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ON THE BOUNDEDNESS OF MULTIPLICATIVE AND POSITIVE FUNCTIONALS

Dedicated to Professor George Szekeres on his 65th birthday

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Abstract

Let A be a complex sequentially complete, locally convex (not necessarily commutative) topological algebra with the defining family $\{p_\alpha\}_{\alpha \in D}$ of seminorms in which (*): for each sequence $x_n \to 0$ there exists $x_m \in \{x_n\}$ such that $x_m^k \to 0$ as $k \to \infty$. Then each multiplicative linear functional on a Fréchet algebra satisfying the above condition (*) is continuous.

These results answer open questions (1) and (2) (Mem. Amer. Math. Soc. **11**, 1953) in the affirmative for Fréchet algebras in which (*) holds. It is also shown that a positive linear functional on such algebras with identity and continuous involution is continuous, thus partially generalizing Shah's result (1959).

1. Introduction

The following problems, initially we understand posed by Mazur but restated by Michael (1953), p. 50, have as yet remained unresolved:

QUESTION (1). Is every multiplicative linear functional on a complex, commutative, complete, metrizable locally m-convex algebra continuous?

QUESTION (2). Is every multiplicative linear functional on a complex commutative, complete, locally m-convex algebra bounded (i.e. maps bounded sets into bounded sets)?

It is clear that an affirmative answer to question (2) implies an affirmative answer to question (1), Michael (1953), p. 50. As a matter of fact, it is known Dixon and Fremlin (1972) that the reverse implication also holds. Under certain

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extra conditions, the answers to both questions are in the affirmative, for example, see Arens (1965), Michael (1953), Zelazko (1960) and the authors, Husain and Ng (1974a), (1974b).

The purpose of this paper is to give another extra condition under which the answers to questions (1) and (2) are in the affirmative. As a matter of fact, we first prove (Theorem 1) that every multiplicative linear functional on a complex, (not necessarily commutative) sequentially complete, locally convex (not necessarily locally m-convex) algebra satisfying an extra condition (see Theorem 1) is bounded. From this we derive affirmative answers to questions (1) and (2) for such algebras (see Corollary 1 and 2). We also partially extend (Theorem 3) a result due to Shah (1959) for positive functionals.

2. Preliminaries

Let A be a complex (or real) algebra, endowed with a topology. A is said to be a topological algebra if A is a topological vector space such that the mapping: $(x, y) \rightarrow xy$ is continuous in both variables together. If the topology of A is Hausdorff, A is said to be a Hausdorff topological algebra. All algebras used in this paper are assumed to be Hausdorff.

If the topology of A is given by a metric, then A is called a metrizable topological algebra. A complete metrizable algebra is called an F-algebra. If the topology of A is given by a family $\{p_\alpha\}_{\alpha \in \Gamma}$ of seminorms, then A is called a locally convex algebra. If each p_α, in addition, is submultiplicative i.e., $p_\alpha(xy) \leqq p_\alpha(x) \times p_\alpha(y)$ for all $x, y \in A$, then A is called locally m-convex.

Γ is countable iff A is a metrizable locally convex algebra. A complete metrizable locally convex algebra is called a B_0-algebra, Zelazko (1973). Sometimes a complete metrizable locally m-convex algebra is called a Fréchet algebra.

A topological algebra is called sequentially complete if each Cauchy sequence in it converges. This notion strictly generalizes the notion of completeness i.e. each Cauchy filter converges.

We define the following notion:

Definition. A topological algebra A will be called *sequential* if for each sequence $\{x_n\}$, $x_n \rightarrow 0$ there exists an element $x_m \in \{x_n\}$ such that $x_m^k \rightarrow 0$ as $k \rightarrow \infty$.

It is clear that each normed algebra is sequential. Also we have:

Proposition 1. *If in a topological algebra A there exists a neighbourhood U of* 0 *such that for all* $x \in U$, $x^m \rightarrow 0$ *as* $m \rightarrow \infty$, *then A is sequential.*

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PROOF. Since for any sequence $x_n \to 0$, $x_n \in U$ for sufficiently large n, it follows that A is sequential by the definition.

PROPOSITION 2. *A sequentially complete, locally m-convex algebra with the identity e is a Q-algebra* (see Michael (1953) p. 77 for definition) *iff there exists a neighbourhood U of 0 such that for all $x \in U$, $x^m \to 0$ as $m \to \infty$. Hence each Q-algebra is sequential.*

PROOF. If A is a Q-algebra, the set W of all invertible elements is open and the function $\lambda \to (e - x/\lambda)^{-1}$ of $\lambda \notin \sigma(x)$, spectrum of x, is holomorphic. Let U be a convex circled neighborhood of 0 such that $e + 2U \subset W$. Then for all $x \in U$,

$$\left(e - \frac{x}{\lambda}\right)^{-1} = e + \sum_{n=1}^{\infty} \left(\frac{x}{\lambda}\right)^n$$

is convergent for $|\lambda| > \frac{1}{2}$. Hence $x^n \to 0$ as $n \to \infty$.

On the other hand, if there exists a neighbourhood U of 0 such that for all $x \in U$, $x^m \to 0$ as $m \to \infty$, then the series $\Sigma_{m=1}^{\infty} x^m$ is convergent because A is sequentially complete and clearly $(e - x)^{-1} = e + \Sigma_{m=1}^{\infty} x^m$ is the inverse of $e - x$. Hence the set of all invertible elements of A has a nonempty interior i.e., A is a Q-algebra.

REMARK. There exist functionally continuous Fréchet algebras which are neither Q-algebras nor sequential. For example, the algebra $C(R)$ of all continuous real-valued functions on the real line R is a functionally continuous, non-Q, Fréchet algebra. If we take the following sequence of continuous functions:

$$f_n(x) = \begin{cases} x - n & \text{if } x \geqq n \\ 0 & \text{if } x \in [-n, n] \\ -(x+n) & \text{if } x \leqq -n \end{cases}$$

then we see that in the metric topology defined by the seminorms: $p_n(f) = \sup_{|x| \leqq n} |f(x)|$, $f_n \to 0$ but there exists no element of $\{f_n\}$ whose powers tend to zero. Hence $C(R)$ is not sequential.

Similarly, the complex Fréchet algebra of entire functions is functionally continuous but neither sequential nor a Q-algebra.

3. Main results

THEOREM. *Let A be a complex, sequential, (not necessarily commutative) locally convex, sequentially complete, algebra with the defining family of $\{p_\alpha\}_{\alpha \in \Gamma}$ of seminorms. Then each multiplicative linear functional f on A is bounded.*

PROOF. We prove the theorem by contradiction. Let B be a bounded subset of A such that $\operatorname{Sup}_{x\in B}|f(x)|=\infty$. Then there is a sequence $\{y_n\}\subset B$ such that $|f(y_n)|\geqq n$ for all $n\geqq 1$. Put $z_n=2n^{-1}y_n$. Then $z_n\to 0$ as $n\to\infty$ (because $\{y_n\}$ is bounded) and $|f(z_n)|\geqq 2$. Since A is sequential, there exists $z\in\{z_n\}$ such that $z^k\to 0$ as $k\to\infty$. Clearly $|f(z)|\geqq 2$.

Suppose there is a positive integer $m\geqq 2$ such that the set $\{z, z^2,\cdots,z^m\}$ is linearly dependent. Then it is easy to see that the algebra E generated by z has dimension at most m. Since each linear functional on a finite-dimensional space is continuous (Bourbaki (1953), Chap. 1, p. 28), the restriction $f\,|\,E$ of f on E is continuous. Since $z^k\to 0$ as $k\to\infty$ and $z^k\in E$, it follows that $f^k(z)=f(z^k)\to 0$ as $k\to\infty$ which is contrary to the fact that $|f(z)|\geqq 2$ and so $|f^k(z)|\geqq 2^k$.

But then the sequence $\{z^k\}$ must be linearly independent. Let E be the algebra of polynomials generated by $\{z^k\}$. Each element $u\in E$ has the unique (because of linear independence of $\{z^k\}$) representation: $u=\Sigma_{i=1}^n\alpha_i z^i$ for some n.
Put $\|u\|=\Sigma_{i=1}^n|\alpha_i|$. Then it is easy to check that $(E,\|\cdot\cdot\|)$ is a normed algebra i.e. E is a normed space in which for all $u,v\in E$, $\|uv\|\leqq\|u\|\,\|v\|$ and $\|z\|=1$. Since $z^k\to 0$ as $k\to\infty$, for each $p_\alpha\in\{p_\alpha\}_{\alpha\in\Gamma}$, $p_\alpha(z^k)\to 0$ as $k\to\infty$. Thus there exists a real number M_α (depending upon α) such that $p_\alpha(z^k)\leqq M_\alpha$ for all $k\geqq 1$. Thus if $u=\Sigma_{i=1}^n\alpha_i z^i$, then

$$(*)\qquad p_\alpha(u)\leqq M_\alpha\sum_{i=1}^n|\alpha_i|=M_\alpha\|u\|$$

for each $u\in E$ and $\alpha\in\Gamma$.

Now let $\hat{E}$ denote the completion of $(E,\|\cdot\cdot\cdot\|)$. Then $\hat{E}$ is a commutative Banach algebra. If $\hat{x}\in\hat{E}$, then there is a Cauchy sequence $\{x_n\}\subset E$ such that $\|x_n-\hat{x}\|\to 0$ as $n\to\infty$. Hence $\{x_n\}$ is a Cauchy sequence under the intial topology by (*) and therefore convergent to $\hat{x}\in A$, because A is sequentially complete. Thus $\hat{E}$ can be set-theoretically identified with a subalgebra of $\bar{E}$ (the closure of E in the initial topology of A). Since E is a commutative Banach algebra and setwise a subalgebra of A, the restriction $f|\hat{E}$ of f, being a multiplicative linear functional on $\hat{E}$ is continuous. Hence $2\leqq|f(z)|\leqq\|z\|=1$ gives a contradiction and the proof is complete.

Since each complete algebra is sequentially complete, we obtain the following result which answers question (2) in the affirmative for a certain class of algebras which are not necessarily locally m-convex, nor commutative.

COROLLARY 1. *Let A be a complex, complete, locally convex (not necessarily locally m-convex, nor commutative) sequential algebra. Then each multiplicative linear functional on A is bounded.*

The following Theorem answers question (1) in the affirmative for a large class of algebras under a certain condition.

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THEOREM 2. *Let A be a sequentially complete, locally convex, born under logical (not necessarily commutative) sequential algebra. Then each multiplicative linear functional on A is continuous.*

PROOF. Since each bounded linear functional on a bornological space is continuous ((Bourbaki (1955), Chap. III, p. 13), the proof is immediate from Theorem 1.

COROLLARY 2. *Let A be a sequential (not necessarily locally m-convex, nor commutative) B_0-algebra. Then each multiplicative linear functional on A is continuous.*

PROOF. This is a particular case of Theorem 2, because each B_0-algebra is a bornological complete algebra.

In particular, Corollary 2 holds for sequential Fréchet algebras.

4. Positive functionals

Let A be an algebra with involution *. A linear fuctional f on A is said to be positive if for all $x \in A$, $f(x^*x) \geqq 0$. It is known Rickart (1960) or Zelazko (1973), p. 120, Theorem 25.8 that each positive functional on a commutative Banach *-algebra is continuous. We partially extend this result in the following:

THEOREM 3. *Let A be a complex, (not necessarily commutative), sequential and complete locally m-convex algebra with the defining family $\{p_\alpha\}_{\alpha \in \Gamma}$ of seminorms such that $p_\alpha(x^*x) = p_\alpha^2(x)$, where * is the continuous involution and e is the identity. Then each positive functional g on A is bounded.*

PROOF. Suppose g is not bounded. Let $g(e) = 1$. As in Theorem 1, there is a sequence $\{y_n\} \subset A$ such that $y_n \to 0$ and $|g(y_n)| \geqq 1$ for all $n \geqq 1$. By Cauchy-Schwarz inequality (cf. Rickart (1960), p. 120), $|g(x^*y)|^2 \leqq g(x^*x)g(y^*y)$ for all $x, y \in A$. Hence we have:

$$\begin{aligned} 1 \leqq |g(y_n)|^2 &= |g(ey_n)|^2 \\ &\leqq |g(e^*e)|\,|g(y^*{}_n y_n)|. \end{aligned}$$

Put $x_n = 2y_n^* y_n / g(e^*e)$ for $n \geqq 1$. Then $x_n \to 0$ and $|g(x_n)| \geqq 2$ for all $n \geqq 1$. Since A is sequential, there exists $x \in \{x_n\}$ such that $x^k \to 0$ as $k \to \infty$. Let $z = \frac{1}{2}x$. Then $z^k = (\frac{1}{2}x)^k \to 0$ as $k \to \infty$ and $|g(z^k)| = |g(z)|^k \geqq 1$ for all $k = 2^n$, $n \geqq 1$. Following the proof of Theorem 1, we show that the algebra E, generated by $\{e, x^k, k \geqq 1, x = x^*\}$ is a commutative Banach *-algebra and * is continuous in E. Since g is a positive functional on A its restriction $g_{|\hat{E}}$ (where $\hat{E}$ is the completion of E under the norm) is continuous. But $z \in E$ and $\|z^k\| \to 0$ as

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$k \to 0$ and so $g(z^k) \to 0$. On the other hand $|g(z^k)| = |g(z)|^k \geqq 1$ for all $k = 2^n$, $n \geqq 1$ gives a contradiction. Hence g is bounded.

REMARK. Theorem 3 partially generalizes Shah's result, Shah (1959). The following Corollary is immediate from Theorem 3. For the commutative case, there is a stronger version in Zelazko (1973), p. 138.

COROLLARY 3. *Let A be a complex, sequential, Fréchet algebra with the defining sequence of seminorms $\{p_n\}$ such that $p_n(x^*x) = p_n^2(x)$ for all $n \geqq 1$ with the identity and continuous involution*. Then each positive functional on A is continuous.*

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