

## BLOCKS WITH A QUATERNION DEFECT GROUP OVER A 2-ADIC RING: THE CASE $\tilde{A}_4$

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**Abstract.** Except for blocks with a cyclic or Klein four defect group, it is not known in general whether the Morita equivalence class of a block algebra over a field of prime characteristic determines that of the corresponding block algebra over a  $p$ -adic ring. We prove this to be the case when the defect group is quaternion of order 8 and the block algebra over an algebraically closed field  $k$  of characteristic 2 is Morita equivalent to  $k\tilde{A}_4$ . The main ingredients are Erdmann's classification of tame blocks [6] and work of Cabanes and Pizarro [4, 5] on perfect isometries between tame blocks.

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**Introduction.** Throughout these notes,  $\mathcal{O}$  is a complete discrete valuation ring with algebraically closed residue field  $k$  of characteristic 2 and with quotient field  $K$  of characteristic 0. According to Erdmann's classification in [6], if  $G$  is a finite group and if  $b$  is a block of  $\mathcal{O}G$  having the quaternion group  $Q_8$  of order 8 as defect group, then the block algebra  $kG\bar{b}$  is Morita equivalent to either  $kQ_8$  or  $k\tilde{A}_4$  or the principal block algebra of  $k\tilde{A}_5$ , where here  $\bar{b}$  is the canonical image of  $b$  in  $kG$ . In the first case the block is nilpotent (cf. [3]), and it follows from Puig's structure theorem of nilpotent blocks in [8] that  $\mathcal{O}G\bar{b}$  is Morita equivalent to  $\mathcal{O}Q_8$ . In the remaining two cases one should expect that  $\mathcal{O}G\bar{b}$  is Morita equivalent to  $\mathcal{O}\tilde{A}_4$  or the principal block algebra of  $\mathcal{O}\tilde{A}_5$ , respectively. We show this to be true in one of these two cases under the assumption that  $K$  is large enough.

**THEOREM A.** *Let  $G$  be a finite group, and let  $b$  be a block of  $\mathcal{O}G$  having a quaternion defect group of order 8. Denote by  $\bar{b}$  the image of  $b$  in  $kG$ . Assume that  $KG\bar{b}$  is split. If  $kG\bar{b}$  is Morita equivalent to  $k\tilde{A}_4$  then  $\mathcal{O}G\bar{b}$  is Morita equivalent to  $\mathcal{O}\tilde{A}_4$ .*

By Cabanes-Pizarro [4, 5], in the situation of Theorem A there is a perfect isometry between the character groups of  $\mathcal{O}G\bar{b}$  and of  $\mathcal{O}\tilde{A}_4$ . Thus Theorem A is a consequence of the following slightly more general Theorem which characterises  $\mathcal{O}G\bar{b}$  in terms of its center, its character group and  $k\tilde{A}_4$ ; see the end of this section for more details regarding the notation.

**THEOREM B.** *Let  $A$  be an  $\mathcal{O}$ -free  $\mathcal{O}$ -algebra such that  $K \otimes_{\mathcal{O}} A$  is split semi-simple and such that  $k \otimes_{\mathcal{O}} A$  is Morita equivalent to  $k\tilde{A}_4$ . Assume that there is an isometry  $\Phi : \mathbb{Z}\mathrm{Irr}_K(A) \cong \mathbb{Z}\mathrm{Irr}_K(\mathcal{O}\tilde{A}_4)$  which maps  $\mathrm{Proj}(A)$  to  $\mathrm{Proj}(\mathcal{O}\tilde{A}_4)$  such that the map sending*

$e(\chi)$  to  $e(\Phi(\chi))$ , for every  $\chi \in \text{Irr}_K(A)$  induces an  $\mathcal{O}$ -algebra isomorphism of the centers  $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$ . Then  $A$  is Morita equivalent to  $\mathcal{O}\tilde{A}_4$ .

Theorem B is in turn a consequence of the more precise Theorem C, describing  $A$  in terms of generators and relations:

**THEOREM C.** *Let  $A$  be a basic  $\mathcal{O}$ -free  $\mathcal{O}$ -algebra such that  $K \otimes_{\mathcal{O}} A$  is split semi-simple and such that  $k \otimes_{\mathcal{O}} A$  is isomorphic to  $k\tilde{A}_4$ . Assume that there is an isometry  $\Phi : \mathbb{Z}\text{Irr}_K(A) \cong \mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  which maps  $\text{Proj}(A)$  to  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  such that the map sending  $e(\chi)$  to  $e(\Phi(\chi))$ , for every  $\chi \in \text{Irr}_K(A)$  induces an  $\mathcal{O}$ -algebra isomorphism of the centers  $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$ . Then  $A$  is isomorphic to the unitary  $\mathcal{O}$ -algebra with set of generators  $\{e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa\}$  of  $A$ , such that  $e_0, e_1, e_2$  are pairwise orthogonal idempotents whose sum is 1 and satisfying the following relations:*

$$\begin{aligned} \beta &= e_0\beta = \beta e_1, \quad \gamma = e_1\gamma = \gamma e_0; \\ \delta &= e_1\delta = \delta e_2, \quad \eta = e_2\eta = \eta e_1; \\ \lambda &= e_2\lambda = \lambda e_0, \quad \kappa = e_0\kappa = \kappa e_2; \\ \beta\delta &= -2\kappa + \kappa\lambda\kappa; \quad \eta\gamma = -2\lambda + \lambda\kappa\lambda; \quad \delta\lambda = -2\gamma + \gamma\beta\gamma; \\ \kappa\eta &= -2\beta + \beta\gamma\beta; \quad \lambda\beta = -2\eta + \eta\delta\eta; \quad \gamma\kappa = -2\delta + \delta\eta\delta; \\ \gamma\beta\delta &= -4\delta + 2\delta\eta\delta; \quad \delta\eta\gamma = -4\gamma + 2\gamma\beta\gamma; \quad \lambda\kappa\eta = -4\eta + 2\eta\delta\eta; \\ \beta\gamma\kappa &= -4\kappa + 2\kappa\lambda\kappa; \quad \eta\delta\lambda = -4\lambda + 2\lambda\kappa\lambda; \quad \kappa\lambda\beta = -4\beta + 2\beta\gamma\beta; \\ \eta\gamma\beta &= -4\eta + 2\eta\delta\eta; \quad \beta\delta\eta = -4\beta + 2\beta\gamma\beta; \quad \delta\lambda\kappa = -4\delta + 2\delta\eta\delta; \\ \lambda\beta\gamma &= -4\lambda + 2\lambda\kappa\lambda; \quad \kappa\eta\delta = -4\kappa + 2\kappa\lambda\kappa; \quad \gamma\kappa\lambda = -4\gamma + 2\gamma\beta\gamma; \\ \beta\delta\lambda\beta &= -8\beta + 4\beta\gamma\beta; \quad \delta\lambda\beta\delta = -8\delta + 4\delta\eta\delta; \quad \lambda\beta\delta\lambda = -8\lambda + 4\lambda\kappa\lambda. \end{aligned}$$

When reduced modulo 2, these relations seem to be more than those occurring in Erdmann’s work [6] over  $k$  (we recall these more precisely in §2); but they are not, since the extra relations over  $k$  can be deduced from those given by Erdmann. We need to add in extra relations over  $\mathcal{O}$  in order to ensure that the algebra we construct is  $\mathcal{O}$ -free of the right rank.

Since  $\mathcal{O}\tilde{A}_4$  fulfills the hypotheses of Theorem C it follows that  $A \cong \mathcal{O}\tilde{A}_4$ , hence Theorem C indeed implies Theorem B. The proof of Theorem C is given at the end of Section 2.

**NOTATION.** If  $A$  is an  $\mathcal{O}$ -algebra such that  $K \otimes_{\mathcal{O}} A$  is split semi-simple, denote by  $\text{Irr}_K(A)$  the set of characters of the simple  $K \otimes_{\mathcal{O}} A$ -modules, viewed as central functions from  $A$  to  $\mathcal{O}$  and denote by  $\text{Irr}_k(k \otimes_{\mathcal{O}} A)$  the set of isomorphism classes of simple  $k \otimes_{\mathcal{O}} A$ -modules. We denote by  $\mathbb{Z}\text{Irr}_K(A)$  the group of characters of  $A$ , and by  $\text{Proj}(A)$  the subgroup of  $\mathbb{Z}\text{Irr}_K(A)$  generated by the characters of the projective indecomposable  $A$ -modules. We denote by  $L^0(A)$  the subgroup of  $\mathbb{Z}\text{Irr}_K(A)$  of all elements which are orthogonal to  $\text{Proj}(A)$  with respect to the usual scalar product in  $\mathbb{Z}\text{Irr}_K(A)$ . For any  $\chi \in \text{Irr}_K(A)$ , we denote by  $e(\chi)$  the corresponding primitive idempotent in  $Z(K \otimes_{\mathcal{O}} A)$ . If  $A = \mathcal{O}G$  for some finite group  $G$  we have the well-known formula

$$e(\chi) = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1})x.$$

We refer to [1, 2] for the concept and basic properties of perfect isometries, and to [9] for general block theoretic background material.

**1. Characters and perfect isometries of  $\mathcal{O}\tilde{A}_4$ .** We identify  $\tilde{A}_4 = Q_8 \rtimes C_3$ . Let  $t$  be a generator of  $C_3$  and let  $y$  be an element of order 4 in  $Q_8$ . Set  $z = y^2$ ; that is,  $z$  is the unique central involution of  $\tilde{A}_4$ . Then the seven elements  $1, z, y, t, t^2, tz, t^2z$  are a complete set of representatives of the conjugacy classes in  $\tilde{A}_4$ .

Let  $\omega$  be a primitive third root of unity in  $\mathcal{O}$ . The character table of  $\tilde{A}_4$  is as follows:

|          | 1 | $z$ | $y$ | $t$         | $t^2$       | $tz$       | $t^2z$     |
|----------|---|-----|-----|-------------|-------------|------------|------------|
| $\eta_0$ | 1 | 1   | 1   | 1           | 1           | 1          | 1          |
| $\eta_1$ | 1 | 1   | 1   | $\omega$    | $\omega^2$  | $\omega$   | $\omega^2$ |
| $\eta_2$ | 1 | 1   | 1   | $\omega^2$  | $\omega$    | $\omega^2$ | $\omega$   |
| $\eta_3$ | 3 | 3   | -1  | 0           | 0           | 0          | 0          |
| $\eta_4$ | 2 | -2  | 0   | $-\omega^2$ | $-\omega$   | $\omega^2$ | $\omega$   |
| $\eta_5$ | 2 | -2  | 0   | $-\omega$   | $-\omega^2$ | $\omega$   | $\omega^2$ |
| $\eta_6$ | 2 | -2  | 0   | -1          | -1          | 1          | 1          |

The algebra  $\mathcal{O}\tilde{A}_4$  has three simple modules  $T_0, T_1, T_2$ , up to isomorphism. Choosing for  $T_0$  the trivial module and after possibly exchanging the notation for  $T_1, T_2$ , the ordinary decomposition matrix of  $\mathcal{O}\tilde{A}_4$  is as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The Cartan matrix of  $\mathcal{O}\tilde{A}_4$  is the product of the decomposition matrix with its transpose, hence equal to

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

Let  $e_0, e_1, e_2$  be primitive idempotents in  $\mathcal{O}\tilde{A}_4$  such that  $\mathcal{O}\tilde{A}_4e_i$  is a projective cover of  $T_i, 0 \leq i \leq 2$ . By the above decomposition matrix, the characters of the projective indecomposable  $\mathcal{O}\tilde{A}_4$ -modules  $\mathcal{O}\tilde{A}_4e_i$  are

$$\begin{aligned} \eta_0 + \eta_3 + \eta_4 + \eta_5, \\ \eta_1 + \eta_3 + \eta_4 + \eta_6, \\ \eta_2 + \eta_3 + \eta_5 + \eta_6, \end{aligned}$$

respectively. Their norm is 4, and the differences of any two different characters of projective indecomposable  $\mathcal{O}\tilde{A}_4$ -modules yields the following further elements in  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  having also norm 4:

$$\begin{aligned} \eta_0 - \eta_1 + \eta_5 - \eta_6, \\ \eta_0 - \eta_2 + \eta_4 - \eta_6, \\ \eta_1 - \eta_2 + \eta_4 - \eta_5. \end{aligned}$$

It is easy to check, that up to signs, these are all the elements in  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  having norm 4.

A self-isometry  $\Phi$  of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  maps every  $\eta_i$  to  $\epsilon_i\eta_{\pi(i)}$  for some signs  $\epsilon_i \in \{1, -1\}$  and a permutation  $\pi$  of  $\{0, 1, \dots, 6\}$ . In other words,  $\Phi$  is determined by the permutation  $\tau$  of the set  $\{1, -1\} \times \{0, 1, \dots, 6\}$  satisfying  $\tau(1, i) = (\epsilon_i, \pi(i))$  and  $\tau(-1, i) = (-\epsilon_i, \pi(i))$  for all  $i, 0 \leq i \leq 6$ . If we write  $i, -i$  instead of  $(1, i), (-1, i)$ , respectively, this becomes  $\tau(i) = \epsilon_i\pi(i)$  and  $\tau(-i) = -\epsilon_i\pi(i)$ , with the usual cancellation rules for signs. In this way, every self-isometry  $\Phi$  of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  gets identified to a permutation of the set of symbols  $\{i, -i \mid 0 \leq i \leq 6\}$ . A perfect self-isometry of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  is a self-isometry which is perfect in the sense of Broué [1]. Any such perfect self-isometry preserves  $\text{Proj}(\mathcal{O}\tilde{A}_4)$ . The next Proposition implies that the converse is true, too.

**PROPOSITION 1.1.** *The group of all perfect self-isometries of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  is equal to the group of all self-isometries of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  which preserve  $\text{Proj}(\mathcal{O}\tilde{A}_4)$ . This group is generated by  $-\text{Id}$  together with the set of permutations*

$$\begin{aligned} &(0, 1, 2)(4, 6, 5), \\ &(1, 2)(4, 5), \\ &(2, -3)(5, -6). \end{aligned}$$

Every algebra automorphism of  $\mathcal{O}\tilde{A}_4$  induces a permutation on  $\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  which is in fact a perfect isometry on  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$ . Since  $\eta_1$  has degree 1, it is an algebra homomorphism from  $\mathcal{O}\tilde{A}_4$  to  $\mathcal{O}$ , and hence the map sending  $x \in \mathcal{O}\tilde{A}_4$  to  $\eta_1(x)x$  is an algebra automorphism of  $\mathcal{O}\tilde{A}_4$  whose inverse sends  $x \in \mathcal{O}\tilde{A}_4$  to  $\eta_2(x)x$ . The following statement is an immediate consequence from the character table of  $\mathcal{O}\tilde{A}_4$ .

**LEMMA 1.2.** *Let  $\gamma$  be the algebra automorphism of  $\mathcal{O}\tilde{A}_4$  defined by  $\gamma(x) = \eta_1(x)x$  for all  $x \in \mathcal{O}\tilde{A}_4$ . The permutation  $\pi$  of  $\{0, 1, \dots, 6\}$  defined by  $\eta_i \circ \gamma = \eta_{\pi(i)}$  is equal to  $\pi = (0, 1, 2)(4, 6, 5)$ .*

The anti-automorphism of  $\mathcal{O}\tilde{A}_4$  sending  $x \in \tilde{A}_4$  to  $x^{-1}$  induces also a permutation of the set  $\text{Irr}_K(\mathcal{O}\tilde{A}_4)$ , and this is also a perfect isometry (this holds for any finite group). This permutation can also be read off the character table.

**LEMMA 1.3.** *Let  $\iota$  be the algebra anti-automorphism of  $\mathcal{O}\tilde{A}_4$  mapping  $x \in \tilde{A}_4$  to  $x^{-1}$ . The permutation  $\pi$  of  $\{0, 1, \dots, 6\}$  defined by  $\eta_i \circ \iota = \eta_{\pi(i)}$  is equal to  $\pi = (1, 2)(4, 5)$ .*

*Proof of 1.1.* The first two permutations are perfect isometries by 1.2 and 1.3, respectively. An easy but painfully long verification shows that the bicharacter sending  $(g, h) \in \tilde{A}_4 \times \tilde{A}_4$  to

$$\begin{aligned} &\eta_0(g)\eta_0(h) + \eta_1(g)\eta_1(h) - \eta_2(g)\eta_3(h) - \eta_3(g)\eta_2(h) \\ &+ \eta_4(g)\eta_4(h) - \eta_5(g)\eta_6(h) - \eta_6(g)\eta_5(h) \end{aligned}$$

is perfect; that is, its value at any  $(g, h)$  is divisible in  $\mathcal{O}$  by the orders of  $C_{\tilde{A}_4}(g)$  and  $C_{\tilde{A}_4}(h)$  and it vanishes if exactly one of  $g, h$  has odd order. Thus the isometry given by the permutation  $(2, -3)(5, -6)$  is perfect. It remains to show that these permutations, together with  $-\text{Id}$ , generate the group of all self-isometries which preserve  $\text{Proj}(\mathcal{O}\tilde{A}_4)$ .

We described above a complete list of all elements in  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  having norm 4. Since the characters of the projective indecomposable modules are in that list, a self-isometry  $\Phi$  of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  preserves  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  if and only if it permutes this set of norm 4 elements.

Let  $\Phi$  be a self-isometry of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  which preserves  $\text{Proj}(\mathcal{O}\tilde{A}_4)$ . Then  $\Phi$  preserves also the group  $L^0(\mathcal{O}\tilde{A}_4)$  of generalised characters which are orthogonal to all characters in  $\text{Proj}(\mathcal{O}\tilde{A}_4)$ . Up to signs, the complete list of elements in  $L^0(\mathcal{O}\tilde{A}_4)$  having norm 3 is

$$\begin{aligned} &\eta_0 + \eta_1 - \eta_4, \quad \eta_0 + \eta_2 - \eta_5, \quad \eta_0 - \eta_3 + \eta_6, \\ &\eta_1 + \eta_2 - \eta_6, \quad \eta_1 - \eta_3 + \eta_5, \quad \eta_2 - \eta_3 + \eta_4. \end{aligned}$$

Up to signs again, the complete list of elements in  $L^0(\mathcal{O}\tilde{A}_4)$  having norm 4 is

$$\begin{aligned} &\eta_0 + \eta_1 + \eta_2 - \eta_3, \\ &\eta_0 - \eta_1 - \eta_5 + \eta_6, \quad \eta_0 - \eta_2 - \eta_4 + \eta_6, \quad \eta_0 + \eta_3 - \eta_4 - \eta_5, \\ &\eta_1 - \eta_2 - \eta_4 + \eta_5, \quad \eta_1 + \eta_3 - \eta_4 - \eta_6, \quad \eta_2 + \eta_3 - \eta_5 - \eta_6. \end{aligned}$$

The first norm 4 element in this list,  $\eta_0 + \eta_1 + \eta_2 - \eta_3$ , is the only norm 4 element which is orthogonal to all other norm 4 elements in  $L^0(\mathcal{O}\tilde{A}_4)$ . Thus  $\Phi$  has to permute the characters  $\eta_0, \eta_1, \eta_2, \eta_3$  amongst each other.

Suppose first that  $\Phi$  fixes  $\eta_3$ . Then, by composing  $\Phi$  with a suitable product of powers of the first two permutations in the statement, we may assume that  $\Phi$  fixes  $\eta_0, \eta_1, \eta_2$  up to signs. By considering the first of the above norm 4 elements in  $L^0(\mathcal{O}\tilde{A}_4)$  we get that  $\Phi$  fixes  $\eta_0, \eta_1, \eta_2$  all with positive signs. By considering the norm 3 elements in  $L^0(\mathcal{O}\tilde{A}_4)$ , it follows that  $\Phi$  fixes also  $\eta_4, \eta_5$  and  $\eta_6$  with positive signs. Thus a self-isometry of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  which preserves  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  and which fixes  $\eta_3$  is in the group generated by the set of two permutations  $(0, 1, 2)(4, 6, 5)$  and  $(1, 2)(4, 5)$ .

Suppose next that  $\Phi$  does not fix  $\eta_3$ . By precomposing  $\Phi$  with a suitable power of  $(0, 1, 2)(4, 6, 5)$  we may assume that  $\Phi$  sends  $\eta_2$  to  $-\eta_3$ . By composing  $\Phi$  with a suitable power of  $(0, 1, 2)(4, 5, 6)$  we may assume that  $\Phi$  fixes  $\eta_0$ , up to a sign. Since  $\Phi$  preserves the norm 4 element  $\eta_0 + \eta_1 + \eta_2 - \eta_3$ , we necessarily have  $\Phi(\eta_0) = \eta_0$ . Then  $\Phi$  maps  $\eta_1$  either to  $\eta_1$  or  $\eta_2$  (with positive signs, again because of that same norm 4 element). In the first case,  $\Phi$  fixes both  $\eta_0, \eta_1$ , and by checking the norm 3 elements in  $L^0(\mathcal{O}\tilde{A}_4)$  one gets  $\Phi = (2, -3)(5, -6)$ . In the second case, again checking on norm 3 elements, one gets  $\Phi = (1, 2, -3)(4, 5, -6)$ , but this is already the product of  $(1, 2)(4, 5)$  and  $(2, -3)(5, -6)$ .  $\square$

**2. The algebra  $A$ .** Let  $A$  be a basic  $\mathcal{O}$ -algebra fulfilling the hypotheses of Theorem B; that is,  $K \otimes_{\mathcal{O}} A$  is split semi-simple,  $k \otimes_{\mathcal{O}} A$  is isomorphic to  $k\tilde{A}_4$ , and there is an isometry  $\mathbb{Z}\text{Irr}_K(A) \cong \mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  mapping  $\text{Proj}(A)$  to  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  and inducing an isomorphism  $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$ . There is a ‘‘compatible choice’’ for these isomorphisms.

**PROPOSITION 2.1.** *There is an algebra isomorphism  $\alpha : k \otimes_{\mathcal{O}} A \cong k\tilde{A}_4$  and an isometry  $\Phi : \mathbb{Z}\text{Irr}_K(A) \cong \mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  mapping  $\text{Proj}(A)$  to  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  with the following properties:*

- (i)  $\Phi$  maps  $\text{Irr}_K(A)$  onto  $\text{Irr}_K(\mathcal{O}\tilde{A}_4)$ ; that is, all signs are  $+1$ .
- (ii) The map sending  $e(\chi)$  to  $e(\Phi(\chi))$  for every  $\chi \in \text{Irr}_K(A)$  induces an isomorphism  $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$ .

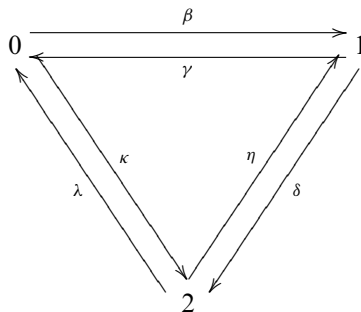
- (iii) For any primitive idempotents  $e \in A$  and  $f \in \mathcal{O}\tilde{A}_4$  and every  $\chi \in \text{Irr}_K(A)$  such that  $\alpha(\bar{e}) = f$  we have  $\chi(e) = \Phi(\chi)(f)$ ; that is,  $A$  and  $\mathcal{O}\tilde{A}_4$  have the same decomposition matrices through  $\alpha$  and  $\Phi$ .

*Proof.* The  $\mathcal{O}$ -rank of  $A$  is 24 and also the sum of the squares of the degrees of the seven irreducible  $K$ -linear characters of  $A$ ; thus every irreducible character of  $A$  has degree smaller than 5. Also, there is no character of degree 4 because  $24 - 4^2 = 8$  cannot be written as a sum of six squares of the degrees of the six remaining characters. But there must be a character of degree 3; if not, 24 would be the sum of seven squares all either 1 or 4, which is not possible. Thus the squares of the degrees of the six remaining characters add up to  $24 - 3^2 = 15$ , and the only way to do this is with three characters of degree 1 and three characters of degree 2.

This proves that the character degrees of the irreducible characters of  $A$  and of  $\mathcal{O}\tilde{A}_4$  coincide for some bijection  $\text{Irr}_K(A) \cong \text{Irr}_K(\mathcal{O}\tilde{A}_4)$ . Since the decomposition matrix of  $A$  multiplied with its transpose yields the Cartan matrix of  $A$  – which is equal to that of  $k\tilde{A}_4$  – the algebra  $A$  has in fact the same decomposition matrix as  $\mathcal{O}\tilde{A}_4$  for a suitable bijection  $\Phi : \text{Irr}_K(A) \cong \text{Irr}_K(\mathcal{O}\tilde{A}_4)$  and the bijection  $\text{Irr}_k(k \otimes_{\mathcal{O}} A) \cong \text{Irr}_k(k\tilde{A}_4)$  induced by  $\alpha$ . Extend  $\Phi$  to a  $\mathbb{Z}$ -linear isomorphism  $\mathbb{Z}\text{Irr}_K(A) \cong \mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$ , still denoted by  $\Phi$ . By construction,  $\Phi$  sends the characters of the projective indecomposable  $A$ -modules to the characters of the projective indecomposable  $\mathcal{O}\tilde{A}_4$ -modules; in particular,  $\Phi$  maps  $\text{Proj}(A)$  to  $\text{Proj}(\mathcal{O}\tilde{A}_4)$ . It remains to see that the map sending  $e(\chi)$  to  $e(\Phi(\chi))$  for every  $\chi \in \text{Irr}_K(A)$  induces an isomorphism  $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$ . For any  $i$ ,  $0 \leq i \leq 6$ , denote by  $\chi_i$  the irreducible character of  $A$  such that  $\Phi(\chi_i) = \eta_i$ . As in the proof of 1.1, we have a distinguished norm 4 element in  $L^0(A)$  which is orthogonal to all other norm 4 elements in  $L^0(A)$ , namely  $\chi_0 + \chi_1 + \chi_2 - \chi_3$ . Thus, if  $\Psi : \mathbb{Z}\text{Irr}_K(A) \cong \mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  is some isometry mapping  $\text{Proj}(A)$  to  $\text{Proj}(\mathcal{O}\tilde{A}_4)$  and inducing an isomorphism  $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$ , then  $\Psi(\chi_0 + \chi_1 + \chi_2 - \chi_3) = \pm(\eta_0 + \eta_1 + \eta_2 - \eta_3)$ . By Proposition 1.1, there is a perfect self-isometry  $\mu$  of  $\mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  such that  $\Phi = \mu \circ \Psi$ . □

REMARK 2.2. If we assume that  $A$  is Morita equivalent to some block algebra with  $Q_8$  as defect group, then Proposition 2.1 follows also from the work of Cabanes and Picaronny in [4, 5].

Since  $k \otimes_{\mathcal{O}} A \cong k\tilde{A}_4$ , the quiver of  $A$  is the same as that of  $k\tilde{A}_4$ , thus of the following form:



Write  $\bar{a}$  for the image of  $a \in A$  in  $\bar{A} = k \otimes_{\mathcal{O}} A \cong k\tilde{A}_4$ . The generators  $\beta, \gamma, \delta, \kappa, \lambda$ ,  $\eta$  can be chosen such that their images in  $\bar{A}$  fulfill the following relations:

$$\begin{aligned}\bar{\beta}\bar{\delta} &= \bar{\kappa}\bar{\lambda}\bar{\kappa}, \\ \bar{\eta}\bar{\gamma} &= \bar{\lambda}\bar{\kappa}\bar{\lambda}, \\ \bar{\delta}\bar{\lambda} &= \bar{\gamma}\bar{\beta}\bar{\gamma}, \\ \bar{\kappa}\bar{\eta} &= \bar{\beta}\bar{\gamma}\bar{\beta}, \\ \bar{\lambda}\bar{\beta} &= \bar{\eta}\bar{\delta}\bar{\eta}, \\ \bar{\gamma}\bar{\kappa} &= \bar{\delta}\bar{\eta}\bar{\delta}\end{aligned}$$

and

$$\bar{\gamma}\bar{\beta}\bar{\delta} = \bar{\delta}\bar{\eta}\bar{\gamma} = \bar{\lambda}\bar{\kappa}\bar{\eta} = 0.$$

In order to determine the algebra structure of  $A$ , we have to “lift” these relations over  $\mathcal{O}$ .

We fix an algebra isomorphism  $\alpha : k \otimes_{\mathcal{O}} A \cong k\tilde{A}_4$  and an isometry  $\Phi : \mathbb{Z}\text{Irr}_K(A) \cong \mathbb{Z}\text{Irr}_K(\mathcal{O}\tilde{A}_4)$  satisfying the conclusions of Proposition 2.1. We denote by  $\chi_i$  the unique irreducible  $K$ -linear character of  $A$  such that  $\Phi(\chi_i) = \eta_i$  for all  $i$ ,  $0 \leq i \leq 6$ .

The characters  $\eta_0, \eta_1, \eta_2, \eta_3$  of  $\mathcal{O}\tilde{A}_4$  have height zero, the characters  $\eta_4, \eta_5, \eta_6$  have height one. Thus, via the isomorphism of the centers induced by  $\Phi$ , it follows that for  $0 \leq i \leq 3$  we have  $8e(\chi_i) \in A$ , and for  $4 \leq j \leq 6$  we have  $4e(\chi_j) \in A$ . We can in fact describe an  $\mathcal{O}$ -basis of  $Z(A)$  in terms of the centrally primitive idempotents  $e(\chi_i)$ . The strategy is now to play off the descriptions of  $Z(k \otimes_{\mathcal{O}} A)$  in terms of the generators in the quiver and of  $Z(A)$  in terms of the centrally primitive idempotents  $e(\chi_i)$ .

LEMMA 2.3. *The following elements of  $Z(k \otimes_{\mathcal{O}} A)$  are all contained in the radical  $J(Z(A))$ :*

$$\begin{aligned}s &= 2e(\chi_4) + 2e(\chi_5) + 2e(\chi_6), \\ z_0 &= 4e(\chi_2) + 4e(\chi_3) + 2e(\chi_4), \\ z_1 &= 4e(\chi_1) + 4e(\chi_3) + 2e(\chi_5), \\ z_2 &= 4e(\chi_0) + 4e(\chi_3) + 2e(\chi_6), \\ y_0 &= 4e(\chi_1) + 4e(\chi_2) + 2e(\chi_4) + 2e(\chi_5), \\ y_1 &= 4e(\chi_0) + 4e(\chi_2) + 2e(\chi_4) + 2e(\chi_6), \\ y_2 &= 4e(\chi_0) + 4e(\chi_1) + 2e(\chi_5) + 2e(\chi_6).\end{aligned}$$

Moreover, for any two different  $i, j$  in  $\{0, 1, 2\}$  the set

$$\{1, z_i, z_j, s, 8e(\chi_3), 4e(\chi_{i+4}), 4e(\chi_{j+4})\}$$

is an  $\mathcal{O}$ -basis of  $Z(A)$ .

*Proof.* In view of Proposition 2.1 we may assume that  $A = \mathcal{O}\tilde{A}_4$  and that  $\chi_i = \eta_i$  for  $0 \leq i \leq 6$ . This is just an explicit verification, using the character table of  $\tilde{A}_4$ . One verifies first that  $z_0 \in A$ . By symmetry, this implies that  $z_1, z_2$  are also in  $A$ . Then  $y_0 = z_0 + z_1 - 8e(\chi_3)$  is in  $A$ , similarly for the  $y_1, y_2$ . An equally easy computation shows that  $s \in A$ . Thus all the given elements belong to  $Z(A)$ . None of these elements is invertible, so they all belong to  $J(Z(A))$  because  $Z(A)$  is local.

In order to see the last statement on the basis of  $Z(A)$ , we may assume that  $i = 0$  and  $j = 1$ . For any  $x \in \tilde{A}_4$  denote by  $\underline{x}$  the conjugacy class sum of  $x$  in  $\mathcal{O}\tilde{A}_4$ . The orthogonality relations imply the well-known formula

$$\underline{x} = \sum_{0 \leq m \leq 6} \frac{\chi_m(x^{-1})}{\chi_m(1)} e(\chi_m).$$

Thus, for the seven conjugacy classes in  $\tilde{A}_4$ , we have

$$\begin{aligned} \underline{1} &= e(\chi_0) + e(\chi_1) + e(\chi_2) + e(\chi_3) + e(\chi_4) + e(\chi_5) + e(\chi_6); \\ \underline{z} &= e(\chi_0) + e(\chi_1) + e(\chi_2) + e(\chi_3) - e(\chi_4) - e(\chi_5) - e(\chi_6); \\ \underline{y} &= 6e(\chi_0) + 6e(\chi_1) + 6e(\chi_2) - 2e(\chi_3); \\ \underline{t} &= 4e(\chi_0) + 4\omega^2 e(\chi_1) + 4\omega e(\chi_2) - 2\omega e(\chi_4) - 2\omega^2 e(\chi_5) - 2e(\chi_6); \\ \underline{t^2} &= 4e(\chi_0) + 4\omega e(\chi_1) + 4\omega^2 e(\chi_2) - 2\omega^2 e(\chi_4) - 2\omega e(\chi_5) - 2e(\chi_6); \\ \underline{tz} &= 4e(\chi_0) + 4\omega^2 e(\chi_1) + 4\omega e(\chi_2) + 2\omega e(\chi_4) + 2\omega^2 e(\chi_5) + 2e(\chi_6); \\ \underline{t^2z} &= 4e(\chi_0) + 4\omega e(\chi_1) + 4\omega^2 e(\chi_2) + 2\omega^2 e(\chi_4) + 2\omega e(\chi_5) + 2e(\chi_6). \end{aligned}$$

We show that they are all in the  $\mathcal{O}$ -linear span of the elements in the set

$$\{1, z_0, z_1, s, 8e(\chi_3), 4e(\chi_4), 4e(\chi_5)\}.$$

Note first that

$$\begin{aligned} z_2 &= 4 \cdot 1 - z_0 - z_1 - s + 8e(\chi_3), \\ 4e(\chi_6) &= 2s - 4e(\chi_4) - 4e(\chi_5) \end{aligned}$$

are in the  $\mathcal{O}$ -linear span of this set. One easily verifies now that

$$\begin{aligned} \underline{z} &= 1 - s, \\ \underline{y} &= 6 \cdot 1 - 3s - 8e(\chi_3), \\ \underline{t} &= \omega z_0 + \omega^2 z_1 + z_2 - 4\omega e(\chi_4) - 4\omega^2 e(\chi_5) - 4e(\chi_6), \\ \underline{t^2} &= \omega^2 z_0 + \omega z_1 + z_2 - 4\omega^2 e(\chi_4) - 4\omega e(\chi_5) - 4e(\chi_6), \\ \underline{tz} &= \omega z_0 + \omega^2 z_1 + z_2, \\ \underline{t^2z} &= \omega^2 z_0 + \omega z_1 + z_2. \end{aligned}$$

This concludes the proof of 2.3. □

The center of  $\bar{A} = k \otimes_{\mathcal{O}} A$  can easily be described in terms of the generators in the quiver of  $A$ :

LEMMA 2.4. *The following set is a  $k$ -basis of  $Z(\bar{A})$ .*

$$\{1, \bar{\beta}\bar{\gamma} + \bar{\gamma}\bar{\beta}, \bar{\kappa}\bar{\lambda} + \bar{\lambda}\bar{\kappa}, \bar{\eta}\bar{\delta} + \bar{\delta}\bar{\eta}, \bar{\beta}\bar{\delta}\bar{\lambda}, \bar{\delta}\bar{\lambda}\bar{\beta}, \bar{\lambda}\bar{\beta}\bar{\delta}\}.$$

*Proof.* Straightforward verification, using  $(\bar{\beta}\bar{\gamma})^2 = \bar{\beta}\bar{\delta}\bar{\lambda}$  and the similar relations for the other elements in the given set. □

PROPOSITION 2.5. *For any primitive idempotent  $e$  in  $A$  we have  $Z(A)e = eAe$ . Moreover,*



- (i) the set  $\{e_0, z_0e_0, z_1e_0, 4e(\chi_4)e_0\}$  is an  $\mathcal{O}$ -basis of  $e_0Ae_0$ .
- (ii) the set  $\{e_1, z_0e_1, z_2e_1, 4e(\chi_4)e_1\}$  is an  $\mathcal{O}$ -basis of  $e_1Ae_1$ ;
- (iii) the set  $\{e_2, z_1e_2, z_2e_2, 4e(\chi_5)e_2\}$  is an  $\mathcal{O}$ -basis of  $e_2Ae_2$ .

*Proof.* Since  $Z(A) \cong Z(\mathcal{O}\tilde{A}_4)$  and  $Z(\bar{A}) \cong Z(k\tilde{A}_4)$ , the canonical map  $A \rightarrow \bar{A}$  maps  $Z(A)$  onto  $Z(\bar{A})$  and hence  $Z(A)e$  onto  $Z(\bar{A})\bar{e}$ . By Nakayama's Lemma, it suffices to show that  $Z(\bar{A})\bar{e} = \bar{e}\bar{A}\bar{e}$ . Now  $\dim_k(\bar{e}\bar{A}\bar{e}) = 4$  by the Cartan matrix, and so we have only to show that  $\dim_k(Z(\bar{A})\bar{e}) = 4$ . By the symmetry of the quiver of  $A$ , we may assume that  $e$  corresponds to the vertex labelled 0. Then the set  $\{\bar{e}, \bar{\beta}\bar{\gamma}, \bar{\kappa}\bar{\lambda}, \bar{\beta}\bar{\delta}\bar{\lambda}\}$  is a  $k$ -basis of  $Z(\bar{A})\bar{e}$  by 2.4; in particular,  $\dim_k(Z(\bar{A})\bar{e}) = 4$  as required. This shows that  $eAe = Z(A)e$ .

In order to prove (i), note that the set

$$\{e_0, z_0e_0z_1e_0, se_0, 8e(\chi_3)e_0, 4e(\chi_4)e_0, 4e(\chi_5)e_0\}$$

generates  $e_0Ae_0$  as  $\mathcal{O}$ -module, by the first statement and by the  $\mathcal{O}$ -basis of  $Z(A)$  described in 2.3. Now we have

$$\begin{aligned} 8e(\chi_3)e_0 &= 2z_0e_0 - 4e(\chi_4)e_0, \\ 4e(\chi_5)e_0 &= 2z_0e_0 - 2z_1e_0 + 4e(\chi_4)e_0, \\ se_0 &= (z_1 - z_0 + 4e(\chi_4))e_0. \end{aligned}$$

Thus the set given in (i) generates  $e_0Ae_0$  as  $\mathcal{O}$ -module, and hence is a basis since the  $\mathcal{O}$ -rank of  $e_0Ae_0$  is 4. The same arguments show (ii) and (iii). □

PROPOSITION 2.6. *We can choose the generators  $\beta, \gamma, \delta, \eta, \lambda, \kappa$  in such a way that*

- (i)  $A\gamma$  is the unique  $\mathcal{O}$ -pure submodule of  $Ae_0$  with character  $\chi_3 + \chi_4$ ;
- (ii)  $A\lambda$  is the unique  $\mathcal{O}$ -pure submodule of  $Ae_0$  with character  $\chi_3 + \chi_5$ ;
- (iii)  $A\eta$  is the unique  $\mathcal{O}$ -pure submodule of  $Ae_1$  with character  $\chi_3 + \chi_6$ ;
- (iv)  $A\beta$  is the unique  $\mathcal{O}$ -pure submodule of  $Ae_1$  with character  $\chi_3 + \chi_4$ ;
- (v)  $A\kappa$  is the unique  $\mathcal{O}$ -pure submodule of  $Ae_2$  with character  $\chi_3 + \chi_5$ ;
- (vi)  $A\delta$  is the unique  $\mathcal{O}$ -pure submodule of  $Ae_2$  with character  $\chi_3 + \chi_6$ .

*Proof.* We are going to prove (i); by the symmetry of the quiver of  $A$  one gets all other statements. Observe first that  $\bar{A}\bar{\gamma}$  is the unique 5-dimensional submodule of  $Ae_0$  with composition factors  $2[S_0], 2[S_1], [S_2]$ . Indeed, the set  $\{\bar{\gamma}, \bar{\beta}\bar{\gamma}, \bar{\eta}\bar{\gamma}, \bar{\gamma}\bar{\beta}\bar{\gamma}, \bar{\beta}\bar{\gamma}\bar{\beta}\bar{\gamma}\}$  is a  $k$ -basis of  $\bar{A}\bar{\gamma}$ , and we have  $\bar{\gamma}, \bar{\gamma}\bar{\beta}\bar{\gamma} \in \bar{e}_0\bar{A}\bar{e}_0$ , yielding the two composition factors isomorphic to  $S_0$ , we have  $\bar{\beta}\bar{\gamma}, \bar{\beta}\bar{\gamma}\bar{\beta}\bar{\gamma} \in \bar{e}_1\bar{A}\bar{e}_0$ , yielding the two composition factors isomorphic to  $S_1$ , and finally  $\bar{\eta}\bar{\gamma} \in \bar{e}_2\bar{A}\bar{e}_0$ , yielding the remaining composition factor isomorphic to  $S_2$ . One checks that there is no other submodule with exactly these composition factors. Now there is exactly one  $\mathcal{O}$ -pure submodule  $U$  of  $Ae_0$  whose reduction modulo  $J(\mathcal{O})$  has composition series  $2[S_0] + 2[S_1] + [S_2]$ , namely the unique  $\mathcal{O}$ -pure submodule of  $Ae_0$  with character  $\chi_3 + \chi_4$ ; this is a direct consequence of the decomposition matrix. One constructs  $U$  as follows: write  $K \otimes_{\mathcal{O}} Ae_0 = X_0 \oplus X_3 \oplus X_4 \oplus X_5$ , where  $X_j$  is the unique submodule of  $K \otimes_{\mathcal{O}} Ae_0$  with character  $\chi_j$  for  $j \in \{0, 3, 4, 5\}$ , and then  $U = Ae_0 \cap (X_3 \oplus X_4)$ . Take now for  $\gamma$  any inverse image in  $U$  of  $\bar{\gamma}$ . Then  $A\gamma \subseteq U$  and  $U \subseteq A\gamma + J(\mathcal{O})U$ . Thus  $A\gamma = U$  by Nakayama's Lemma. □

COROLLARY 2.7. *If the generators  $\beta, \gamma, \delta, \eta, \lambda, \kappa$  are chosen such that they fulfill the conclusions of 2.6 then, with the notation of 2.3, the following hold.*

- (i)  $y_0\delta = y_0\eta = 0$ .

- (ii)  $y_1\lambda = y_1\kappa = 0$ .
- (iii)  $y_2\gamma = y_2\beta = 0$ .

PROPOSITION 2.8. *We can choose the generators  $\beta, \gamma, \delta, \eta, \lambda, \kappa$  such that the following hold:*

$$\begin{aligned} \beta\gamma &= z_0e_0 = 4e(\chi_3)e_0 + 2e(\chi_4)e_0; \\ \gamma\beta &= z_0e_1 = 4e(\chi_3)e_1 + 2e(\chi_4)e_1; \\ \delta\eta &= z_2e_1 = 4e(\chi_3)e_1 + 2e(\chi_6)e_1; \\ \eta\delta &= z_2e_2 = 4e(\chi_3)e_2 + 2e(\chi_6)e_2; \\ \lambda\kappa &= z_1e_2 = 4e(\chi_3)e_2 + 2e(\chi_5)e_2; \\ \kappa\lambda &= z_1e_0 = 4e(\chi_3)e_0 + 2e(\chi_5)e_0; \\ \beta\delta\lambda &= \kappa\eta\gamma = 8e(\chi_3)e_0; \\ \delta\lambda\beta &= \gamma\kappa\eta = 8e(\chi_3)e_1; \\ \lambda\beta\delta &= \eta\gamma\kappa = 8e(\chi_3)e_2. \end{aligned}$$

*Proof.* In view of the decomposition matrix of  $A$  we have  $e_0 = e(\chi_0)e_0 + e(\chi_3)e_0 + e(\chi_4)e_0 + e(\chi_5)e_0$ . Moreover, the elements  $e(\chi_0)e_0, e(\chi_3)e_0, e(\chi_4)e_0, e(\chi_5)e_0$  are  $K$ -linearly independent because they are pairwise orthogonal idempotents in  $K \otimes_{\mathcal{O}} A$ . Similar statements hold for  $e_1, e_2$ .

We assume a choice of generators fulfilling 2.6. We have  $A\beta\gamma \subseteq A\gamma$ , and the submodule  $A\gamma$  of  $Ae_0$  has character  $\chi_3 + \chi_4$  by 2.6. Thus  $\beta\gamma$  is a  $K$ -linear combination of  $e(\chi_3)e_1$  and  $e(\chi_4)e_1$ . But also  $\beta\gamma$  is an  $\mathcal{O}$ -linear combination of the basis elements  $e_0, z_0e_0, z_1e_0, 4e(\chi_4)e_0$  given in 2.5 in which none of  $\chi_1, \chi_5$  shows up. Therefore  $\beta\gamma$  is in fact an  $\mathcal{O}$ -linear combination of the elements  $z_0e_0, 4e(\chi_4)e_0$ ; say

$$\beta\gamma = (\mu_0z_0e_0 + 4\nu_0e(\chi_4))e_0 = (4\mu_0e(\chi_3) + 2(\mu_0 + 2\nu_0)e(\chi_4))e_0$$

for some coefficients  $\mu_0, \nu_0 \in \mathcal{O}$ . Hence

$$(\beta\gamma)^2 = (16\mu_0^2e(\chi_3) + 4(\mu_0 + 2\nu_0)^2e(\chi_4))e_0.$$

Now  $(\bar{\beta}\bar{\gamma})^2 \neq 0$ , and therefore  $\mu_0 \in \mathcal{O}^\times$ . Set now

$$a_0 = 1 + \nu_0\mu_0^{-1}y_0.$$

Since  $y_0 \in J(Z(A))$  by 2.3 we have  $a_0 \in Z(A)^\times$ . A trivial verification, comparing coefficients, shows that we have

$$\beta\gamma = \mu_0z_0a_0e_0.$$

Since  $\gamma = e_1\gamma = \gamma e_0$ , multiplying this with  $\gamma$  on the left yields

$$\gamma\beta\gamma = \mu_0z_0a_0e_1\gamma.$$

Now both  $\gamma\beta$  and  $\mu_0z_0a_0e_1$  are contained in the pure submodule  $A\beta$  of  $Ae_1$  with character  $\chi_3 + \chi_4$ , by 2.6 and the nature of the element  $z_0$ . Right multiplication by  $\gamma$  on this submodule is therefore injective (the annihilator of  $\gamma$  in  $Ae_1$  is the pure submodule

with character  $\chi_1 + \chi_6$ ). Hence the previous equality implies also the equality

$$\gamma\beta = \mu_0 z_0 a_0 e_1.$$

In an entirely analogous way one finds scalars  $\mu_1, \mu_2 \in \mathcal{O}^\times$  such that, setting  $a_1 = 1 + v_1 \mu_1^{-1} y_1$  and  $a_2 = 1 + v_2 \mu_2^{-1} y_2$ , one gets the equalities

$$\begin{aligned} \delta\eta &= \mu_2 z_2 a_2 e_1, & \eta\delta &= \mu_2 z_2 a_2 e_2, \\ \lambda\kappa &= \mu_1 z_1 a_1 e_2, & \kappa\lambda &= \mu_1 z_1 a_1 e_0. \end{aligned}$$

Moreover, the equalities in 2.7 imply the following equalities:

$$\begin{aligned} a_0\delta &= \delta, & a_0\eta &= \eta, \\ a_1\lambda &= \lambda, & a_1\kappa &= \kappa, \\ a_2\gamma &= \gamma, & a_2\beta &= \beta. \end{aligned}$$

If we replace now  $\beta$  by  $a_0^{-1}\beta$ , this is not going to change the properties stated in 2.6 and also this is not changing the relations over  $k$  of the quiver. Similarly, we can replace  $\delta$  by  $a_2^{-1}\delta$  and  $\lambda$  by  $a_1^{-1}\lambda$ . Then the generators  $\beta, \gamma, \delta, \eta, \lambda, \kappa$  still fulfill 2.6, and in addition, we have now the following equalities:

$$\begin{aligned} \beta\gamma &= \mu_0 z_0 e_0, & \gamma\beta &= \mu_0 z_0 e_1, \\ \delta\eta &= \mu_2 z_2 e_1, & \eta\delta &= \mu_2 z_2 e_2, \\ \lambda\kappa &= \mu_1 z_1 e_2, & \kappa\lambda &= \mu_1 z_1 e_0. \end{aligned}$$

We have to get rid of the scalars  $\mu_0, \mu_1, \mu_2$ . Since  $\chi_3$  is the only character appearing in the characters of all projective indecomposable  $A$ -modules we have

$$\beta\delta\lambda = 8\mu e(\chi_3)e_0$$

for some  $\mu \in \mathcal{O}$ . Then actually  $\mu \in \mathcal{O}^\times$  because  $\bar{\beta}\bar{\delta}\bar{\lambda} \neq 0$ . Moreover,  $\beta\delta\lambda\beta = 8\mu e(\chi_3)\beta$ , and hence also

$$\delta\lambda\beta = 8\mu e(\chi_3)e_1.$$

The same argument applied again yields

$$\lambda\beta\delta = 8\mu e(\chi_3)e_2.$$

Applying this argument to the arrows in the quiver in the opposite direction implies that there is  $\mu' \in \mathcal{O}^\times$  such that

$$\begin{aligned} \kappa\eta\gamma &= 8\mu' e(\chi_3)e_0, \\ \eta\gamma\kappa &= 8\mu' e(\chi_3)e_2, \\ \gamma\kappa\eta &= 8\mu' e(\chi_3)e_1. \end{aligned}$$

Now  $\bar{\beta}\bar{\delta}\bar{\lambda} = \bar{\kappa}\bar{\lambda}\bar{\kappa} = \bar{\kappa}\bar{\eta}\bar{\gamma}$ , and hence  $\mu' = \mu(1 + v)$  for some  $v \in J(\mathcal{O})$ . Note that we can always multiply any of the generators by any scalar in  $1 + J(\mathcal{O})$  without modifying the relations over  $k$ . Thus, if we replace  $\kappa$  by  $(1 + v)\kappa$ , we may assume that  $\mu' = \mu$ .

Since the set  $\{\kappa, \kappa\lambda\kappa\}$  is an  $\mathcal{O}$ -basis of  $e_0Ae_2$ , we can write

$$\beta\delta = a\kappa + b\kappa\lambda\kappa$$

for some unique scalars  $a, b \in \mathcal{O}$ . Multiplying this by  $\lambda$  yields

$$8\mu e(\chi_3)e_0 = \beta\delta\lambda = a\kappa\lambda + b(\kappa\lambda)^2 = (a\mu_1z_1 + b\mu_1^2z_1^2)e_0.$$

By comparing the coefficients at  $e(\chi_3)e_0$  and  $e(\chi_5)e_0$  of the left and right expression in this equality, we get the equations

$$\begin{aligned} 8\mu &= 4a\mu_1 + 16b\mu_1^2, \\ 0 &= 2a\mu_1 + 4b\mu_1^2. \end{aligned}$$

An easy computation shows that  $b = \frac{\mu}{\mu_1^2}$ . Moreover, since  $\bar{\beta}\bar{\delta}\bar{\lambda} = (\bar{\kappa}\bar{\lambda})^2$  we have  $\bar{a} = 0$  and  $\bar{b} = 1_k$ , hence  $b = \frac{\mu}{\mu_1^2} \in 1 + J(\mathcal{O})$ . By repeating the same argument we find also that the coefficients  $\frac{\mu}{\mu_0^2}, \frac{\mu}{\mu_2^2}$  are in  $1 + J(\mathcal{O})$ .

Next, we compute  $\beta\delta\lambda\kappa\eta\gamma$  in two different ways: on one hand we have

$$(\beta\delta\lambda)(\kappa\eta\gamma) = 64\mu^2e(\chi_3)e_0,$$

and on the other hand we have

$$\beta(\delta(\lambda\kappa)\eta)\gamma = \mu_0\mu_1\mu_2z_0z_1z_2e(\chi_3)e_0 = 64\mu_0\mu_1\mu_2e(\chi_3)e_0.$$

Together we get

$$\mu^2 = \mu_0\mu_1\mu_2.$$

Thus  $\frac{\mu}{\mu_0^2} \frac{\mu}{\mu_1^2} = \frac{\mu_2}{\mu_0\mu_1} \in 1 + J(\mathcal{O})$ . Similarly,  $\frac{\mu_1}{\mu_0\mu_2}, \frac{\mu_0}{\mu_1\mu_2} \in 1 + J(\mathcal{O})$ . But then also  $\frac{\mu_1\mu_2}{\mu_0} \frac{\mu_1}{\mu_0\mu_2} = \frac{\mu_1^2}{\mu_0^2} \in 1 + J(\mathcal{O})$ . Since  $2 \in J(\mathcal{O})$  this implies that  $\frac{\mu_1}{\mu_0} \in 1 + J(\mathcal{O})$ . But then actually  $\mu_2 = \frac{\mu_1\mu_2}{\mu_0} \frac{\mu_0}{\mu_1} \in 1 + J(\mathcal{O})$ . Similarly,  $\mu_0, \mu_1 \in 1 + J(\mathcal{O})$ . So we can replace  $\beta$  by  $\mu_0^{-1}\beta$ , or equivalently, we can assume that  $\mu_0 = 1$ . Similarly, we can assume that  $\mu_1 = \mu_2 = 1$ . Then  $\mu^2 = 1$ . If  $\mu = -1$  we multiply all generators by  $-1$ ; since  $2 \in J(\mathcal{O})$ , this does not change the relations over  $k$ , but it does change the sign of any of the above expressions  $\beta\delta\lambda$  etc. involving three generators. Therefore, we can also assume that  $\mu = 1$ . □

We can now prove Theorem C from the introduction.

*Proof of Theorem C.* We assume a choice of generators of  $A$  fulfilling Proposition 2.8. We show that  $A$  satisfies the relations given in Theorem C. Those in the first three lines are obvious. Since the set  $\{\kappa, \kappa\lambda\kappa\}$  is an  $\mathcal{O}$ -basis of  $e_0Ae_2$ , we can write

$$\beta\delta = a\kappa + b\kappa\lambda\kappa$$

for some unique scalars  $a, b \in \mathcal{O}$ . Multiplying this by  $\lambda$  yields

$$8e(\chi_3)e_0 = \beta\delta\lambda = a\kappa\lambda + b(\kappa\lambda)^2 = (4a + 16b)e(\chi_3)e_0 + (2a + 4b)e(\chi_5)e_0.$$

By comparing the coefficients at  $e(\chi_3)e_0$  and  $e(\chi_5)e_0$  of the left and right expression in this equality, we get the equations

$$\begin{aligned}8 &= 4a + 16b, \\ 0 &= 2a + 4b.\end{aligned}$$

Thus the coefficients  $a, b$  have values

$$a = -2, \quad b = 1,$$

and from this we get the following relation in the statement of Theorem C:

$$\beta\delta = -2\kappa + \kappa\lambda\kappa.$$

In exactly the same way we get the following five relations in the Theorem:

$$\begin{aligned}\eta\gamma &= -2\lambda + \lambda\kappa\lambda, \\ \delta\lambda &= -2\gamma + \gamma\beta\gamma, \\ \kappa\eta &= -2\beta + \beta\gamma\beta, \\ \lambda\beta &= -2\eta + \eta\delta\eta, \\ \gamma\kappa &= -2\delta + \delta\eta\delta.\end{aligned}$$

A similar technique is going to yield the remaining relations: write  $\gamma\beta\delta = c\delta + d\delta\eta\delta$  for some unique  $c, d \in \mathcal{O}$ ; as before, this is possible since  $\{\delta, \delta\eta\delta\}$  is an  $\mathcal{O}$ -basis of  $e_1Ae_2$ . Multiplying by  $\eta$  yields

$$\gamma\beta\delta\eta = c\delta\eta + d(\delta\eta)^2 = cz_2e_1 + dz_2^2e_1.$$

The left side is equal to  $(\gamma\beta)(\delta\eta) = z_0z_2e_1$ , so comparing coefficients yields now

$$\begin{aligned}16 &= 4c + 16d, \\ 0 &= 2c + 4d,\end{aligned}$$

and this implies  $c = -4$  and  $d = 2$ . Thus we get indeed

$$\gamma\beta\delta = -4\delta + 2\delta\eta\delta$$

as claimed. The remaining relations of this type follow in exactly the same way.

Now consider the last three relations. Write  $\beta\delta\lambda\beta = r\beta + s\beta\gamma\beta$ , for  $r, s \in \mathcal{O}$ . Then  $\beta\delta\lambda\beta\gamma = r\beta\gamma + s\beta\gamma\beta\gamma$ . So

$$32e(\chi_3)e_0 = (4r + 16s)e(\chi_3)e_0 + (2r + 4s)e(\chi_4)e_0$$

which yields  $s = 4$  and  $r = -8$ . The remaining two relations follow in exactly the same way. Thus  $A$  satisfies all relations given in Theorem C.

Let  $\tilde{A}$  be the  $\mathcal{O}$ -algebra described by the generators and relations given in Theorem C. There is a surjective algebra morphism from  $\tilde{A}$  to  $A$ . In order to show that  $\tilde{A}$  and  $A$  are isomorphic it suffices therefore to show that the cardinality of a minimal generating set for  $\tilde{A}$  as an  $\mathcal{O}$ -module is at most 24. Thus it suffices to check that the set

$$\begin{aligned}\mathcal{S} := \{ &e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa, \beta\gamma, \gamma\beta, \delta\eta, \eta\delta, \lambda\kappa, \kappa\lambda, \\ &\beta\gamma\beta, \gamma\beta\gamma, \delta\eta\delta, \eta\delta\eta, \lambda\kappa\lambda, \kappa\lambda\kappa, \beta\delta\lambda, \delta\lambda\beta, \lambda\beta\delta\}\end{aligned}$$

spans  $\tilde{A}$  as  $\mathcal{O}$ -module. This is an easy consequence of the given relations; we give some details for the convenience of the reader. Let

$$\mathcal{G} = \{e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa\}.$$

From the given relations it is immediate that for any two elements  $x, y$  of  $\mathcal{G}$ ,  $xy$  is in the  $\mathcal{O}$ -span of  $\mathcal{S}$ . Thus it suffices to show that for any two elements  $x, y$  of  $\mathcal{G} - \{e_0, e_1, e_2\}$  and any element  $u$  of  $\mathcal{S} - \{e_0, e_1, e_2, \beta, \gamma, \delta, \eta, \lambda, \kappa\}$ ,  $xu$  and  $uy$  are in the  $\mathcal{O}$ -span of  $\mathcal{S}$ . From the given relations we may also assume that  $u$  is one of  $\beta\gamma\beta, \gamma\beta\gamma, \delta\eta\delta, \eta\delta\eta, \lambda\kappa\lambda, \kappa\lambda\kappa$  or one of  $\beta\delta\lambda, \delta\lambda\beta, \lambda\beta\delta$ .

First, note that the relations  $\kappa\eta = -2\beta + \beta\gamma\beta$  and  $\delta\lambda = -2\gamma + \gamma\beta\gamma$  give that  $\kappa\eta\gamma = \beta\delta\lambda$ . Similarly, we get  $\eta\gamma\kappa = \lambda\beta\delta$  and  $\gamma\kappa\eta = \delta\lambda\beta$ .

Now suppose  $u = \beta\gamma\beta$ . Then we may assume that  $x$  is one of  $\gamma$  or  $\lambda$  and that  $y$  is one of  $\gamma$  or  $\delta$ . The relation  $\kappa\eta = -2\beta + \beta\gamma\beta$  gives  $\gamma\kappa\eta = -2\gamma\beta + \gamma\beta\gamma\beta$ , hence  $\gamma\beta\gamma\beta$  is in the  $\mathcal{O}$ -span of  $\mathcal{S}$ . The relation  $\kappa\eta = -2\beta + \beta\gamma\beta$  also gives  $\lambda\kappa\eta = -2\lambda\beta + \lambda\beta\gamma\beta$ . It follows from the relation  $\lambda\kappa\eta = -4\eta + 2\eta\delta\eta$  that  $\lambda\beta\gamma\beta$  is in the  $\mathcal{O}$ -span of  $\mathcal{S}$ . We show similarly that  $\beta\gamma\beta\gamma$  and  $\beta\gamma\beta\delta$  are in the  $\mathcal{O}$ -span of  $\mathcal{S}$ .

The cases  $u = \gamma\beta\gamma, \delta\eta\delta, \eta\delta\eta, \lambda\kappa\lambda, \kappa\lambda\kappa$  are handled analogously.

Now suppose  $u = \beta\delta\lambda$ . Then we may assume that  $x$  is one of  $\lambda$  or  $\gamma$  and  $y$  is one of  $\beta$  or  $\kappa$ . The relation  $\lambda\beta\delta\lambda = -8\lambda + 4\lambda\kappa\lambda$  shows that  $\lambda\beta\delta\lambda$  is in the  $\mathcal{O}$ -span of  $\mathcal{S}$ . From the relation  $\gamma\beta\delta = -4\delta + 2\delta\eta\delta$  we get  $\gamma\beta\delta\lambda = -4\delta\lambda + 2\delta\eta\delta\lambda$ . From  $\gamma\kappa = -2\delta + \delta\eta\delta$ , we get  $\delta\eta\delta\lambda = \gamma\kappa\lambda + 2\delta\lambda$ . Hence  $\delta\eta\delta\lambda$  is in the  $\mathcal{O}$ -span of  $\mathcal{S}$ , and so is  $\gamma\beta\delta\lambda$ . We argue similarly to show that  $\beta\delta\lambda\beta$  and  $\beta\delta\lambda\kappa$  are in the  $\mathcal{O}$ -span of  $\mathcal{S}$ .

The cases  $u = \delta\lambda\beta$  and  $u = \lambda\beta\delta$  are handled in the same fashion.  $\square$

**REMARK 2.9.** An interesting consequence of 2.5 is the structure of  $eAe$  for any primitive idempotent  $e$  in  $A$ . We have an  $\mathcal{O}$ -algebra isomorphism

$$eAe \cong \mathcal{O}[X, Y]/\langle X^2 - Y^2 - 2(X - Y), XY - 2X^2 + 4X \rangle;$$

indeed, we may assume that  $e = e_0$ , and then the assignment  $X \mapsto z_0e_0, Y \mapsto z_1e_0$  induces the required isomorphism. In particular, we have an isomorphism of  $k$ -algebras

$$\bar{e}A\bar{e} \cong k[X, Y]/\langle X^2 - Y^2, XY \rangle.$$

This is, by Erdmann [6, III.1, III.3], up to isomorphism the unique 4-dimensional symmetric  $k$ -algebra which is not isomorphic to the group algebra of the Klein four group. One might be tempted to ask whether any symmetric  $\mathcal{O}$ -algebra is the endomorphism algebra of some projective module of some block algebra.

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