(\infty)}(K_0, K_1, \ldots, K_m)$ as bodies whose distance functions are

$$\lim_{p\to\infty} M_p(F_0, F_1, \ldots, F_m), \lim_{p\to\infty} S_p(F_0, F_1, \ldots, F_m).$$

Since these limits are equal $\dot{M}^{(\infty)}(K_0, K_1, \ldots, K_m)$, $\dot{S}^{(\infty)}(K_0, K_1, \ldots, K_m)$ are the same body. This is the convex body whose distance function is max $(F_0, F_1, \ldots, F_m)$. $\partial\dot{M}^{(\infty)}(K_0, K_1, \ldots, K_m)$ has the polar co-ordinate equation $\rho = \min\,(f_0, f_1, \ldots, f_m)$ if $\partial K_i$ has the equation $\rho = f_i(u)$. Clearly

$$\dot{M}^{(\infty)}(K_0, K_1, \ldots, K_m) = K_0 \cap K_1 \cap \ldots \cap K_m.$$

We always have $\dot{S}^{(p)}(K_0, K_1) \subset K_i$ since

$$\sqrt[p]{[F_0^p(x) + F_1^p(x)]} > F_i(x)$$

for $x \neq 0$.

The bodies $\dot{S}^{(p)}(K_0, K_1)$ and $\dot{K}_\vartheta^{(p)}$ are not translation-invariant in the sense displayed by the usual Minkowski sum $K_0 + K_1$. In the case of Minkowski sums, if $K_i$ is translated by the addition of a vector $t_i$ to each vector in $K_i$, then $K_0 + K_1$ is translated by the addition of the vector $t_0 + t_1$. It can be proved that, in general, there is no such translation vector for $\dot{S}^{(p)}(K_0, K_1)$ or $\dot{K}_\vartheta^{(p)}$. For this reason we must distinguish bodies which differ by a translation.

A measure of deviation between the two convex bodies is defined as follows. Let $E$ be the sphere of radius one, centred at $Q$. For $1 \leqq p < \infty$, consider those numbers $\lambda > 0$ such that $\dot{S}^{(p)}(K_0, \lambda E) \subseteq K_1$ and $\dot{S}^{(p)}(K_1, \lambda E) \subseteq K_0$. We define $\dot{\delta}^{(p)}(K_0, K_1)$ to be the greatest lower bound of the numbers $1/\lambda$. In terms

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of distance functions, if $F_i(x)$ is the distance function of $K_i$, $\dot{\delta}^{(p)}(K_0, K_1)$ is the greatest lower bound of numbers $1/\lambda = \mu$ such that

$$\sqrt[p]{[F_0^p(x) + \mu^p||x||^p]} \geqq F_1(x)$$

and

$$\sqrt[p]{[F_1^p(x) + \mu^p||x||^p]} \geqq F_0(x).$$

Since such function $F_i(x)$ is continuous and bounded over $||x|| = 1$, we have

$$\dot{\delta}^{(p)}(K_0, K_1) = \max \sqrt[p]{|F_0^p(u) - F_1^p(u)|},$$

the maximum being taken over the sphere of directions $u$. Clearly $\dot{\delta}^{(p)}(K_0, K_1) \geqq 0$ with equality if and only if $F_0(x) \equiv F_1(x)$, that is $K_0 = K_1$. Further $\dot{\delta}^{(p)}(K_0, K_1) = \dot{\delta}^{(p)}(K_1, K_0)$. The deviation satisfies a triangle inequality:

$$\dot{\delta}^{(p)}(K_0, K_2) \leqq \dot{\delta}^{(p)}(K_0, K_1) + \dot{\delta}^{(p)}(K_1, K_2).$$

For let

$$\mu_1 = \dot{\delta}^{(p)}(K_0, K_1),$$
$$\mu_2 = \dot{\delta}^{(p)}(K_0, K_2),$$
$$\mu_3 = \dot{\delta}^{(p)}(K_1, K_2).$$

Then

$$\mu_2 = \max \sqrt[p]{|F_0^p(u) - F_2^p(u)|} \leqq \max \sqrt[p]{[|F_0^p(u) - F_1^p(u)| + |F_1^p(u) - F_2^p(u)|]}$$
$$\leqq \max \sqrt[p]{|F_0^p(u) - F_1^p(u)|} + \max \sqrt[p]{|F_1^p(u) - F_2^p(u)|} = \mu_1 + \mu_3,$$

all the maxima being taken over the unit sphere of directions $u$.

For $p = \infty$, we define $\dot{\delta}^{(\infty)}(K_0, K_1)$ to be

$$\max_{||u||=1} \left(\max_{(0,1)} [F_0(u), F_1(u)]\right)$$

if $K_0$ and $K_1$ are not identical and take $\dot{\delta}^{(\infty)}(K_0, K_0) = 0$. $\dot{\delta}^{(\infty)}(K_0, K_1)$ is thus the reciprocal of the radius of the largest sphere centred at $Q$ which lies in $K_0 \cap K_1$. We may alternately describe $\dot{\delta}^{(\infty)}(K_0, K_1)$ as max $(1/\nu_0, 1/\nu_1)$ where $\nu_i E$ is the largest sphere centred at $Q$ contained in $K_i$. Clearly $\dot{\delta}^{(\infty)}(K_0, K_1) = \dot{\delta}^{(\infty)}(K_1, K_0)$ and $\dot{\delta}^{(\infty)}(K_0, K_1) \geqq 0$ with equality if and only if $K_0 = K_1$. This deviation satisfies a triangle inequality:

$$\dot{\delta}^{(\infty)}(K_0, K_2) \leqq \dot{\delta}^{(\infty)}(K_0, K_1) + \dot{\delta}^{(\infty)}(K_1, K_2).$$

If $K_0 = K_2$, this follows from the non-negativity of the deviation. If $K_0 = K_1$ or $K_1 = K_2$, there is obvious equality. Otherwise, using the numbers $\nu_0$, $\nu_1$, $\nu_2$ defined above, we have

$$\max\left(\frac{1}{\nu_0}, \frac{1}{\nu_2}\right) \leqq \max\left(\frac{1}{\nu_0}, \frac{1}{\nu_1}, \frac{1}{\nu_2}\right) < \max\left(\frac{1}{\nu_0}, \frac{1}{\nu_1}\right) + \max\left(\frac{1}{\nu_1}, \frac{1}{\nu_2}\right)$$

which proves the assertion.

Thus, for $1 \leqq p \leqq \infty$, the deviations $\dot{\delta}^{(p)}(K_0, K_1)$ satisfy the requirements for a metric in the space of convex bodies. For the remainder of the section, deviations will be considered only for $1 \leqq p < \infty$.

Let $K$ be a convex body with distance function $F(x)$. We denote by $\hat{K}$ the polar reciprocal of $K$ with respect to the unit sphere $E$ centred at $Q$. The support function with respect to $Q$ of $\hat{K}$ is defined as follows. Let $x$ be any point other than $Q$, $z$ a vector from $Q$ in the direction of $x$ which terminates at the support plane of $\hat{K}$ normal to $x$. The support function of $\hat{K}$ is $||z|| \cdot ||x||$. Since $K$ and $\hat{K}$ are polar reciprocals with respect to $E$, if $y$ is the vector from $Q$ having the same direction as $x$ and terminating at $\partial K$, we have $||y|| \cdot ||z|| = 1$. Hence the support function of $\hat{K}$ is $||x||/||y|| = F(x)$. Further, if $H(x)$ is the distance function of $\hat{K}$, then $H(x)$ is the support function of $K$. If $Q$ is an interior point of $K$, it is an interior point of $\hat{K}$. Consider the convex body $\dot{K}_\vartheta^{(p)}$; its polar reciprocal $\hat{\dot{K}}_\vartheta^{(p)}$ has

$$\sqrt[p]{[(1-\vartheta)F_0^p(x) + \vartheta F_1^p(x)]}$$

as its support function. This support function is the $p$th mean of the support functions of $\hat{K}_0$ and $\hat{K}_1$. In particular for $p = 1$, $\hat{\dot{K}}_\vartheta^{(p)}$ is the usual Minkowski mean $(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1$. More generally $\hat{\dot{K}}_\vartheta^{(p)}$ is the convex body denoted by $\hat{K}_\vartheta^{(p)}$ called the $p$th mean of $\hat{K}_0$, $\hat{K}_1$ in **(2)**. Similarly $\hat{\dot{S}}^{(p)}(K_0, K_1) = S^{(p)}(\hat{K}_0, \hat{K}_1)$.

It is convenient to express these notions in terms of the space $\mathscr{K}_p$ of convex bodies $K$ with metric $\dot{\delta}^{(p)}$ and the space $\hat{\mathscr{K}}_p$ of convex bodies $\hat{K}$ with metric $\delta^{(p)}$ introduced in **(2)**. There $\delta^{(p)}(\hat{K}_0, \hat{K}_1)$ was defined as the greatest lower bound of numbers $\mu$ such that

$$\sqrt[p]{[F_0^p(x) + \mu^p ||x||^p]} \geqq F_1(x)$$

and

$$\sqrt[p]{[F_1^p(x) + \mu^p ||x||^p]} \geqq F_0(x)$$

where $F_i(x)$ is the support function of $\hat{K}_i$. Polar reciprocation with respect to $E$ is an involutary mapping $R_p: \mathscr{K}_p \to \hat{\mathscr{K}}_p$. Under this mapping $p$th dot-means correspond to $p$th means.

We have directly from the definitions of $\dot{\delta}^{(p)}$ and $\delta^{(p)}$ that $\dot{\delta}^{(p)}(K_0, K_1) = \delta^{(p)}(\hat{K}_0, \hat{K}_1)$. Therefore $R_p$ is a homeomorphism. In **(2)** it was shown that the metrics $\delta^{(p)}$ are topologically equivalent and so it follows also for the metrics $\dot{\delta}^{(p)}$.

We summarize.

THEOREM 1. *Polar reciprocation with respect to E furnishes a homeomorphism* $\mathscr{K}_p \to \hat{\mathscr{K}}_p$, *for* $1 \leqq p < \infty$ *and for each such* $p$ *and* $q$ *satisfying* $1 \leqq q < \infty$, $\mathscr{K}_p$ *is homeomorphic to* $\mathscr{K}_q$.

Let $E_m$ $(1 \leqq m < n)$ be an $m$-dimensional linear subspace of the Euclidean $n$-space which contains $Q$. The distance function of $K \cap E_m$ in $E_m$ is the restriction of the distance function of $K$ to vectors in $E_m$. Hence in $E_m$ we have

$$\dot{S}^{(p)}(K_0, K_1) \cap E_m = \dot{S}^{(p)}(K_0 \cap E_m, K_1 \cap E_m).$$

This is the dual of the following result. Let $K^*$ be the projection of $K$ onto $E_m$; then

$$S^{(p)}(K_0^*, K_1^*) = [S^{(p)}(K_0, K_1)]^*.$$

We have further

$$S^{(p)}(K_0 \cap E_m, K_1 \cap E_m) \subseteq S^{(p)}(K_0, K_1) \cap E_m$$

and, as the dual of this result

$$\dot{S}^{(p)}(K_0^*, K_1^*) \supseteq [\dot{S}^{(p)}(K_0, K_1)]^*.$$

The latter follows from the former with the observations that if $F^*$ is the support function of $\hat{K} \cap E_m$ then it is the distance function of $K^*$, and by the first inclusion

$$\sqrt[p]{[(F_0^*)^p + (F_1^*)^p]} \leqq (\sqrt[p]{[F_0^p + F_1^p]})^*.$$

**3. Dependence of the means on their parameters.** The $p$th dot-means $\dot{K}_\vartheta^{(p)}$ depend continuously on $p$, $\vartheta$, $K_0$ and $K_1$ in the following sense. Let $S$ be the space of elements $(p, \vartheta, K_0, K_1)$ where $1 \leqq p \leqq P < \infty$, $0 \leqq \vartheta \leqq 1$, $K_i$ in $\mathscr{K}_1$ with the distance $d(e, e')$ between elements $e = (p, \vartheta, K_0, K_1)$ and $e' = (p', \vartheta', K_0', K_1')$ defined as $|p - p'| + |\vartheta - \vartheta'| + \dot{\delta}^{(1)}(K_0, K_0') + \dot{\delta}^{(1)}(K_1, K_1')$. By Theorem 1, the deviation $\dot{\delta}^{(1)}$ can be replaced by any of the deviations $\dot{\delta}^{(q)}$, $\delta^{(q)}$ for finite $q \geqq 1$. Further let $K(e)$ be the $p$th dot-mean $\dot{K}_\vartheta^{(p)}$ associated with element $e$. $K(e)$ is continuous in $e$, that is if $\{e_n\}$ is any sequence of elements of $S$ for which

$$\lim_{n\to\infty} d(e_n, e) = 0,$$

we have

$$\lim_{n\to\infty} \dot{\delta}^{(1)}(K(e_n), K(e)) = 0.$$

To demonstrate this continuity, we first remark that the algebraic function

$$f(p,\vartheta, a_0, a_1) = \sqrt[p]{[(1-\vartheta)a_0^p + \vartheta a_1^p]}$$

has no singularities for $(p, \vartheta, a_0, a_1)$ satisfying $0 < A \leqq a_i \leqq B < \infty$, $0 \leqq \vartheta \leqq 1$, $1 \leqq p \leqq P < \infty$ and so is uniformly continuous for such $(p, \vartheta, a_0, a_1)$. Suppose that $\{F_{0n}(x)\}$ and $\{F_{1n}(x)\}$ converge to $F_0(x)$ and $F_1(x)$ uniformly for $||x|| = 1$ and further satisfy $A \leqq F_{in}(x) \leqq B$. Then it is easily shown that $\{f(p_n, \vartheta_n, F_{0n}(x), F_{1n}(x))\}$ is a sequence converging to $f(p, \vartheta, F_0(x), F_1(x))$ uniformly for $||x|| = 1$, where $\{p_n\}$ and $\{\vartheta_n\}$ converge to $p$ and $\vartheta$ and satisfy $1 \leqq p_n \leqq P$, $0 \leqq \vartheta_n \leqq 1$.

The convergence of a sequence of elements $e_n = (p_n, \vartheta_n, K_{0n}, K_{1n})$ of $S$ to element $e$ of $S$ implies

$$\lim_{n\to\infty} \dot{\delta}^{(1)}(K_i, K_{in}) = 0$$

which in turn is equivalent to the convergence of the associated sequences of distance functions $\{F_{in}(x)\}$ to $F_i(x)$ uniformly for $||x|| = 1$. Moreover, since all the bodies in the sequences $\{K_{in}\}$ as well as the limit bodies $K_i$ are in $\mathscr{K}_1$ we know that there is a sphere $(1/A)E$ containing each $K_i$ and $K_{in}$, and a sphere $(1/B)E$ contained in each $K_i$ and $K_{in}$. From this it follows that $0 < A \leqq F_{in}(x) \leqq B < \infty$. Thus, by the preceding paragraph, the convergence of $\{e_n\}$ to $e$ entails the convergence of $\{f(p_n, \vartheta_n, F_{0n}(x), F_{1n}(x))\}$ to $f(p, \vartheta, F_0(x), F_1(x))$ uniformly for $||x|| = 1$. This is to say that

$$\lim_{n\to\infty} \dot{\delta}^{(1)}(K(e_n), K(e)) = 0$$

as asserted.

We next examine inclusion relations among the means $\dot{K}_\vartheta^{(p)}$. Since

$$\sqrt[p]{[(1-\vartheta)F_0^p(x) + \vartheta F_1^p(x)]} \leqq \sqrt[q]{[(1-\vartheta)F_0^q(x) + \vartheta F_1^q(x)]}$$

for $1 \leqq p < q \leqq \infty$ with equality if and only if $F_0(x) = F_1(x)$, we have $\dot{K}_\vartheta^{(p)} \supseteq \dot{K}_\vartheta^{(q)}$ with equality if and only if $K_0 = K_1$. Thus the means are either constant if $K_0 = K_1 = \dot{K}_\vartheta^{(p)}$ or are strictly monotonic decreasing in $p$ from $\dot{K}_\vartheta^{(1)}$ to $K_0 \cap K_1$.

Finally consider the family $\{\dot{K}_\vartheta^{(p)}\}$ for fixed $p$ and varying $\vartheta$. For $p = \infty$, it is geometrically obvious that the family is convex by which we mean that

$$K_{\vartheta'}^{(p)} \subseteq (1-\vartheta)\dot{K}_{\vartheta_0}^{(p)} + \vartheta\dot{K}_{\vartheta_1}^{(p)}$$

where $\vartheta' = (1-\vartheta)\vartheta_0 + \vartheta\vartheta_1$. But this is true for all $p$ satisfying $1 \leqq p \leqq \infty$. In virtue of the monotonicity in $p$ discussed in the preceding paragraph, it is enough to show the asserted convexity for $p = 1$.

We make a further reduction of the problem. Since

$$\hat{\dot{K}}_\vartheta^{(1)} = (1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1,$$

we have

$$\begin{aligned}\hat{\dot{K}}_{\vartheta'}^{(1)} &= [(1-\vartheta')\hat{K}_0 + \vartheta'\hat{K}_1]^\wedge \\ &= [(1-\vartheta)[(1-\vartheta_0)\hat{K}_0 + \vartheta\,\hat{K}_1] + \vartheta[(1-\vartheta_1)\hat{K}_0 + \vartheta_1\hat{K}_1]]^\wedge,\end{aligned}$$

and

$$(1-\vartheta)\dot{K}_{\vartheta_0}^{(1)} + \vartheta\dot{K}_{\vartheta_1}^{(1)} = (1-\vartheta)[(1-\vartheta_0)\hat{K}_0 + \vartheta_0\hat{K}_1]^\wedge + \vartheta[(1-\vartheta_1)\hat{K}_0 + \vartheta_1\hat{K}_1]^\wedge.$$

Set $K = (1-\vartheta_0)\hat{K}_0 + \vartheta_0\hat{K}_1$ and $K' = (1-\vartheta_1)\hat{K}_0 + \vartheta_1\hat{K}_1$. In terms of $K$, $K'$ we must prove that $[(1-\vartheta)K + \vartheta K']^\wedge \subseteq (1-\vartheta)\hat{K} + \vartheta\hat{K}'$.

On a ray $r$ from $Q$ let $x$ be on $\partial K$, $x'$ on $\partial K'$. Then $x_\vartheta = (1-\vartheta)x + \vartheta x'$ is a point, in general interior, of the Minkowski sum $(1-\vartheta)K + \vartheta K'$. Let $\Pi$, $\Pi'$, $\Pi_\vartheta$ be the polar planes of $x$, $x'$, and $x_\vartheta$. These planes are orthogonal to $r$ and meet $r$ in points $z$, $z'$, and $z_\vartheta$. $\Pi$ and $\Pi'$ are support planes of $K$ and $K'$. $\Pi_\vartheta$ is a plane exterior to $[(1-\vartheta)K + \vartheta K']^\wedge$ unless $x_\vartheta$ happens to be a boundary point of $(1-\vartheta)K + \vartheta K'$, in which case $\Pi_\vartheta$ is a support plane of $[(1-\vartheta)K$

$+ \vartheta K']^\wedge$. Let $\bar{z} = (1-\vartheta)z + \vartheta z'$. The plane $\Pi$, orthogonal to $r$ through $\bar{z}$ is a support plane of $(1-\vartheta)\hat{K} + \vartheta\hat{K}'$.

If we can show that $z_\vartheta \leqq \bar{z}$, it will follow that $\bar{\Pi}$ is either exterior to $[(1-\vartheta)K + \vartheta K']^\wedge$ or coincides with $\Pi_\vartheta$ if $z_\vartheta = \bar{z}$. Since $r$ is arbitrary, this will prove that

$$[(1-\vartheta)K + \vartheta K']^\wedge \subseteq (1-\vartheta)\hat{K} + \vartheta\hat{K}'.$$

We have from the polarity relations:

$$||z|| \cdot ||x|| = ||z'|| \cdot ||x'|| = ||z_\vartheta|| \cdot ||x_\vartheta|| = 1.$$

Hence

$$||z_\vartheta|| = \frac{\dfrac{(1-\vartheta)}{||z||} \cdot ||z|| \cdot ||x|| + \dfrac{\vartheta}{||z'||} \cdot ||z'|| \cdot ||x'||}{\dfrac{(1-\vartheta)}{||z||} \cdot ||x_\vartheta|| + \dfrac{\vartheta}{||z'||} \cdot ||x_\vartheta||}$$

$$= \frac{||(1-\vartheta)x + \vartheta x'||}{||x_\vartheta|| \cdot \left(\dfrac{1-\vartheta}{||z||} + \dfrac{\vartheta}{||z'||}\right)}.$$

In the last step, we have utilized the collinearity of $Q$, $x$, and $x'$. Continuing:

$$||z_\vartheta|| = \frac{1}{\dfrac{(1-\vartheta)}{||z||} + \dfrac{\vartheta}{||z'||}} \leqq (1-\vartheta)||z|| + \vartheta||z'|| = ||\bar{z}||$$

where the collinearity of $Q$, $z$, $z'$, and $z_\vartheta$ has been used. In the inequality of the arithmetic and harmonic means, there is equality if and only if $||z|| = ||z'||$, from which we conclude that the original inclusion is an equality if and only if $K = K'$.

This argument proves the convexity of $\{\dot{K}_\vartheta^{(p)}\}$. The family is linear if and only if

$$\dot{K}_{\vartheta_0}^{(p)} = \dot{K}_{\vartheta_1}^{(p)}$$

which means $K_0 = K_1$.

This completes the proof of our next theorem.

THEOREM 2. *The family* $\{\dot{K}_\vartheta^{(p)}\}$ *depends continuously on* $(p, \vartheta, K_0, K_1)$ *for* $1 \leqq p \leqq P < \infty$, $0 \leqq \vartheta \leqq 1$, $K_i$ in $\mathscr{K}_1$. *It is strictly monotonic decreasing in* $p$ *for* $1 \leqq p \leqq \infty$ *and convex in* $\vartheta$.

An immediate consequence of Theorem 2 is as follows. Let $W_{(s)}(K)$ denote the $s$th cross-sectional measure of $K$, that is, the mixed volume

$$V(\underbrace{K, \ldots, K}_{(n-s)}; \underbrace{E, \ldots, E}_{s})$$

for $s = 0, 1, \ldots, n-1$. The measures $W_{(s)}(K)$ are well known to be monotonic in $K$, that is if $K \subseteq K'$ then $W_{(s)}(K) \leqq W_{(s)}(K')$ (cf. **(1)**, p. 50). Hence

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$$W_{(s)}(\dot{K}_\vartheta^{(p)}) \geqq W_{(s)}(\dot{K}_\vartheta^{(q)})$$

when $1 \leqq p \leqq q \leqq \infty$, with equality if and only if $K_0$ and $K_1$ are identical. Thus $W_{(s)}(\dot{K}_\vartheta^{(p)})$ is monotonic decreasing in $p$ and, in virtue of Theorem 2, continuous in that parameter. In particular, the intersection $K_0 \cap K_1$ has minimal cross-sectional measures and $\dot{K}_\vartheta^{(p)}$ has maximal. This latter family of bodies might well be called the set of weighted harmonic means of $K_0$ and $K_1$ in view of the next remarks.

A special instance of the convexity of the family $\dot{K}_\vartheta^{(1)}$ is

$$\dot{K}_\vartheta^{(1)} = [(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^{\wedge} \subseteq (1-\vartheta)K_0 + \vartheta K_1.$$

In the inclusion, there is equality if and only if $K_0 = K_1$. This may be viewed as the analogue, for convex bodies, of the theorem of the arithmetic and harmonic means for positive numbers. Indeed, the latter may be looked upon as a special case of the former in which $K_0$ and $K_1$ are centrally symmetric bodies in a one-dimensional Euclidean space, the centre of symmetry being the common interior point $Q$. A similar observation is valid regarding the monotonicity of the means $\dot{K}_\vartheta^{(p)}$ in $p$ for fixed $\vartheta$.

The results of these last two paragraphs give us the inequalities

$$W_{(s)}(\dot{K}_\vartheta^{(p)}) \leqq W_{(s)}((1-\vartheta)K_0 + \vartheta K_1)$$

for $1 \leqq p \leqq \infty$ with equality if and only if $K_0 = K_1$. The next section furnishes an improvement on this result for the case $s = 0$, that is for the volume functional.

**4. A dual Brunn–Minkowski theorem.** For fixed $p$ satisfying $1 \leqq p < \infty$, let $V(\dot{K}_\vartheta^{(p)}) = V_\vartheta$ be the volume of $\dot{K}_\vartheta^{(p)}$ where $0 \leqq \vartheta \leqq 1$. Since $\dot{K}_\vartheta^{(p)}$ contains an interior point $Q$, $V_\vartheta > 0$. The distance function of $\dot{K}_\vartheta^{(p)}$ is

$$F_\vartheta(x) = \sqrt[p]{[(1-\vartheta)F_0^p(x) + \vartheta F_1^p(x)]}.$$

Let

$$\bar{K}_i = \frac{1}{V_i^{1/n}} K_i;\ V(\bar{K}_i) = 1.$$

Set

$$\bar{F}_{\vartheta'}(x) = \sqrt[p]{[(1-\vartheta')\bar{F}_0^p(x) + \vartheta'\bar{F}_1^p(x)]}$$

where $\bar{F}_i(x) = V_i^{1/n}F_i(x)$ is the distance function of $\bar{K}_i$. Finally, let $\bar{V}_{\vartheta'}$ be the volume of that convex body whose distance function is $\bar{F}_{\vartheta'}(x)$. Since $F_\vartheta(x) = \bar{F}_{\vartheta'}(x)/\mu$, where

$$\mu = 1 \Big/ \sqrt[p]{\left[\frac{(1-\vartheta)}{V_0^{p/n}} + \frac{\vartheta}{V_1^{p/n}}\right]},\ \vartheta' = \vartheta\mu^p/V_1^{p/n},$$

we have $V_\vartheta^{1/n} = \mu\bar{V}_{\vartheta'}^{1/n}$.

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The polar co-ordinate formula for the volume of a convex body gives

$$\bar{V}_{\vartheta'} = \frac{1}{n}\int_{\partial E}\left[\frac{1}{\bar{F}_{\vartheta'}(u)}\right]^n dw,$$

where $dw$ is the differential of surface area of the unit sphere $E$ centred at $Q$.

For the integrand we have

$$1 \Big/ \sqrt[p]{\left[\frac{(1-\vartheta')}{\left(\frac{1}{\bar{F}_0(u)}\right)^p} + \frac{\vartheta'}{\left(\frac{1}{\bar{F}_1(u)}\right)^p}\right]} \leqq \sqrt[n]{\left[(1-\vartheta')\left(\frac{1}{\bar{F}_0(u)}\right)^n + \vartheta'\left(\frac{1}{\bar{F}_1(u)}\right)^n\right]}$$

with equality if and only if $\bar{F}_0(u) = \bar{F}_1(u)$. Therefore

$$\bar{V}_{\vartheta'} \leqq \frac{1}{n}\int_{\partial E}\left[\frac{(1-\vartheta')}{(\bar{F}_0(u))^n} + \frac{\vartheta'}{(\bar{F}_1(u))^n}\right] dw = (1-\vartheta')V(\bar{K}_0) + \vartheta' V(\bar{K}_1) = 1.$$

There is equality if and only if $\bar{K}_0 = \bar{K}_1$. This gives as the analogue of the Brunn–Minkowski theorem: $V_\vartheta{}^{1/n} \leqq \mu$. There is equality if and only if $K_0 = \lambda K_1$, $\lambda = (V_0/V_1)^{1/n}$, the centre of homothety being at $Q$.

If $p = \infty$, we have $K_0 \cap K_1 \subseteq K_i$ and so $V(K_0 \cap K_1) \leqq \min(V_0, V_1)$. Clearly there is equality if and only if one of the bodies $K_i$ is a subset of the other. The volume functional is monotonic under set inclusion and so, by Theorem 2, $V(K_0 \cap K_1) \leqq V(\dot{K}_\vartheta{}^{(p)})$ for $1 \leqq p < \infty$ with equality if and only if $K_0 = K_1$.

We collect these results in our last theorem.

THEOREM 3.

$$V^{1/n}(K_0 \cap K_1) \leqq V^{1/n}(\dot{K}_\vartheta^{(p)}) \leqq 1 \Big/ \sqrt[p]{\left[\frac{(1-\vartheta)}{V^{p/n}(K_0)} + \frac{\vartheta}{V^{p/n}(K_1)}\right]},$$

*for $1 \leqq p < \infty$. There is equality on the left if and only if $K_0 = K_1$ and on the right if and only if $K_0 = \lambda K_1$ with centre of homothety at $Q$. Further*

$$V^{1/n}(K_0 \cap K_1) = V^{1/n}(K_\vartheta^{(\infty)}) \leqq \min(V^{1/n}(K_0), V^{1/n}(K_1))$$

*with equality on the right if and only if $K_0 = K_1$.*

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*Washington State University*