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NECESSARY AND SUFFICIENT CONDITIONS FOR HYPOELLIPTICITY FOR A CLASS OF CONVOLUTION OPERATORS

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ABSTRACT. In this paper the Corwin's conjecture is proved, which says that if d is a function analytic near ∞, then the hypoellipticity of the convolution operator A_d, defined by $\widehat{A_d u} = d\hat{u}$ for every $u \in S'(\mathbb{R}^n)$, implies that $P(x)/\log x \longrightarrow \infty$ as $x \longrightarrow \infty$, where $P(x)$ is the distance from $x \in \mathbb{R}^n$ to the set of complex zeros of d.

0. **Introduction.** The main purpose of this paper is to solve the two open questions announced by L. Corwin in [1] which are concerned with the hypoellipticity for certain left invariant differential operators on (H)-groups and for a class of convolution operators.

Let us recall some notions and facts related to the open questions before giving the description of them.

The 2-step nilpotent Lie group G with Lie algebra g is said to be an *(H)-group* if for every nonzero element ℓ of g_2, g_2 being the center of g, there is (up to equivalence) a unique irreducible representation π_ℓ such that for all $Z \in g_2$, $\pi_\ell(Z) = 2\pi i \ell(Z)\ell$. We say that G is a *good* (H)-group if g has a sub-algebra h which is polarizing for all $f \in g^*$ with $f(g_2) \neq 0$.

Let G be an (H)-group and L be a left invariant differential operator on G such that L_m, the part of highest homogeneous degree, is elliptic in the generating directions on G. Theorem 1.1 in [1] says that if G is a good (H)-group, then the hypoellipticity for $L \Leftrightarrow$ the condition (a):

(a) $\lim_{\lambda_0 \to \infty} P(\lambda_0)^{-1} \log|\lambda_0| = 0$, where $P(\lambda_0)$ is the distance from $\lambda_0 \in g_2^*$ to the nearest point $\ell \in (g_2)_C^*$ such that $\pi_\ell(L^*L)$ has non-trivial kernel. It is also shown in the same theorem that the implication $\Leftarrow$ holds for all Lie groups of type (H).

The open question 2 in [1] is then the following:

Is the hypothesis of goodness necessary in Theorem 1.1?

Then another open question is closely related to the above one. Denote by $d(\ell)$ the product of small eigenvalues of $\rho^{-m}\pi_\ell(L^*L)$ with large $\ell \in g_2^*$, where (ρ, θ) is the spherical coordinate of ℓ. As pointed out in [1], d is analytic near ∞, that is, $d(1/t, \theta)$ can be extended to a function holomorphic in a complex neighborhood of $\{0\} \times S^{n-1}$, where $S^{n-1} \subset \mathbb{R}^n$ is the unit sphere and n is the dimension of g_2^*. Let A_d be the map $E' \longrightarrow S'$

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such that $A_d(u)^\wedge = du^\wedge$ for all $u \in E'$ where $^\wedge$ denotes Fourier transform. E' is the space of distributions with compact supports.

Theorem B in [1] states that the micro-locally hypoellipticity for the operator L^*L (or L) is equivalent to that of the operator A_d. Note that the function $p(\lambda_0)$ in condition (a) is just the distance from $\lambda_0 \in \mathbb{R}^n$ to the closed set $\{\lambda : \lambda \in \mathbb{C}^n, d(\lambda) = 0\}$. Thus the author of [1] propounded a more general question (the open question 5 in [1]).

If d is a function which is analytic at ∞ but is not necessarily related to a differential operator L, is condition (a) equivalent to the hypoellipticity of A_d? A positive answer would, of course, give an answer to question 2 as well.

In the present paper we will give a positive answer to the open question (5) and hence a negative one to the question (2).

Our approach is as follows:

In order to apply the Hörmander's method (see [2], pp. 354–355), we concretely construct a distribution space ϕ' satisfying the conditions:

(i) $S' \subset \phi' \subset D'$ with weakly continuous embedding and the domain of A_d can be continuously extended from E' (or S') to ϕ'.

(ii) For each $m \in I_+$, ϕ' contains all the distributions with the forms $\sum_1^\infty a_j \exp(iz_jx)$ where $a_j \in C$ such that $\sum_1^\infty |a_j| < +\infty$; $z_j \in \mathbb{C}^n$ satisfying $|\operatorname{Im} z_j| \le m \log|\operatorname{Re} z_j|$ $|z_j| \ge R$ for given $R > 0, j = 1, 2, \ldots$.

We say that the operator A_α is *ϕ'-hypoelliptic* if the implication $u \in \phi' A_\alpha(u) \in C^\infty(\omega) \Rightarrow u \in C^\infty(\omega)$ holds for every open set $\omega \subset \mathbb{R}^n$.

By using the Hörmander's argument cited above, we can easily prove that ϕ'-hypoellipticity for A_d implies condition (a). The next step is then to prove that ϕ'-hypoellipticity for A_d is equivalent to that in the original sense in [1]. To do so we have to make a careful study on the functions analytic near ∞. As a result we obtain Proposition 1.5, which enables us to show the equivalence, and whose proof is based on several lemmas which seem interesting in their own right.

1. **Asymptotic analysis on functions analytic near ∞.**

Denote by (ρ, θ) the spherical coordinate of $x \in \mathbb{R}^n$ and by S^{n-1} the unit sphere in $\mathbb{R}^n$. We recall (see [1]) that the function f defined for all (ρ, θ) with large ρ is analytic near ∞ if $f(1/s, \theta)$ (where $s = 1/\rho$) extends to a function holomorphic in a complex neighborhood of $\{0\} \times S^{n-1}$.

By the same reason stated at page 5 in [1], we shall add the conditions that $f \in C^\infty(\mathbb{R}^n)$ to the above definition in what follows.

LEMMA 1.1. *Suppose that f is analytic near ∞. Then:*

(i) There are constants $k > 0$ and $R > 0$ such that f extends to an analytic and bounded function on the region $G_{k,R}$ where

$$G_{k,R} = \{z : z = x + iy \in \mathbb{C}^n, |y| \le k|x|, |x| \ge R\}. \tag{1-1}$$

(ii) For every $\alpha \in I_+^n$, there is a constant $C_\alpha > 0$ such that

$$|\partial^\alpha f(x)| \le C_\alpha (1 + |x|)^{-|\alpha|} \quad \textit{for all } x \in \mathbb{R}^n. \tag{1-2}$$

PROOF. By the definition of f we see that there are constants $\delta > 0$ and $r > 0$ such that f is analytic in the region:

$$V = \{z : z \in \mathbb{C}^n, D(z/|z|, S^{n-1}) < \delta, |z| > r\}$$

where $D(z/|z|, S^{n-1})$ denotes the distance from $z/|z|$ to S^{n-1}.

By simple calculation it follows that $D(z/|z|, S^{n-1}) = 2\big(1 - |x|/(|x|^2+|y|^2)^{1/2}\big)$ which implies $G_{k,R} \subset V$ provided $0 < k \leq \frac{1}{4}\big(1 - (1 - \delta/4)^2\big)^{1/2}(1 - \delta/4)^{-1}$ and $R > r$ so f is analytic on $G_{k,R}$ as well.

To prove the boundedness of f, we assume that f is unbounded on $G_{k,R}$. Then there is a sequence $(z_j)_{j=1}^{\infty}$, with $z_j \in G_{k,R}$, such that $|z_j| \longrightarrow \infty, f(z_j) \longrightarrow \infty$ and $z_j/|z_j| \longrightarrow z^0$ $(j \longrightarrow \infty)$ where z^0 satisfies $D(z^0, S^{n-1}) \leq \delta/2$. But on the other hand, we have for j large enough, $f(z_j) = \sum_0^\infty a_\ell(z_j/|z_j|)\,|z_j|^{-\ell}$, where all a_ℓ are analytic functions in U_δ, the neighborhood of $S^{n-1} : U_\delta = \{z : z \in C, D(z, S^{n-1}) < \delta\}$. Hence we have $\mathrm{Lim}_{j\to\infty} f(z_j) = a_0(z^0)$ which is a contradiction to the assumption $\mathrm{Lim}_{j\to\infty} f(z_j) = \infty$, so $f(z)$ is bounded on $G_{k,R}$.

We now turn to assertion (ii).

Denote by $\rho(x)$ the distance from $x \in \mathbb{R}^n$ to the boundary of $G_{k,R}$. It is easy to check that $\rho(x) \geq k(1 + k^2)^{-1/2}|x|$ provided $|x| \geq R' = \big(k + (1 + k^2)^{1/2}\big)(1 + k^2)R$. By using the Cauchy inequality for analytic functions, we obtain that if $|x| \geq R'$ then $|\partial^\alpha f(x)| \leq M\alpha!\big(\rho(x)\big)^{-|\alpha|} \leq M\alpha!\big((1 + k^2)^{1/2}k^{-1}\big)^{|\alpha|}|x|^{-|\alpha|}$. Therefore we see that (1-2) holds.

LEMMA 1.2. *Suppose that f is analytic near ∞, and that $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \equiv 1$ in some neighborhood of origin. Let $\varphi_1 = 1 - \varphi$. Then $\varphi_1 f^\wedge \in S$, S being the Schwartz Space of rapidly decreasing functions.*

PROOF. Denote by Δ the Laplace operator on $\mathbb{R}^n$. It follows from (1-2) that $\Delta^p f \in L_1(\mathbb{R}^n)$, if the integer $p > n/2$. Hence $(\Delta^p f)^\wedge \in L_\infty(\mathbb{R}^n)$, which yields that

$$\varphi_1 f^\wedge(\xi) = \varphi_1(\xi)|\xi|^{-2p} \int \exp(-i\xi x)(-\Delta)^p f(x)\, dx. \tag{1-3}$$

For $\alpha \in I_+^n$, $\beta \in I_+^n$, by using (1-2) and and (1-3) with $P > (n + |\alpha| + |\beta|)/2$, we get $|\xi^\alpha \partial^\beta(\varphi_1 f^\wedge)(\xi)| \leq C_{\alpha,\beta}$ for some constant $C_{\alpha,\beta} > 0$, which means that $\varphi_1 f^\wedge \in S$.

LEMMA 1.3. *Let $\Omega_{m,R} = \{z : z = x+iy \in C, |y| \leq m\log(|x|/R), |x| \geq R\}$ for $m \geq 0$, $R > 0$. Let $\alpha = \big(1 + \exp(m/R)\big)^{-1}$. If $f(z)$ is analytic and bounded on $\Omega_{2m,R}$, then*

$$\sup_{\Omega_{m,R}} |f(z)| \leq \max\Big\{\sup_{\Omega_{0,R}} |f(z)|, \big(\sup_{\Omega_{0,R}} |f(z)|\big)^\alpha \big(\sup_{\Omega_{2m,R}} |f(z)|\big)^{1-\alpha}\Big\}. \tag{1-4}$$

PROOF. Let $\Omega_{m,R}^+ = \Omega_{m,R} \bigcap \{(x,y) : (x,y) \in \mathbb{R}^2, x \geq 0, y \geq 0\}$, that is,

$$\Omega_{m,R}^+ = \{z : z = x + iy \in C, 0 \leq y \leq m\log(x/R), x \geq R\}.$$

Set $F_\varepsilon(z) = \log|f(z)\cdot\exp(-\varepsilon z)| = \log|f(z)| - \varepsilon x$ for $\varepsilon > 0$.

It is easy to see that $F_\varepsilon(x+iy)$ is subharmonic in $\Omega_{2m,R}$ (see [3]). Note that $\lim_{x\to+\infty} F_\varepsilon(x+iy) = -\infty$. By the maximum principle for subharmonic functions (see Section 12 of [4]) we see that the function $M_\varepsilon(\rho) = \mathrm{Sup}_{x\geq R} F_\varepsilon\big(x, \rho\log(x/R)\big)$ can not have a local maximum in $(0, 2m)$. Hence it follows that there is a constant $\rho_\varepsilon \in [0, 2m]$ such that $M_\varepsilon(\rho)$ is decreasing on $[0, \rho_\varepsilon]$, while increasing on $[\rho_\varepsilon, 2m]$, based on which we can prove that

$$M_\varepsilon(m) \leq \max\{M_\varepsilon(0), \alpha M_\varepsilon(0) + (1-\alpha)M_\varepsilon(2m)\}. \tag{1-5}$$

In fact, it is obvious that (1-5) holds as $m \leq \rho_\varepsilon \leq 2m$, since $M_\varepsilon(m) \leq M_\varepsilon(0)$. We now consider the case that $0 \leq \rho_\varepsilon \leq m$. Let $\phi(x,y) = y\big(\log(x/R)\big)^{-1}$ and

$$\begin{aligned}
&\psi_\varepsilon(x,y)\\
&= M_\varepsilon(2m)\Big(\exp\big(-\varphi(x,y)/R\big) - \exp(-\rho_\varepsilon/R)\Big)\big(\exp(-2m/R) - \exp(-\rho_\varepsilon/R)\big)^{-1}\\
&\quad + M_\varepsilon(\rho_\varepsilon)\Big(\exp(-2m/R) - \exp\big(-\varphi(x,y)/R\big)\Big)\big(\exp(-2m/R) - \exp(-\rho_\varepsilon/R)\big)^{-1}
\end{aligned}$$

By calculation, noting that $M_\varepsilon(\rho)$ is increasing on $[\rho_\varepsilon, 2m]$, we see that $\Delta\psi_\varepsilon \leq 0$ provided $\rho_\varepsilon < \varphi(x,y) < 2m$ and $x > R$. Therefore $F_\varepsilon - \psi_\varepsilon$ is subharmonic in the region $U = \{(x,y) \in \mathbb{R}^2 : \rho_\varepsilon < \varphi(x,y) < 2m\}$.

Note that $\psi_\varepsilon(x,y) = M_\varepsilon(\rho_\varepsilon)$ and $F_\varepsilon(x,y) \leq M_\varepsilon(\rho_\varepsilon)$ if $y = \rho_\varepsilon\log(x/R)$ and that $\psi_\varepsilon(x,y) = M_\varepsilon(2m)$ and $F_\varepsilon(x,y) \leq M_\varepsilon(2m)$ if $y = 2m\log(x/R)$. We get that $F_\varepsilon - \phi_\varepsilon \leq 0$ on the boundary of U. So, by the classical Phragmen-Lindelöf principle we see that $F_\varepsilon - \psi_\varepsilon \leq 0$ in U. Thus we obtain that $M_\varepsilon(m) \leq \alpha_\varepsilon M_\varepsilon(\rho_\varepsilon) + (1-\alpha_\varepsilon)M_\varepsilon(2m)$ where $\alpha_\varepsilon = \big(\exp(-2m/R) - \exp(-m/R)\big)\,\big(\exp(-2m/R) - \exp\big(-\rho_\varepsilon(R)\big)$. Since $0 \leq \rho_\varepsilon < m$, $\alpha_\varepsilon \geq \big(1 + \exp(m/R)\big)\exp(-m/R)^{-1} = \alpha$. We then have $M_\varepsilon(m) \leq \alpha M_\varepsilon(\rho_\varepsilon) + (1 - \alpha_\varepsilon)M_\varepsilon(2m) \leq \alpha M_\varepsilon(\rho_\varepsilon) + (1-\alpha)M_\varepsilon(2m) \leq \alpha M_\varepsilon(0) + (1-\alpha)M_\varepsilon(2m)$. Letting $\varepsilon \to 0$ in the above inequalities we obtain (1-5). From the definition of F_ε and (1-5), it follows that

$$\operatorname*{Sup}_{\Omega^+_{m,R}} |f(z)| \leq \max\Big\{\operatorname*{Sup}_{\Omega^+_{m,R}} |f(z)|, \big(\operatorname*{Sup}_{\Omega^+_{m,R}} |f(z)|\big)^\alpha \big(\operatorname*{Sup}_{\Omega^+_{m,R}} |f(z)|\big)^{1-\alpha}\Big\}. \tag{1-4'}$$

In the same way, we can establish estimations with the same form as (1-4′) for the other three subregions of $\Omega_{m,R}$: $\Omega_{m,R}\bigcap\{(x,y) \in \mathbb{R}^2, x > 0, y \leq 0\}$; $\Omega_{m,R}\bigcap\{(x,y) \in \mathbb{R}^2, x < 0, y \geq 0\}$; and $\Omega_{m,R}\bigcap\{(x,y) \in \mathbb{R}^2, x < 0, y \leq 0\}$. Putting everything together, we have obtained (1-4).

LEMMA 1.4. *Suppose that $f(z)$ is analytic on $G_{2m,R}$ and f satisfies $|f(z)| \leq C_1(1+|x|)^h$ for $z = x+iy \in \Omega_{2m,R}$ and $|f(z)| \leq C_0(1+|x|)^\ell$ for $x \in R$, with $|x| > R$, where $\ell \leq h$.*

Then there is a constant $C > 0$ such that

$$|f(z)| \leq C(1+(x))^{(\ell+h)/2} \quad \text{for } z = x+iy \in \Omega_{m,R}. \tag{1-6}$$

PROOF. Let $g(z) = (1+z)^{-\ell}\exp\big(iz(h-\ell)/(2m)\big)f(z)$ for $z \in \Omega^+_{2m,R}$. It is easy to see that $g(z)$ is bounded on $\partial\Omega^+_{2m,R}$. Note that $\Omega^+_{2m,R} \subset \{(x,y) \in \mathbb{R}^2, 0 \leq y \leq$

$2m(x-R)/R, x \geq 0\}$ which is a sector in $\mathbb{R}^2$. Hence by Phragmen-Lindelöf Principle (see [4], in particular, the Remarks (ii), p. 96) we have $|g(z)| \leq C'$ on $\Omega^+_{2m,R}$, for another constant C'. Therefore we get $|f(z)| \leq C'(1+x)^{\ell} \exp\big((h-\ell)/(2m)\big)$ on $\Omega^+_{2m,R}$, from which we obtain $|f(z)| \leq C(1+x)^{(h+\ell)/2}$ on $\Omega^+_{m,R}$ since $0 \leq y \leq m\log(x/R)$.

Analogously we can establish the estimate (1-6) for other three subdomains of $\Omega_{m,R}$ stated in the proof of Lemma 1.3. Thus we have shown that (1-6) holds on $\Omega_{m,R}$.

PROPOSITION 1.5. *Let* $\Omega_{m,R} = \{z : z = x+iy \in \mathbb{C}^n, |y| \leq m\log(|x|/R), |x| \geq R\}$. *Suppose that* $f(z)$ *is analytic on* $\Omega_{4m\sqrt{n},R}$ *and satisfies the following conditions:*

$$|f(z)| \leq C\prod_1^n (1+|x_j|)^{h_j} \text{ on } \Omega_{4m\sqrt{n},R} \tag{1-7}$$

$$|f(z)| \leq C\prod_1^n (1+|x_j|)^{\ell_j} \text{ on } \Omega_{0,R} \text{ where } \ell_j \leq h_j. \tag{1-8}$$

Then

$$|f(z)| \leq C'\prod_i^n (1+|x_j|)^{\alpha^n\ell_j+(1-\alpha^n)h_j} \text{ on } \Omega_{m,R} \tag{1-9}$$

where $\alpha = \big(1+\exp(2m/r)\big)^{-1}$.

PROOF. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. It follows from the compactness of S^{n-1} that there are a positive integer $N(n)$ and points $\theta_j \in S^{n-1}$, $1 \leq j \leq N$, such that

$$S^{n-1} = \bigcup_1^N S_j^{n-1} \text{ where } S_j^{n-1} = \{\theta : \theta \in S^{n-1}, |\theta-\theta_j| \leq 1/(2\sqrt{n})\}.$$

Let $\Omega^{(j)}_{m,R} = \{z : z = x+iy \in \Omega_{m,R}, x/|x| \in S_j^{n-1}\}$, $1 \leq j \leq N$.

We see that if the estimate (1-9) holds on $\Omega^{(j)}_{m,R}$ for each j, then it holds on $\Omega_{m,R}$ as well. Moreover, for every j, there is a real orthogonal transformation A_j such that:

$$A_j\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad A_j\theta_j = \theta^* \text{ where } \theta^* = (1,1,\ldots,1)/\sqrt{n} \in S^{n-1}.$$

We then define T_j as follows:

$$T_j\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad T_j(z) = A_jx + iA_jy \quad \text{for all } z = x+iy \in \mathbb{C}^n.$$

It is clear that $\Omega_{m,R}$ is invariant under transformation T_j, $1 \leq j \leq N$. Hence, from the C-R conditions for analytic functions, it follows that function $F_j(z) = f(T_j^{-1}z)$ is analytic on $\Omega_{4m\sqrt{n},R}$ and satisfies (1-7), (1-8). Let

$$\Omega^*_{m,R} = \Big\{z : z = x+iy \in \Omega_{m,R}, \big|x/|x| - \theta^*\big| \leq 1/(2\sqrt{n})\Big\}. \tag{1-10}$$

It is easy to see that $\Omega^*_{m,R}$ is the image of $\Omega^*_{m,R}$ under the transformation T_j. Therefore we see that if $F_j(z)$ satisfies (1-9) on $\Omega_{m,R}$ then $f(z)$ satisfies it on $\Omega_{m,R}$ and vice versa.

Taking all things together, we have shown that the only thing needed to do is to establish (1-9) on $\Omega^*_{m,R}$ for f.

Let

$$\tilde{\Omega}_{m,R} = \{z : z = x + iy \in \mathbb{C}^n, |y_h| \le \log(|x_h|/R), |x_h| \ge R, 1 \le k \le n\} \tag{1-11}$$

in which $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

From definition it follows that

$$\Omega^*_{m,2R\sqrt{n}} \subset \tilde{\Omega}_{m,R} \subset \tilde{\Omega}_{2m,R} \subset \Omega_{2m\sqrt{n},R}. \tag{1-12}$$

Let $F_1(z) = (1+z)^{-h_1} f(z, x_2, \ldots, x_n)$ in which $(x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$ is fixed and $|x_h| \ge R$, $2 \le h \le n$.

Note that $(z, x_2, \ldots, x_n) \in \Omega_{4m\sqrt{n},R}$ if $z = x + iy$, $|y| \le 4m \log(|x|/R)$ and $|x| \ge R$. We then get from (1-7), (1-8) that

$$|F_1(z)| \le C \prod_2^n (1 + |x_j|)^{h_j}$$

where $z = x + iy \in \mathbb{C}$, $|y| \le 4m \log(|x|/R)$, $|x| \ge R$ and that $|F_1(x)| \le C \prod_2^n (1 + |x|)^{\ell_j}$, $x \in \mathbb{R}$.

Applying Lemma 1.3 to the function F_1, we then obtain

$$\operatorname*{Sup}_{\substack{|y| \le 2m \log(|x|/R) \\ |x| \ge R}} \{|F_1(z)|\} \le C \prod_2^n (1 + |x_j|)^{\alpha \ell_j + (1-\alpha) h_j}.$$

Hence, we have

$$|f(z_1, x_2, \ldots, x_n)| \le C(1 + |x_1|)^{h_1} \prod_2^n (1 + |x_j|)^{\alpha \ell_j + (1-\alpha) h_j}, \tag{1-13}$$

where $z_1 = x_1 + iy_1 \in \mathbb{C}$, $|y_1| \le 2m \log(|x_1|/R)$, $|x_j| \ge R$, $1 \le j \le n$.

By applying Lemma 1.4 to the function $(1 + z_1)^{h_1} F_1(z)$, noting (1-8) and (1-13), we get

$$|f(z_1, x_2, \ldots, x_n)| \le C_1 (1 + |x_1|)^{(\ell_1 + h_1)/2} \prod_2^n (1 + |x_j|)^{\alpha \ell_j + (1-\alpha) h_j},$$

where $|y_1| \le n \log(|x_1|/R)$, $|x_1| \ge R$, $1 \le j \le n$.

Note that $\alpha = \big(1 + \exp(2m/R)\big)^{-1}$ and $\ell_1 \le h_1$; hence $(h_1 + \ell_1)/2 \le \alpha \ell_1 + (1-\alpha) h_1$. We then obtain

$$|f(z, x_2, \ldots, x_n)| \le C_1 \prod_1^n (1 + |x_1|)^{\alpha \ell_j + (1-\alpha) h_j}. \tag{1-13'}$$

where $z_1 = x_1 + iy_1$, $|y_j| \le m \log(|x_1|/R)$.

We now consider the function $F_2(z) = (1 + z_2)^{-h_2} f(z_1, z, x_3, \ldots, x_n)$, where $z_1 \in \mathbb{C}$, $|y_1| \le m \log(|x_j|/R)$, $|x_j| \ge R$, $j = 1, 3, \ldots, n$.

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By using the same argument as above from (1-8) and (1-13′) we get

$$|f(z_1,z_2,x_3,\ldots,x_n)| \le C_2 \prod_1^n (1+|x_j|)^{\alpha^2\ell_j+(1-\alpha^2)h_j}$$

where $|y_j| \le m\log(|x_j|/R), j = 1,2, |x_k| \ge R, 1 \le k \le n$.

Continuing to do so in this way, we then finally obtain

$$|f(z_1,z_2,\ldots,z_n)| \le C_n \prod_1^n (1+|x_j|)^{\alpha^n\ell_j+(1-\alpha^n)h_j}, \quad Z = (z_1,z_2,\ldots,z_n) \in \tilde{\Omega}_{m,R}$$

which implies that (1-9) holds on $\Omega^*_{m,R}$ since $\tilde{\Omega}_{m,2R\sqrt{n}} \subset \tilde{\Omega}_{m,R}$ and f is analytic on $\Omega^*_{m,R}$. Therefore we have proved the proposition.

PROPOSITION 1.6. *Suppose that $d \in C^\infty(\mathbb{R}^n)$ which extends to an analytic and bounded function on the region $G_{k,R}$ where*

$$G_{k,R} = \{z : z = x+iy \in \mathbb{C}^n, |y| \le k|x|, |x| \ge R\}. \tag{1-14}$$

Let φ_1 be the function given in Lemma 1.2, $f = (\varphi, d^\vee)^\wedge$ where $\vee$ denotes the inverse Fourier transform. Then

(1) $f \in S(\mathbb{R}^n)$ and f can be extended to an analytic function on $G_{k,R}$.

(2) For every integer $m \ge 0$ and $N \ge 0$, there is a constant $C_{m,N} > 0$ such that

$$|f(z)| \le C_{m,N}(1+|x|)^{-N} \quad \textit{for } z \in G_{k,R} \cap \Omega_{m,R}. \tag{1-15}$$

PROOF. It follows from Lemma 1.2 that $f \in S$. Note that $f = d - (\varphi d^\vee)^\wedge$ where $\varphi = 1 - \varphi_1 \in C_0^\infty$. We then get from well-known P-W-S theorem that $(\varphi d^\vee)^\wedge$ is an entire function on $\mathbb{C}^n$. Hence f is analytic on $G_{k,R}$.

Take $R' > 0$ large enough such that $\Omega_{2(m+1)\sqrt{n},R'} \subset G_{k,R}$. From the P-W-S theorem and the boundedness of d, it follows that there are constants $h(m)$ and $C_m > 0$ such that

$$|f(z)| \le C_m \prod_1^n (1+|x_j|)^h \quad \text{for } z \in \Omega_{2(m+1)\sqrt{n},R'}. \tag{1-16}$$

For given N and h, we take ℓ such that $\alpha^n\ell + (1-\alpha^n)h \le -N$ where $\alpha = \Big(1+\exp\big(2(m+1)/R'\big)\Big)^{-1}$. Because $f \in S$, there is a constant $C'_{m,N} > 0$ such that

$$|f(x)| \le C'_{m,N} \prod_1^n (1+|x_j|)^\ell \quad \text{for } x \in \Omega_{0,R'}. \tag{1-17}$$

Applying Proposition 1.6 and noting (1-6), (1-7), we have

$$|f(x)| \le C'_{m,N}(1+|x|)^{-N} \quad \text{for } z \in \Omega_{m+1,R'}. \tag{1-18}$$

Note that if $|x| \ge R'' = (R')^{m+1}R^{-m}$ then $m\log(|x|/R) \le (m+1)\log(|x|/R')$. We then see that $|f(z)| \le C'_{m,N}(1+|x|)^{-N}$ if $z \in \Omega_{m,R}$ and $|x| \ge R''$, which implies that (1-15) holds with some constant $C_{m,N} \ge C'_{m,n}$.

2. **Distribution space ϕ' and ϕ'-hypoellipticity.** Suppose that d is analytic near ∞, which is analytic and bounded on $G_{k,R}$ (see Lemma 1.1).

Let

$$\Gamma_m = G_{k,R} \cap \Omega_{m,R} \text{ for fixed } R > 0, \quad m = 1, 2, \ldots . \tag{2-1}$$

Let Ψ be the Fréchet space defined as follows:

A function $\psi \in \Psi$, if $\psi \in S$ and ψ can be extended to a function analytic and bounded on Γ_m for every positive integer m. The topology of Ψ is given by the sequence of norms:

$$|\psi|_m = \operatorname*{Sup}_{\substack{|\alpha+\beta|\leq m \\ x\in\mathbb{R}^n}} \{|x^\alpha \partial^\beta \psi(x)|\} + \operatorname*{Sup}_{\Gamma_m} |\psi(z)|, \quad m = 1, 2, \ldots . \tag{2-2}$$

Define the space ϕ as follows: $\Phi = \{\varphi : \varphi \in S, \hat{\varphi}' \in \Psi\}$ in which the topology is given by $\|\varphi\|_m = |\varphi^\wedge|_m$ where $|\ |_m$ is defined as in (2-2), $m = 1, 2, \ldots$.

LEMMA 2.1. *For every compact set $K \subset \mathbb{R}^n$, $C_0^\infty(k) \subset \Phi$ and the embedding is continuous.*

PROOF. Because Fourier transform is a continuous map from C_0^∞ to S, it follows that for every $\ell \in N$, there are constants $p_1(\ell) \in N$ and $C_1(\ell) > 0$ such that

$$\operatorname*{Sup}_{\substack{|\alpha+\beta|\leq \ell \\ x\in\mathbb{R}^n}} |x^\alpha \partial^\beta \varphi(x)| \leq C_1(\ell) \operatorname*{Sup}_{\substack{|\alpha|\leq p_1 \\ x\in\mathbb{R}^n}} |\partial^\alpha \varphi(x)| \quad \text{for all } \varphi \in C_0^\infty(k). \tag{2-3}$$

Take $K \subset \{x : x \in \mathbb{R}^n, |x| \leq b\}$ and $p_2 \in N$ such that $p_2 \geq b\ell/2$; we then have

$$\begin{aligned}\operatorname*{Sup}_{\Gamma_\ell} |\varphi^\wedge(z)| &= \operatorname*{Sup}_{\Gamma_\ell} |(z_1^2 + \cdots + z_n^2)^{p_2} \int_K e^{-iz\xi} (-\Delta)^{p_2} \varphi(\xi)\, d\xi \\ &\leq C_1 \operatorname*{Sup}_{\substack{\mathbb{R}^n \\ |A|\leq P_2}} |\partial^\alpha \varphi(\xi)| \operatorname*{Sup}_{|x|\geq R} \{(1+|x|)^{-2p_2} (|x|/R)^{b\ell}\}.\end{aligned}$$

Hence, we have

$$\operatorname*{Sup}_{\Gamma_\ell} |\varphi^\wedge(z)| \leq C_2(\ell) \operatorname*{Sup}_{\substack{x\in\mathbb{R}^n \\ |\alpha|\leq p_2}} |\partial^\alpha \varphi(x)|. \tag{2-4}$$

From (2-3) and (2-4), we get

$$\|\varphi\|_\ell \leq C(\ell) \operatorname*{Sup}_{\substack{x\in\mathbb{R}^n \\ |\alpha|\leq p(\ell)}} |\partial^\alpha \varphi(x)| \quad \text{for all } \varphi \in C_0^\infty(K),$$

which shows that the lemma is true.

It is not hard to see that

$$C_0^\infty \subset \Phi \subset S \tag{2-5}$$

and each imbedding of them is continuous. Hence it follows that

$$S' \subset \Phi' \subset D', \tag{2-6}$$

with weakly continuous imbeddings, where Φ' is the dual space of Φ.

We define the Fourier transform on Φ': $\Phi' \longrightarrow \Psi'$ as follows: for $u \in \Phi'$,

$$(u^\wedge, \psi) = (u, \psi^\wedge) \quad \text{for all } \psi \in \Psi. \tag{2-7}$$

By definition, it is clear that the Fourier transform is topologically isomorphic from Φ to Ψ, and hence from Φ' to Ψ'.

We say that a function f defined on R^n is a *multiplicator of space* Ψ, if for every $\psi \in \Psi, f\psi \in \Psi$ and the map $M_f: \psi \longrightarrow f\psi$ is continuous on Ψ.

Now, let f be a multiplicator of Ψ. We can then define the operator $A_f: \Phi' \longrightarrow \Phi'$ as follows:

$$A_f(u)^\wedge = f \cdot u^\wedge \quad \text{for every } u \in \Phi'. \tag{2-8}$$

It is obvious that A_f is continuous from Φ to Φ and from Φ' to Φ'.

We say that the operator A_f is Φ'-*hypoelliptic* if for every open set $\omega \subset \mathbb{R}^n$, the following implication holds:

$$u \in \Phi',\ A_f(u) \in C^\infty(\omega) \Rightarrow u \in C^\infty(\omega) \tag{2-9}$$

where $u \in C^\infty(\omega)$, u being an element of Φ', means that for every $\alpha \in I_+^n$, there is a function $u_\alpha \in C(\omega)$ such that $\big(u, (-\partial)^\alpha\varphi\big) = (u_\alpha, \varphi)$ for all $\varphi \in C_0^\infty(\omega)$. Let E' be the space of distributions with compact support.

We say A_f is E'-*hypoelliptic* if for every open set $\omega \subset \mathbb{R}^n$, the following implication holds:

$$u \in E',\ A_f(u) \in C^\infty(\omega) \Rightarrow u \in C^\infty(\omega). \tag{2-10}$$

The definition of S'-hypoellipticity for A_f is analogous.

It is easy to see that if $d \in C^\infty(\mathbb{R}^n)$ and d extends to a bounded analytic function on $G_{k,R}$, then d is a multiplicator of Ψ', so the operator A_d is well defined.

PROPOSITION 2.2. *Suppose that d is analytic near ∞ and is bounded and analytic on $G_{k,R}$. Then A_d is E'-hypoelliptic if and only if it is Φ'-hypoelliptic.*

PROOF. If A_d is Φ'-hypoelliptic, then from (2-6) and the fact that $E' \subset S'$, it follows that A_d is E'-hypoelliptic.

We now assume that A_d is E'-hypoelliptic. Let $u \in \Phi'$ which satisfies the condition $A_d(u) \in C^\infty(\omega)$; we want to prove that $u \in C^\infty(\omega)$. Without loss of generality, we assume that ω is a bounded open set.

Let φ and φ_1 be the functions given in Lemma 1.1. Let $f_0 = (\varphi d^\vee)^\wedge, f_1 = (\varphi_1 d^\vee)^\wedge$. Take h such that $h \in C_0^\infty$ and $h \equiv 1$ in ω. We then have

$$A_d(hu) = A_d(u) - A_{f_0}(u_1) - A_{f_1}(u_1), \quad \text{where } u_1 = (1-h)u. \tag{2-11}$$

It is clear that $A_{f_0}(u_1) = (\phi d^{\vee}) * u_1$. By Lemma 1.1 it follows that $\varphi d^{\wedge} \in C^{\infty}(\mathbb{R}^n \setminus \{0\}) \cap E'(\mathbb{R}^n)$. From Theorem 7.1 of [5] we know that the operator A_{f_0} is pseudolocal. Hence $A_{f_0}(u_1) \in C^{\infty}(\omega)$ because $u_1 = (1-h)u \equiv 0$ in ω.

We next prove $A_{f_1}(u_1) \in C^{\infty}(\omega)$.

We take a constant a such that $\omega \subset \omega_a = \{x : x \in \mathbb{R}^n, |x| \leq a\}$; let $b > a$. Denote by $F_\alpha(x)$ the function $(-ix)^\alpha f_1(x)$ for $\alpha \in I_+^n$. We then have for each $v \in C_0^\infty(\omega_b)$

$$\left(A_{f_1}(u_1), (-\partial)^\alpha v\right) = \left(f_1 u_1^\wedge [(-\partial)^\alpha v]^\wedge\right) = (u_1^\wedge, F_\alpha v^\wedge). \tag{2-12}$$

Since $u_1^\wedge \in \Psi'$, by definition it follows that there are constants $\ell \in N$ and $C > 0$, which are independent of α and v, such that

$$|(u_1^\wedge, F_\alpha v^\wedge)| \leq C|F_\alpha v^\wedge|_\ell, \tag{2-13}$$

where $|\ |_\ell$ is defined in (2-2).

Note that $F_\alpha(x) = (-ix)^\alpha(\varphi_1 d^\vee)^\wedge$. It follows from Proposition 1.6 that $F_\alpha \in S$. Hence there is $C_\alpha > 0$ such that

$$|F_\alpha v^\wedge|_\ell \leq C_\alpha |v^\wedge|_\ell = C_\alpha \|v\|_\ell. \tag{2-14}$$

By Lemma 2.1, it follows that there are $\ell' \in N$ and $C' > 0$, which are independent upon α and v, such that

$$\|v\|_\ell \leq C' \sup_{\substack{x\in\mathbb{R}^n \\ |\beta|\leq \ell}}, |\partial^\beta v(x)|. \tag{2-15}$$

Therefore, putting (2-12)–(2-15) together, we get

$$|(A_{f_1}(u_1), (-\partial^\alpha)v| \leq C'_\alpha \sup_{\substack{\mathbb{R}^n \\ |\beta|\leq \ell'}} |\partial^\beta v(x)|, \quad \text{for all } v \in C_0^\infty(\omega_b)$$

which means that for each $\alpha \in I_+^n$

$$\partial^\alpha[A_{f_1}(u_1)] \in [C_0^{\ell'}(\omega_r)]' \text{ with } \ell' \text{ independent of } \alpha. \tag{2-16}$$

Take $p \in N$ such that $2p \geq 2n + \ell + \ell'$. Let E_p be a fundamental solution of the operator Δ^p. It is a well-known result that $E_p \in C^{\ell'}(\mathbb{R}^n)$.

Let $H_\alpha = \partial^\alpha\left(A_{f_1}(u_1)\right)$. Then $H_\alpha \in [C_0^{\ell'}(\omega_p)]'$ because of (2-16).

Take h_0 such that $h_0 \in C_0^\infty$ and $h_0 \equiv 1$ in some neighborhood of the origin.

Let $g = \Delta^p(h_0 E_p) - \delta$, δ being the Dirac δ-function. It is not hard to see that $g \in C_0^\infty$. Moreover, we have

$$H_\alpha = H_\alpha * \delta = H_\alpha * \left(\Delta^p(h_0 E_p)\right) - H_\alpha * g,$$

that is,

$$H_\alpha = (\Delta^p H_\alpha) * (h_0 E_p) - H_\alpha * g. \tag{2-17}$$

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By (2-16) it follows that H_α and $\Delta^p H_\alpha$ belong to $\left(C_0^{\ell'}(\omega_r)\right)'$. From (2-17) and the fact $E_p \in C^{\ell'}(\mathbb{R}^n)$ we see that $H_\alpha \in C(\omega_a) \subset C(\omega)$ provided the support of h_0 is taken to be small enough. Hence we have $\partial^\alpha\left(A_{f_1}(u_1)\right) \in C(\omega)$ for all $\alpha \in I_+^n$ which means $A_{f_1} \in C^\infty(\omega)$.

Let us return to (2-11). Note that $A_d(u) \in C^\infty(\omega)$ by our assumption. We have then shown that $A_d(hu) \in C^\infty(\omega)$. Since $hu \in E'$, it follows from the definition of E'-hypoellipticity that $hu \in C^\infty(\omega)$, and hence $u \in C^\infty(\omega)$.

Putting all things together, we have proved that the E'-hypoellipticity for A_d implies the Φ'-hypoellipticity for A_d, which completes the proof of the proposition.

REMARK. It follows from Proposition 2.2 that the Φ'-hypoellipticity for A_d does not depend upon the constants k and R appearing in the definition of the space Φ, since the E'-hypoellipticity does not depend upon them.

PROPOSITION 2.3. *Suppose that d is analytic near ∞ and that the operator A_d is Φ'-hypoelliptic. Then*

(2-18) $$|\operatorname{Im} z| / \log |z| \longrightarrow \infty \text{ if } z \longrightarrow \infty \text{ on the surface } d(z) = 0.$$

PROOF. Suppose that there is a positive integer m_0 and a sequence of points $\{z_j\}_1^\infty$, with $d(z_j) = 0$, such that

(2-19) $$|z_j| \longrightarrow \infty, \quad (j \longrightarrow \infty)$$

and

(2-20) $$|y_j| \le (m_0/2) \log |z_j|, \quad \text{where } \operatorname{Re} z_j = x_j,\ \operatorname{Im} z_j = y_j.$$

By (2-19) and (2-20), we see that there is a subsequence of $\{z_j\}_1^\infty$, denoted by $\{z_j'\}_1^\infty$, such that $z_j'/|z_j'| \longrightarrow z^*$, $(j \longrightarrow \infty)$, where $|z^*| = 1$ and $|\operatorname{Re} z^*| \ne 0$. So there is a constant $k' > 0$ such that $|y_j'| \le k'|x_j'|$, from (2-20) and which we see that $|y_1'| \le m_0 \log |x_j'|/R$ for j large enough.

Therefore we can assume that $\{z_j\}_1^\infty \subset \Gamma_{m_0} = G_{k,R} \cap \Omega_{m_0,R}$ where $G_{k,R}$ is given by Lemma 1.1.

Let $\ell_1 = \left\{\{a_j\}_1^\infty, a_j \in \mathbb{C}, \Sigma_1^\infty |a_j| < \infty\right\}$. Denote by a the element $\{a_j\}_1^\infty$ of ℓ_1. Let $|a| = \Sigma_1^\infty |a_j|$.

For any given $a \in \ell_1$, let $u(t) = \Sigma_1^\infty a_j \exp(-ixz_j)$ which is defined as an element of Φ' in the following fashion:

(2-20′) $$(u, \varphi) = \sum_1^\infty a_j\left(\exp(-ixz_j), \varphi(x)\right) = \sum_1^\infty a_j \varphi^\wedge(z_j) \quad \text{for all } \varphi \in \Phi.$$

From the fact that $\{z_j\}_1^\infty \subset \Gamma_0$, we see that

(2-21) $$|(u, \varphi)| \le |a|\, \|\varphi\|_{m_0}, \quad \text{for } \varphi \in \Phi$$

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which means that $u \in \Phi'$.

Moreover we have $A_d(u) = 0$. In fact, we have that for $\varphi \in \Phi$,

$$\big(A_d(u), \varphi\big) = (du^\wedge, \varphi^\wedge) = \big(u, (d\varphi^\wedge)^\vee\big) = \sum_1^\infty a_j d(z_j)\varphi(z_j) = 0.$$

Therefore we get from Φ'-hypoellipticity for A_d that $u \in C^\infty(\mathbb{R}^n)$. Thus we obtain a map $T\colon \ell_1 \longrightarrow C^\infty(\mathbb{R}^n)$ as follows: for each $a \in \ell_1$, $T(a) = \sum_1^\infty a_j \exp(-ixz_j) = u$. By (2-21) it is not hard to show that T is a closed and therefore a continuous map $\ell_1 \longrightarrow C^\infty(\mathbb{R}^n)$. Hence $|\partial_k u(0)| \le C|a|$ for all $a \in \ell_1$, $k = 1, 2, \ldots, n$, that is,

$$\Big|\sum_1^\infty a_j z_j^{(k)}\Big| \le C|a|, \tag{2-22}$$

where $z_j^{(k)}$ is the k-th component of z_j.

From (2-22) we see that $\{z_j\}_1^\infty \in \ell_1^* = \ell_\infty$, and hence $|z_j| \le C'$, $j = 1, 2, \ldots$, which is contrary to the hypothesis $z_j \longrightarrow \infty$. The contradiction proves the proposition.

3. **Main theorem.**

THEOREM 3.1. *Suppose that d is a function analytic near ∞. Then the following statements are equivalent:*

(1) the operator A_d is E'-hypoelliptic,

(2) A_d is Φ'-hypoelliptic,

(3) A_d is S'-hypoelliptic,

(4) $\operatorname{Lim}_{|x|\to\infty}\big(P(x)/\log|x|\big) = \infty$ where $P(x)$ is distance from $x \in \mathbb{R}^n$ to the set of complex zeros of d,

(5) $|\operatorname{Im} z|/\log|z| \longrightarrow \infty$ if $z \longrightarrow \infty$ on the surface $d(z) = 0$.

PROOF. From Proposition 2.2 and formula (2-6) it follows that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Proposition 2.3 states that (2) $\Rightarrow$ (5). Moreover from [1] (see pp. 1–15 of [1]) we know that (4) $\Rightarrow$ (1). Therefore the only thing we need to do is to show (5) $\Rightarrow$ (4). To do so, we assume that there are $x_j \in \mathbb{R}^n$, $(j = 1, 2, \ldots)$ such that $P(x_j) \le C\log|x_j|$ $(j = 1, 2, \ldots)$ and $|x_j| \longrightarrow \infty$ $(j \longrightarrow \infty)$. By the definition of $P(x_j)$ there are $w_j = u_j + iv_j$ satisfying $d(w_j) = 0$ and $P(x_j) = (|x_j - u_j|^2 + v_j^2)^{1/2}$ $(j = 1, 2, \ldots)$. Hence we have $|u_j| \ge |x_j| - |x_j - u_j| \ge |x_j| - P(x_j) \ge |x_j| - C\log|x_j| \ge |x_j|^{1/2}$ for j large enough. Therefore we have $|v_j|/\log|w_j| \le |v_j|/\log|u_j| \le 2|v_j|/\log|x_j| \le 2C|v_j|/p(x_j) \le 2C$, that is, $|\operatorname{Im} w_j|/\log|w_j| < 2C$ for large j, which is a contradiction to statement (5). Hence it follows that (5) $\Rightarrow$ (4).

From this theorem and Theorem 1.1 of [1], we easily get the following consequence:

CONSEQUENCE 3.1. Let G be a (H)-group with Lie algebra g. Suppose that L is a left invariant differential operator on G such that the term of highest homogeneous degree is elliptic in the generating directions on G. Denote by g_2 the center of g. Fix a Euclidean norm $|\cdot|$ on $(g_2)^*_C$. Then the following statements are equivalent:

(a) The distance $P(\lambda_0)$ from $\lambda_0 \in g_2^*$ to the nearest point $\ell \in (g_2)_C^*$ such that $\pi_\ell(L^*L)$ has no trivial kernel satisfies

$$P(\lambda_0)^{-1} \log|\lambda_0| \longrightarrow 0 \quad \text{as } \lambda_0 \longrightarrow \infty.$$

(b) L is microlocally hypoelliptic.

(c) L^*L is microlocally hypoelliptic.

(d) L^*L is hypoelliptic.

It is clear that Theorem 3.1 gives the positive answer to the open question (5) in [1], while Consequence 3.2 gives the negative one to the question (2) in [1].

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