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FOUR INTEGERS WHOSE TWELVE QUOTIENTS SUM TO ZERO

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1. Introduction. This paper is devoted to the study of sets of four unequal integers n_i such that the twelve quotients n_i/n_j $(i \neq j)$ of pairs of distinct members sum to zero. (Without the restriction $i \neq j$ the condition would be equivalent to $(\sum n_i)(\sum 1/n_i) = 0$, of no great interest.) Constructions for these sets are given, and relations between them are studied. It is found that each set belongs to an orbit of six related sets, and that each such orbit is related to four neighbours, each of which is another orbit of six sets. A study is made of the graph formed by assigning a node to each orbit of six solutions and joining it to the nodes assigned to its four neighbours. This appears to comprise one component containing an infinity of cycles together with an infinite forest of infinite trees.

A context in which this problem arises is a special case of Fermat's triple equations. As I find no recent account of triple equations, and the earlier accounts do not mention this special case, I begin with a brief account of Fermat's method for solving triple equations and of the circumstances of its failure. This will keep the present account self-contained, and will obviate any need for the reader to seek out the earlier accounts which are not too readily accessible. A different treatment, not following Fermat's method but exhibiting the same cases of failure, was given in [**6**]. It included a brief discussion of sets of four integers whose twelve quotients sum to zero, but this was limited in scope as the construction given in the present Section 7 was not then available.

2. Fermat's triple equations. Fermat's original formulation of triple equations was in a note to Diophantus' *Arithmetica* vi.24 (in the numbering of [**2**]; vi.22 in that of [**4**]), in that notorious margin which was inadequate to contain his proof of his "last theorem". This note was published by S. de Fermat, in his second edition of [**2**], together with a more extended account by J. de Billy (French translation in [**3**], see also [**4**, **9**]). I do not reproduce Fermat's notation.

Fermat was seeking rational numbers r which make $1 + ar$, $1 + br$, $1 + cr$ rational squares simultaneously, for given fixed rational a, b, c. These latter are usually taken as integers, since they can clearly be

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multiplied by a constant scale factor with no significant effect on the problem. In Fermat's and some of Billy's examples, the expressions to be made square had squares of integers in place of the 1's. Such expressions can always be rescaled to adjust these to 1, for example $p^2 + ar$ is a rational square whenever $1 + ap^{-2}r$ is, and we shall suppose this done.

Fermat began by setting $r = 2s + as^2$, so as to have $1 + ar = (1 + as)^2$, and he thus required solutions of

$$(1) \quad \begin{cases} 1 + 2bs + abs^2 = v^2 \\ 1 + 2cs + acs^2 = w^2 \end{cases}$$

a double equation. He then examined the difference

$$v^2 - w^2 = (b - c)(2s + as^2).$$

He factorized this as $(v + w)(v - w)$, separating the right hand side into factors which are linear in s, so as to express v and w linearly in s. In order to obtain a linear equation for s, he required that the squares of these expressions for v and w, when equated to the expressions (1), should match them either in the terms independent of s or in the terms in s^2.

3. First factorization. We set

$$v + w = 2 + as,$$

$$v - w = s(b - c),$$

so

$$v = 1 + \frac{1}{2}s(a + b - c),$$

$$w = 1 + \frac{1}{2}s(c + a - b),$$

from which we obtain

$$v^2 = 1 + s(a + b - c) + \frac{1}{4}s^2(a + b - c)^2.$$

Equating this to $v^2 = 1 + 2bs + abs^2$ from (1), we obtain $s = 0$ or

$$s = \frac{4(b + c - a)}{a^2 + b^2 + c^2 - 2(bc + ca + ab)}.$$

(The same expression is obtained from w^2.)

If any of a, b, c is the sum of the other two, this factorization leads only to trivial solutions with $r = 0$, as was noted by Billy. We call this the *first exceptional case*. Billy noted that there may nevertheless be solutions with non-zero r in this case; for example $1 + 5r$, $1 + 16r$, $1 + 21r$ are all made squares by setting $r = 3$. But there is no non-zero r for which

$1 + r$, $1 + 2r$, $1 + 3r$ are all squares, Fermat having observed that there cannot be four unequal squares in arithmetic progression.

This factorization leads to no solution at all if

$$a^2 + b^2 + c^2 - 2(bc + ca + ab) = 0,$$

i.e.,

$$\sqrt{a} \pm \sqrt{b} \pm \sqrt{c} = 0.$$

We call this the *second exceptional case*; it seems to have been noted only recently [**6**]. As in the first exceptional case, there may be solutions with non-zero r in this case. Such solutions are shown in Section 6 to correspond to sets of four unequal integers whose twelve quotients sum to zero, the main topic of this paper. Pocklington [**8**] has shown that there is no non-zero r for which $1 + r$, $1 + 4r$, $1 + 9r$ are all squares.

There are variant forms of this second exceptional case in which this factorization leads only to a solution in which one of $1 + ar$, $1 + br$, $1 + cr$ is zero. For example $1 + 5r$, $1 + 8r$, $1 + 9r$ are all made squares only by $r = 0$ or $r = -1/9$, since otherwise the non-zero square $1 + 9r$ could be rescaled to 1 and the squares $1 + 8r$, $1 + 5r$, 1 would then correspond to squares $1 + r'$, $1 + 4r'$, $1 + 9r'$, with

$$r' = -r/(1 + 9r),$$

contrary to Pocklington's result quoted above.

In all other cases, this factorization gives a non-zero r_1 such that

$$1 + ar_1 = u^2,\ 1 + br_1 = v^2,\ 1 + cr_1 = w^2,$$

with $uvw \neq 0$. Fermat then went on to find a further solution $r_2 = r_1 + r$ by solving similarly the problem of making $u^2 + ar$, $v^2 + br$, $w^2 + cr$ squares simultaneously, which is equivalent to making $1 + au^{-2}r$, $1 + bv^{-2}r$, $1 + cw^{-2}r$ squares simultaneously. It can be shown that such a derived problem never presents an exceptional case, so we can repeat this operation to obtain further solutions (cf. Section 12).

4. Second factorization. In the special case that a, b, c are perfect squares α^2, β^2, γ^2, we factorize

$$v^2 - w^2 = (\beta^2 - \gamma^2)(2s + \alpha^2 s^2)$$

as

$$v + w = \alpha s(\beta + \gamma) + 2(\beta + \gamma)/\alpha,$$

$$v - w = \alpha s(\beta - \gamma),$$

so

$$v = \alpha\beta s + (\beta + \gamma)/\alpha,$$

$$w = \alpha\gamma s + (\beta + \gamma)/\alpha,$$

from which we obtain

$$v^2 = (\beta + \gamma)^2/\alpha^2 + 2\beta(\beta + \gamma)s + \alpha^2\beta^2 s^2.$$

Equating this to

$$v^2 = 1 + 2\beta^2 s + \alpha^2\beta^2 s^2$$

from (1), we obtain s infinite or

$$s = \frac{\alpha^2 - (\beta + \gamma)^2}{2\alpha^2\beta\gamma}.$$

(Again we obtain the same expression from w^2.)

This leads only to trivial solutions with $r = 0$ if $\alpha \pm \beta \pm \gamma = 0$, the second exceptional case of Section 3. It leads only to a solution with one of $1 + \alpha^2 r$, $1 + \beta^2 r$, $1 + \gamma^2 r$ zero if α, β, γ are the three sides of a right-angled triangle. This is a particular solution of a special case of the first exceptional case of Section 3. For example $1 + 9r$, $1 + 16r$, $1 + 25r$ are all made squares by setting $r = -1/25$. We class this special solution as trivial. J. H. E. Cohn (private communication) has shown that there is no other non-zero r solving these equations. In certain other examples there are additional solutions. One such is $1 + 8^2 r$, $1 + 15^2 r$, $1 + 17^2 r$, which are all made squares by setting $r = 91/99^2$ or $627/119^2$ as well as by $r = -1/17^2$.

Except in these special cases, this factorization gives a solution in non-zero squares, and Fermat's method can then be applied to obtain further solutions. Fermat's method can also be applied whenever we have a non-trivial solution in an exceptional case.

5. First exceptional case. Here it is convenient to restate the problem in a homogeneous form, requiring $t^2 + ar$, $t^2 + br$, $t^2 + (a + b)r$ to be integer squares, say u^2, v^2, w^2 respectively. We thus have

$$2t^2 + (a + b)r = t^2 + w^2 = u^2 + v^2.$$

An integer which has two distinct expressions as the sum of two squares is the product of two sums of two squares, so we write

$$t^2 + w^2 = u^2 + v^2 = (\alpha^2 + \beta^2)(\gamma^2 + \delta^2),$$

and obtain

$$t = \alpha\gamma + \beta\delta,\ u = \alpha\gamma - \beta\delta,$$

$$v = \alpha\delta + \beta\gamma,\ w = \alpha\delta - \beta\gamma.$$

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From this we deduce

$$\frac{b}{a} = \frac{t^2 - v^2}{t^2 - u^2} = \frac{(\alpha\gamma + \beta\delta)^2 - (\alpha\delta + \beta\gamma)^2}{(\alpha\gamma + \beta\delta)^2 - (\alpha\gamma - \beta\delta)^2} = \frac{\alpha^2 - \beta^2}{2\alpha\beta} \cdot \frac{\gamma^2 - \delta^2}{2\gamma\delta},$$

so b/a has to be expressible as the product of two ratios of the form $(\alpha^2 - \beta^2)/2\alpha\beta$, the ratios of the perpendicular sides of rational right-angle triangles. (An equivalent condition was obtained by Schaewen [**9**].) Some examples of possible and impossible values for b/a are given in [**1**, **6**]. Billy's example corresponds to

$$\frac{5}{16} = \frac{3^2 - 2^2}{2.3.2} \cdot \frac{2^2 - 1^2}{2.2.1}.$$

If b/a is the square of a ratio $(\alpha^2 - \beta^2)/2\alpha\beta$, this expression leads only to the trivial solution with $w = 0$. There will be non-trivial solutions in this case if b/a can be expressed both as the square of a ratio $(\alpha^2 - \beta^2)/2\alpha\beta$ and as a product of two unequal ratios of this form. Some examples of such values for b/a are given in [**6**, **5**]. The numerical examples in Section 4 depend on

$$\left(\frac{4^2 - 1^2}{2.4.1}\right)^2 = \frac{13^2 - 8^2}{2.13.8} \cdot \frac{14^2 - 1^2}{2.14.1} = \frac{19^2 - 8^2}{2.19.8} \cdot \frac{22^2 - 3^2}{2.22},$$

and are the simplest such solutions.

This first exceptional case will not be developed further here.

6. Second exceptional case: relation to the present problem. Here also it is convenient to restate the problem in a homogeneous form, requiring $t^2 + \alpha^2 r$, $t^2 + \beta^2 r$, $t^2 + \gamma^2 r$ to be integer squares, say u^2, v^2, w^2 respectively, where the signs of α, β, γ are chosen to satisfy the symmetrical relation $\alpha + \beta + \gamma = 0$. We discount as trivial solutions with $r = 0$ or $t = 0$. Unfortunately there seems to be no simple criterion for the existence of non-trivial solutions for given α, β, γ, corresponding to that for b/a in Section 5.

We have

$$t^2 + w^2 - u^2 - v^2 = 2\alpha\beta r,$$

so

$$(t^2 + w^2 - u^2 - v^2)^2 = 4\alpha^2\beta^2 r^2 = 4(u^2 - t^2)(v^2 - t^2),$$

i.e.,

$$(u^2 + v^2 + w^2 - t^2)^2 = 2(u^4 + v^4 + w^4 - t^4).$$

This equation is conveniently studied by applying the transformation

$$\begin{aligned} 2n_1 &= t + u + v + w, & 2t &= n_1 + n_2 + n_3 + n_4, \\ 2n_2 &= t + u - v - w, & 2u &= n_1 + n_2 - n_3 - n_4, \\ 2n_3 &= t - u + v - w, & 2v &= n_1 - n_2 + n_3 - n_4, \\ 2n_4 &= t - u - v + w, & 2w &= n_1 - n_2 - n_3 + n_4, \end{aligned}$$

under which it takes the symmetrical form

$$\sum n_i^2 n_j n_k = 0,$$

or equivalently

$$\sum n_i/n_j = 0,$$

the problem of our title. In these and later summations, the subscripts range over 1, 2, 3, 4, and the summations are over all possible terms in which no two subscripts take equal values.

We now express α, β, γ in terms of the n_i. Let

$$\bar{\alpha} = n_1 n_2 + n_3 n_4, \bar{\beta} = n_1 n_3 + n_4 n_2, \bar{\gamma} = n_1 n_4 + n_2 n_3.$$

Then

$$\bar{\beta}\bar{\gamma} + \bar{\gamma}\bar{\alpha} + \bar{\alpha}\bar{\beta} = \sum n_i^2 n_j n_k = 0.$$

We have

$$\begin{aligned} \frac{\alpha^2}{\beta^2} &= \frac{t^2 - u^2}{t^2 - v^2} = \frac{(t - u)(t + u)}{(t - v)(t + v)} = \frac{(n_1 + n_2)(n_3 + n_4)}{(n_1 + n_3)(n_4 + n_2)} \\ &= \frac{\bar{\beta} + \bar{\gamma}}{\bar{\alpha} + \bar{\gamma}} = \frac{\bar{\alpha}\bar{\beta} + \bar{\alpha}\bar{\gamma}}{\bar{\beta}\bar{\alpha} + \bar{\beta}\bar{\gamma}} \quad \frac{\bar{\alpha}}{\bar{\beta}} = \frac{\bar{\beta}\bar{\gamma}}{\bar{\alpha}\bar{\gamma}} \cdot \frac{\bar{\beta}}{\bar{\alpha}} = \frac{\bar{\beta}^2}{\bar{\alpha}^2}, \end{aligned}$$

so $\alpha^2\bar{\alpha}^2 = \beta^2\bar{\beta}^2, = \gamma^2\bar{\gamma}^2$ similarly. Since

$$\bar{\beta}\bar{\gamma} + \bar{\gamma}\bar{\alpha} + \bar{\alpha}\bar{\beta} = 0$$

is equivalent to

$$\bar{\alpha}^{-1} + \bar{\beta}^{-1} + \bar{\gamma}^{-1} = 0,$$

this determines the signs, giving

$$\alpha\bar{\alpha} = \beta\bar{\beta} = \gamma\bar{\gamma},$$

i.e.,

$$\alpha(n_1 n_2 + n_3 n_4) = \beta(n_1 n_3 + n_4 n_2) = \gamma(n_1 n_4 + n_2 n_3),$$

which is the required expression.

Replacing the n_i by integers proportional to their reciprocals leads to another solution with the same values of α, β, γ, since

$$n_1^{-1} n_2^{-1} + n_3^{-1} n_4^{-1} = (n_3 n_4 + n_1 n_2)/n_1 n_2 n_3 n_4,$$

etc. Changing the signs of u, v, w also leads to another solution having the same values of α, β, γ; this is equivalent to replacing each n_i by

$$\frac{1}{2}(n_1 + n_2 + n_3 + n_4) - n_i.$$

Performed alternately, these transformations yield an orbit of six solutions, as we may see thus.

Since only ratios n_i/n_j are involved, we may replace the integers n_i by rational numbers

$$x_i = 2n_i/(n_1 + n_2 + n_3 + n_4).$$

This gives $\sum x_i = 2$, and corresponds to replacing t, u, v, w by 1, u/t, v/t, w/t. Replacing each x_i by $1 - x_i$ preserves this sum, and so does replacing each x_i by $1/x_i$ since

$$(\sum x_i)(\sum 1/x_i) = \sum_{i \neq j} x_i/x_j + \sum x_i/x_i = 0 + 4 = 4.$$

So we see that these operations transform each x_i through the cycle of values

$$x_i \to \frac{1}{x_i} \to 1 - \frac{1}{x_i} \to \frac{x_i}{x_i - 1} \to \frac{1}{1 - x_i} \to 1 - x_i \to x_i.$$

Thus solutions of the original triple equations in this second exceptional case come in sets of three, since we do not regard solutions differing only in the signs of u, v, w as distinct. Some numerical examples are given in later sections.

Define a *triad* to be a set of three integers with sum zero, such as (α, β, γ) above, where we shall assume that any common factor has been removed. We shall usually enclose triads in round brackets.

Write $x_i = -a/b$, and define c to complete the triad (a, b, c) with sum zero. Then the transforms of x_i in this cycle are

$$x_i = -\frac{a}{b} \to -\frac{b}{a} \to -\frac{c}{a} \to -\frac{a}{c} \to -\frac{b}{c} \to -\frac{c}{b} \to -\frac{a}{b},$$

so they are the negatives of the six ratios of pairs of integers from the triad (a, b, c). The six solutions forming an orbit are thus characterized if we specify the appropriate four ordered triads with sums zero. Each member of the orbit of solutions comprises the negatives of the ratios of ordered pairs taken from corresponding positions in these ordered triads.

According to context, orbits of solutions will be specified as sets of four triads as above, six tetrads of x_i or of n_i, or sets of three solutions of the corresponding triple equations.

7. Generation of new orbits of solutions from a known orbit. Before going on to give numerical examples, I give a construction for obtaining new solutions from old. We have seen that a set of three solutions of the triple equations

$$t^2 + r(\alpha^2, \beta^2, \gamma^2) = u^2, v^2, w^2,$$

for a triad (α, β, γ), is associated with an orbit of six solutions of $\sum x_i/x_j = 0$ characterized by four triads (a, b, c). I now describe how to construct a new orbit of solutions, in which the role of the triad (α, β, γ) is exchanged with that of one of the triads (a, b, c).

Suppose an orbit of solutions is set out as an array, in which each row comprises a tetrad of integers n_i, n_j, n_k, n_l forming a solution, and each column corresponds to a triad (a, b, c). Label the columns i, j, k, l, and let us select a column l to correspond to a chosen triad (a, b, c), so each n_l lies in this column. From the other integers n_i, n_j, n_k completing a row, we form a triad (a', b', c') from the differences

$$a':b':c' = n_j - n_k:n_k - n_i:n_i - n_j.$$

Since replacing each n by $\frac{1}{2} \sum n_i - n$ only changes the signs of the differences, and does not alter $a':b':c'$, the orbit of six tetrads yields only three significantly different triads. With these we associate the original triad (α, β, γ) in the permutation which associates the reciprocals of

$$n_l n_i + n_j n_k, \quad n_l n_j + n_k n_i, \quad n_l n_k + n_i n_j$$

with

$$n_j - n_k, \qquad n_k - n_i, \qquad n_i - n_j$$

respectively.

I state that these four triads characterize a new orbit of solutions, in which the new numbers $n_1 n_2 + n_3 n_4$, $n_1 n_3 + n_4 n_2$, $n_1 n_4 + n_2 n_3$ are reciprocally proportional to the triad (a, b, c) chosen from the old orbit. Unfortunately I know of no short or elegant or perspicuous proof that this construction always achieves the result stated. It can be proved by brute force, calculating the new triads in terms of the old and exhibiting that these also generate sets of x_i which satisfy $\sum x_i/x_j = 0$. A worked example is given in Section 8.

This construction is self-reciprocal, in the following sense. If from the new orbit of solutions we choose the n_l corresponding to the triad (α, β, γ) from the old orbit of solutions, and repeat the construction of forming triads from the differences $n_j - n_k$, $n_k - n_i$, $n_i - n_j$, then we reconstruct the original orbit of solutions. Let us call such orbits of solutions *adjacent*. With certain exceptions noted in the next section, each orbit is adjacent to four others, each of which in turn is adjacent to three new orbits in addition to the original orbit.

I have written a simple interactive computer program to calculate a new orbit as a set of four triads from a given set of triads of which one has been selected. At each stage the user selects one of the current set of triads, and this is used to specify the next set of current triads. Apart from needing care with the handling of large integers, the programming presents no problems. Many of the results given in later sections were first discovered with the use of this program as a tool of exploration.

8. Some numerical solutions. We begin with trivial solutions in which the x_i are not all different. The most trivial has $x_i = 1, 1, 1, -1$, and almost as trivial is that with $x_i = 1, 1, x, -x$ with $x \neq 0, \pm 1$. These do not belong to orbits, as the operation $x_i \to 1 - x_i$ introduces zero values which are impossible in proper solutions of $\sum x_i/x_j = 0$, and which make the subsequent operation $x_i \to 1/x_i$ impossible.

Next we consider semi-trivial solutions, corresponding to triple equations with $(\alpha, \beta, \gamma) = (1, 1, -2)$, so two of the equations are identical. We thus require simultaneous solution of the two equations $t^2 + r = u^2$, $t^2 + 4r = w^2$. Write $t = p + q$, $w = p - q$; then $r = -pq$, and we have to satisfy

$$p^2 + pq + q^2 = u^2.$$

Let (l, m, n) be an arbitrary triad of integers with sum zero, and set

$$p = l^2 - m^2, q = m^2 - n^2.$$

Then we have

$$p^2 + pq + q^2 = (mn + nl + lm)^2,$$

and our equations are satisfied. Cyclic permutation of l, m, n generates a set of three solutions with the same value of $u = mn + nl + lm$, and these correspond to an orbit of six solutions of $\sum n_i/n_j = 0$, each with two equal values of n_i. The corresponding triads (a, b, c) have the values (l, m, n), $(m - n, n - l, l - m)$, and $(m^2 - n^2, n^2 - l^2, l^2 - m^2)$, the last being taken twice, corresponding to the equal values of n_i. Replacement of l, m, n by their differences $m - n, n - l, l - m$ leads only to the same triads, after removal of common factors, otherwise distinct triads (l, m, n) lead to distinct orbits. In the simplest numerical example, the triads are $(1, 2, -3)$, $(-5, 4, 1)$ and $(5, -8, 3)$ twice. The corresponding sets of n_i are

−4	10	5	5
−10	4	8	8
15	1	−3	−3
1	15	−5	−5
2	−12	8	8
12	−2	3	3,

where a common factor 2 has been left in rows 2 and 5 to clarify the relations between rows (in any two adjacent rows either the sum or the product of corresponding terms is constant). These correspond to the following solutions of the (repetitive) triple equations

$$3^2 + 40(1^2, 1^2, 2^2) = 7^2, 7^2, 13^2,$$

$$5^2 + 24(1^2, 1^2, 2^2) = 7^2, 7^2, 11^2,$$

$$8^2 - 15(1^2, 1^2, 2^2) = 7^2, 7^2, 2^2.$$

The construction of Section 7 is unsuccessful if we choose one of the unrepeated triads from this orbit, as the presence of equal values of n_i corresponding to the repeated triad leads only to the trivial triads $(0, 1, -1)$ which do not correspond to orbits of solutions. (But the three such triads, together with $(-2, 1, 1)$, may be said to correspond to the trivial solution $n_i = 1, 1, 1, -1$, with (α, β, γ) indeterminate.)

Choice of one of the repeated triads is successful. From the first row we get the triad $(10 - 5, 5 - (-4), (-4) - 10)$, i.e., $(5, 9, -14)$. From the third row we get $(1 - (-3), (-3) - 15, 15 - 1)$, equivalent to $(2, -9, 7)$. From the fifth row we get $((-12) - 8, 8 - 2, 2 - (-12))$, equivalent to $(-10, 3, 7)$. These three triads, together with the triad $(1, 1, -2)$, give the following orbit of solutions:

−5	2	30	−9
−18	45	3	−10
28	−35	7	20
5	−4	20	7
9	18	−6	7
14	7	−21	18.

The corresponding triple equation solutions are

$$9^2 + 7(3^2, 5^2, 8^2) = 12^2, 16^2, 23^2,$$

$$10^2 + 21(3^2, 5^2, 8^2) = 17^2, 25^2, 38^2,$$

$$14^2 - 3(3^2, 5^2, 8^2) = 13^2, 11^2, 2^2.$$

For general l, m, n, the orbit of solutions is characterized by the triads

$$(2nl + 2lm - mn, 3mn, -2(mn + nl + lm)),$$

$$(2lm + 2mn - nl, 3nl, -2(mn + nl + lm)),$$

$$(2mn + 2nl - lm, 3lm, -2(mn + nl + lm)),$$

$$(1, 1, -2),$$

where the integers in these triads may have removable common factors 2 or 3, and we have

$$(\alpha, \beta, \gamma) = (m^2 - n^2, n^2 - l^2, l^2 - m^2).$$

Again, replacement of l, m, n by their differences $m - n$, $n - l$, $l - m$ leads only to the same triads. These orbits can be used to construct further orbits by the construction of Section 7. Some numerical examples of the triads in these orbits are displayed in Section 9.

The solutions in the orbits obtained above were constructed in the following different way in **[6]**. Suppose that we have a set of four n_i forming a solution, and let us fix three of them. The condition

$$\sum n_i^2 n_j n_k = 0$$

is quadratic in the fourth, and this in general gives a new solution along with our known solution. When this quadratic has either a trivial root (zero or infinite) or two equal roots, we obtain solutions belonging to the orbits with

$$(\alpha, \beta, \gamma) = (m^2 - n^2, n^2 - l^2, l^2 - m^2)$$

above. If its roots are rational and unequal, then the two solutions belong to orbits which are both adjacent to an orbit one of whose triads is

$$(a, b, c) = (n_j - n_k, n_k - n_i, n_i - n_j),$$

where n_i, n_j, n_k are those fixed in the solutions. (Another orbit adjacent to this last orbit has this triad for its (α, β, γ).) These solutions thus belong to orbits which are separated by two steps of the construction of Section 7. The construction of **[6]** is also applicable to the trivial solution with $x_i = 1, 1, x, -x$, mentioned at the beginning of this section. If we fix $x_i = 1, 1, x$ and solve the quadratic for the fourth member, then its second root gives a solution belonging to an orbit with $(\alpha, \beta, \gamma) = (1, 1, -2)$ described above.

9. A graph of orbits of solutions. The foregoing results enable us to represent orbits of solutions and the relations between them as a labelled graph. The node corresponding to an orbit is labelled with the triad (α, β, γ), and it is joined to four adjacent nodes, each of which is labelled with one of the triads (a, b, c) which characterize the orbit. The reciprocity of the construction of Section 7 implies that each of these nodes also corresponds to an orbit of solutions, and the joins are undirected. The only exceptions to this statement are those nodes associated with solutions in which the n_i are not all unequal, as described in Section 8; here the correspondence between the graph and the solutions is imperfect. Otherwise this graph portrays concisely the triads involved in orbits of solutions and the adjacency relations between orbits.

The numerical solutions given in Section 8 appear in Figure 1. In view of its special status described in Section 11, we call this component the *principal component* of the graph. Except for the nodes (1, −1, 0) which are terminal, and the nodes (−3, 1, 2) and (1, −5, 4) adjacent to them, each node in this component corresponds to an orbit of solutions characterized by the four triads labelling the adjacent nodes. In view of its special position adjacent to two nodes with identical labels and neighbours and two nodes not corresponding to orbits, we call the node labelled (1, 1, −2) the *root* of this component, and similarly the nodes labelled (1, 1, −2) corresponding to the other members of the parametric family of orbits given in Section 8. A component of the graph containing a root will be called a *rooted component*.

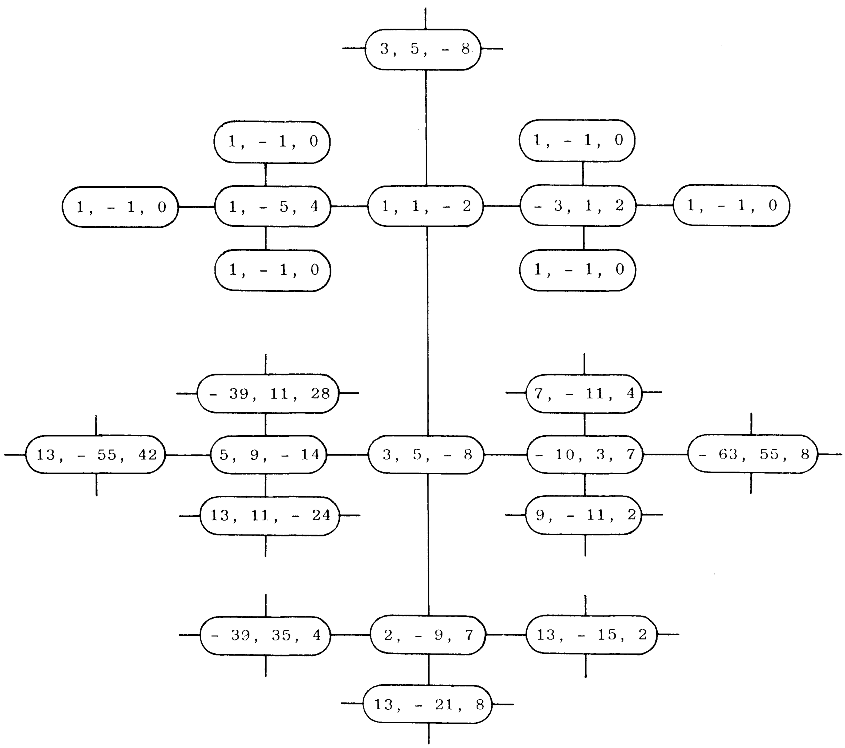

Figure 1

Part of the principal component of the graph of orbits of solutions. The infinite branches are symmetrical about the root node (1, 1, −2), with the two nodes (3, 5, −8) having identical neighbours.

The following observations are based on numerical studies of this graph, but they are wholly conjectural, and I see no way of proving them. Most significantly, I see no way of proving

(9.1). There are solutions which do not belong to the same component of the graph, i.e., the graph has separate components.

Specifically

(9.2). Nodes labelled (1, 1, −2) belong to distinct components, i.e., no component contains more than one root.

(9.3). Except for the principal component, which has cycles described in Section 11, all components, whether rooted or not, are trees. Thus the graph comprises a principal component including an infinity of cycles (Section 11) together with an infinite forest of infinite trees.

I repeat that these are conjectures which I see no way of proving.

10. Infinite paths in the graph. Examination of the principal component of the graph, part of which is displayed in Figure 1, shows that it includes a path beginning

$$(0, 1, -1), (1, 2, -3), (1, 1, -2), (3, 5, -8),$$

$$(2, 7, -9), (8, 13, -21), (1, 10, -11), (21, 34, -55), \dots,$$

in which alternate triads are triads from the Fibonacci sequence 1, 2, 3, 5, 8, 13, 21, 34, 55, ... (In the remaining sections, elements of triads are usually arranged in order of magnitude, not in the systematic ordering corresponding to the formulation of the x_i which is less perspicuous here.) This is one of a general family of paths whose description follows.

Let p_i, q_i satisfy

$$p_i^2 + p_i q_i - q_i^2 = -N^2,$$

where either $N = 1$ or N is divisible only by prime factors of the forms $10n \pm 1$ (prime factors may be repeated), and let

$$p_{i+1} = p_i + q_i, \; q_{i+1} = q_i + p_{i+1}.$$

Then we have also

$$p_{i+1}^2 + p_{i+1} q_{i+1} - q_{i+1}^2 = -N^2,$$

so we have an infinite sequence of pairs p_i, q_i satisfying this relation for each admissible N. For $N = 1$ the sequence $p_i, q_i, p_{i+1}, q_{i+1}, \dots$ is the Fibonacci sequence. When N is 1 or a prime or a prime power, there is only one primitive sequence (p_i, q_i coprime); for other N there are distinct primitive sequences. We may see this in the following way. The pairs of elements q_i, p_{i+1} of each sequence satisfy

$$q_i^2 + q_i p_{i+1} - p_{i+1}^2 = N^2.$$

Suppose we have

$$a^2 + ab - b^2 = N_1^2 \quad \text{and} \quad c^2 + cd - d^2 = N_2^2,$$

with N_1, N_2 coprime, and let

$$A = ac + bd, \qquad B = ad + bc + bd,$$
$$C = ac + bc - bd, \quad D = ad - bc.$$

Then we have

$$A^2 + AB - B^2 = C^2 + CD - D^2 = N_1^2 N_2^2,$$

but it may be shown that the pairs A, B and C, D belong to different sequences.

To each solution of

$$p_i^2 + p_i q_i - q_i^2 = -N^2,$$

there corresponds an orbit of solutions of our problem with

$$(\alpha, \beta, \gamma) = (p_i, q_i, -p_{i+1})$$

and with neighbours in the graph labelled

$$(2q_i, -q_i + 2N, -q_i - 2N),$$

$$(2N, p_i - N, -p_i - N),$$

$$(2N, p_{i+1} - N, -p_{i+1} - N),$$

$$\left(q_i N, \frac{1}{2} q_i (q_i - N) - N^2, -\frac{1}{2} q_i (q_i + N) + N^2\right),$$

where one or more of these triads may have removable common factors 2. The verification of this statement is straightforward but tedious. The node labelled

$$(2N, p_{i+1} - N, -p_{i+1} - N)$$

is adjacent to both $(p_i, q_i, -p_{i+1})$ and $(p_{i+1}, q_{i+1}, -p_{i+2})$, and so these nodes are linked to form an infinite path with labels of these two types alternating. For $N = 1$ this path has a trivial beginning and continues as given at the beginning of this section. For $N > 1$ the paths extend indefinitely in both directions without becoming trivial.

The number of distinct paths for each N is $2^{\nu - 1}$, where ν is the number of distinct primes dividing N. (For $N = 1$ we have $\nu = 0$, and we may count the one-way infinite path as half a path.)

The triads $(q_i, p_{i+1}, -q_{i+1})$, corresponding to the other alternate triads of the sequence $p_i, q_i, p_{i+1}, q_{i+1}, \ldots$, do not correspond to orbits of solutions forming any such path. Thus there is not a path containing the other alternate Fibonacci triads $(2, 3, -5)$, $(5, 8, -13)$, etc.

11. Cycles in the graph. The node labelled

$$(2q_i, -q_i + 2N, -q_i - 2N)$$

is adjacent to the node labelled $(p_i, q_i, -p_{i+1})$ in the path for this N, but does not itself lie in this path. In the sequence $p_i, q_i, p_{i+1}, q_{i+1}, \ldots$ every third q_i is twice an odd number, all other q_i being odd, and for these even values of q_i this label is reduced to

$$\left(\frac{1}{2}q_i, \frac{1}{4}(-q_i + 2N), -\frac{1}{4}(q_i + 2N)\right).$$

This node lies in another path, for which

$$N' = |p_i + p_{i+1}|$$

when q_i is odd, or

$$N' = \frac{1}{4}|p_i + p_{i+1}|$$

when q_i is even. q_i is odd in one path, and its double appears in the other. Except for these parity differences, the relationship between the paths is reciprocal. Let us call such paths *adjacent*.

In each path for $N > 1$ there is a *minimum node*

$$(p_m, q_m, -p_{m+1})$$

for which $|q_m| < N$, since

$$N^2 = q_i^2 - p_i p_{i+1},$$

and there is only one node for which p_i, p_{i+1} are of opposite sign. For this node in the path for N, the adjacent path has $N' < N$, since

$$N' \leqq |p_m + p_{m+1}| < |p_m - p_{m+1}| = |q_m| < N.$$

Thus these paths all lie in the same component of the graph, since we can always follow a path to its minimum node and then move to the adjacent path which has smaller N, continuing in this way via paths for ever-decreasing N until we reach the path for $N = 1$.

Additionally, in those paths in which q_m is odd, either q_{m-1} or q_{m+1} is even, and we can see similarly that for the corresponding node the adjacent path also has $N' < N$, and this N' differs from the previous N'. There is thus an alternative route from this path to the path for $N = 1$, and this exhibits a cycle in the graph. With these exceptions, all other paths adjacent to that for N have $N' > N$.

We see easily that there are an infinity of these cycles. In each path every third q_i is even, and, except only for $i = m - 1$ or m or $m + 1$ as noted above, the corresponding node is adjacent to a node with odd $q_{m'} = \frac{1}{2}q_i$ in a path with $N' > N$. This must be the minimum node in that path, since its adjacent path has $N < N'$. One of $q_{m'\pm 1}$ is even, and the nodes corresponding to this and to $q_{m'}$ form part of a cycle as above. We thus have an infinity of paths in which q_m is odd, each of which contains part of a cycle in which it has the greatest value of N of any path forming part of the cycle.

I conjecture that cycles of this form and their compounds are the only cycles in the graph, numerical search having failed to identify any other. This distinctive property of containing these paths and cycles is the justification for calling this component the *principal component*.

The simplest cycle found may be described as follows. The first non-trivial node in the path for $N = 1$ corresponding to q_i even is $(21, 34, -55)$ with $q_i = 34$. It is adjacent to the node $(-8, 17, -9)$, the minimum node in the path for $N = 19$, part of which is

$$\dots, (11, 27, -38), (-8, 17, -9), (5, 14, -19),$$
$$(9, 26, -35), (8, 19, -27), \dots.$$

In this path, the node $(9, 26, -35)$ has $q_i = 26$, and it is adjacent to the node $(-16, 13, 3)$ in the path for $N = 11$, part of which is

$$\dots, (5, 22, -27), (-16, 13, 3), (4, 7, -11),$$
$$(-3, 10, -7), (2, 9, -11), \dots.$$

In this path the minimum node is $(-3, 10, -7)$, which is adjacent to $(3, 5, -8)$ in Figure 1. We thus have the cycle

$$(3, 5, -8), (2, 7, -9), (8, 13, -21), (1, 10, -11),$$
$$(21, 34, -55); (8, 9, -17), (5, 14, -19), (9, 26, -35);$$
$$(3, 13, -16), (4, 7, -11), (3, 7, -10); (3, 5, -8),$$

where the semicolons indicate the changes from one path to another.

The existence of an infinity of cycles in this principal component makes it one of great complexity, some idea of which is indicated in Figure 2. In this, the paths are simplified by retaining only those nodes which are adjacent to nodes of other paths, and labelling each node with the value of N for the path it is on.

This principal component is not planar. An example of a sub-factor graph homomorphic to the complete bipartite "utilities" graph $K_{3,3}$ is obtained in the following way. One of the triples of nodes is obtained by amalgamating the nodes on each of the paths for $N = 1, 31, 41$. Two of the other triple are obtained by amalgamating the nodes on each of the paths for $N = 11, 19$, and the final node is the amalgam of all nodes on the paths for $N = 71, 29, 121, 79$. Similarly a sub-factor graph homomorphic to the complete graph K_5 on five nodes is obtained by amalgamating into five nodes the nodes on all the paths for $N = 19, 1, 29$, $N = 11, 61$, $N = 41, 109, 59$, $N = 31, 71, 149$, $N = 101, 79, 121$. Either of these sub-factor graphs characterizes the graph as non-planar. Neither is unique.

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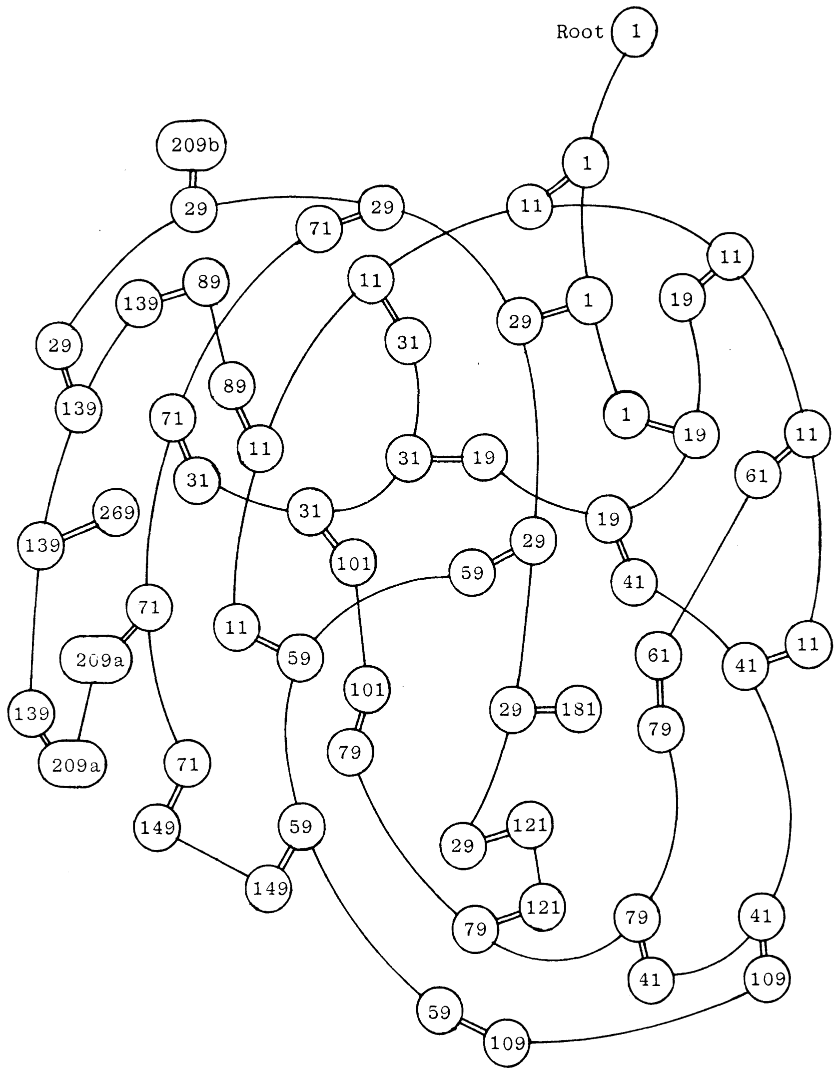

Figure 2
Simplified part of the principal component of the graph. All paths extend indefinitely, in both directions except for $N = 1$. The paths 209a and 209b are different ($209 = 11.19$).

12. Orbits of solutions with the same (α, β, γ). Finding solutions of Fermat's triple equations is equivalent to finding rational points on a plane curve of genus 1. Following **[9]** we may solve one of the equations (1) generally, obtaining a quadratic expression for s, and by inserting this in the other equation (1) we obtain a quartic expression to be made square. Another way of obtaining a quartic expression to be made square is given in **[6]**. This problem is readily transformed into one of finding rational points on a plane cubic curve. These rational points form an Abelian group which, by Mordell's theorem [7], is finitely generated.

Some of the generators are of finite order. In the first exceptional case (Section 5) these generate the group $C_2 \times C_4$ of order 8, extended to the group $C_2 \times C_8$ of order 16 in the particular case that b/a is the ratio of the squares of the perpendicular sides of a rational right-angled triangle. In the second exceptional case (Section 6) and those equivalent to it (Section 3), they generate the group $C_2 \times C_6$ of order 12. When neither of these exceptional cases obtains, Fermat's method of solution gives a generator of infinite order, and so there are an infinity of rational solutions to the triple equations. Whether or not we are considering an exceptional case, there may be additional generators, necessarily of infinite order and leading to an infinity of rational solutions to the equations.

For example, the solutions in Section 8 for $(\alpha, \beta, \gamma) = (3, 5, -8)$ lead to an infinity of further solutions with these values, of which the next simplest are

$$1001^2 + 1782960(3^2, 5^2, 8^2) = 4129^2, 6751^2, 10729^2,$$

$$8075^2 + 1105104(3^2, 5^2, 8^2) = 8669^2, 9635^2, 11659^2,$$

$$13248^2 - 1616615(3^2, 5^2, 8^2) = 12687^2, 11623^2, 8488^2.$$

These correspond to a node in the graph labelled $(3, 5, -8)$ whose neighbours are labelled

$$(91, 323, -414), (77, 475, -552),$$

$$(-1001, 425, 576), (143, -1615, 1472).$$

The large size of the integers involved is typical of solutions obtained in this way. This part of the graph of solutions is believed not to belong to any of the rooted components containing the parametric solutions of Section 8, numerical examination indicating that the size of (α, β, γ) increases (though not monotonically) in all directions from this neighbourhood.

The existence of an infinity of solutions for given (α, β, γ) (when there are any at all) implies that there are an infinity of nodes in the graph bearing this same label. These show no evident relation to one another in the graph, even though there are several examples of two or three nodes in the same component bearing the same label. (Here we do not regard as different the two equivalent occurrences of each label on either side of the root of a rooted component. The root has two identical neighbours, such as $(3, 5, -8)$ in Figure 1, and its removal separates the non-trivial part of the component into two identical infinite components, which we do not regard as distinct.) Consider, for example, the path for $N = 11$ in Section 11, which extends to $(4, 7, -11)$, $(-3, 10, -7)$, $(2, 9, -11)$, $(7, 17, -24)$, (13, 22, −35), (24, 41, −65), (11, 27, −38), (65, 106, −171), (11, 80, −91), ... Here (11, 27, −38) duplicates a label in the path for $N = 19$ (Section 11). Two neighbours of (4, 7, −11) are labelled (3, 13, −16) and (13, 20, −33), each of which has a neighbour labelled (9, 26, −35); this is the closest pair of duplicates so far noticed (ignoring reflections in roots as above). The fourth neighbour of (4, 7, −11) is labelled (11, 80, −91), duplicating a label in the path for $N = 11$ just given. A third occurrence of this label in this component is joined by the sequence (11, 80, −91), (29, 156, −185), (9, 20, −29), (11, 15, −26) to the node (8, 13, −21) in the path for $N = 1$. Some other labels have triple occurrences in this principal component, but none is known with four occurrences.

13. Some further numerical examples. It is found by inspection of sets of solutions to triple equations that the product of the three values of r in each set of solutions is always a negative square. Indeed, I can prove the general result

$$r_1 r_2 r_3 = -(2t_1 t_2 t_3/\alpha\beta\gamma)^2,$$

where r_i, t_i are the values in the set of three equations

$$t_i^2 + r_i(\alpha^2, \beta^2, \gamma^2) = u_i^2, v_i^2, w_i^2.$$

Unfortunately I have no specially simple or elegant version so I do not give a proof here. More specifically, it appears that the three values of r are the negatives of the products in pairs of three integers, two positive and one negative, but at present I have not proved this.

It is possible for the three values of r to be squares individually (one a negative square), and even for the values to be ± 1, as in the example

$$168^2 + (425^2, 576^2, 1001^2) = 457^2, 600^2, 1015^2,$$

$$660^2 + (425^2, 576^2, 1001^2) = 785^2, 876^2, 1199^2,$$

$$1105^2 - (425^2, 576^2, 1001^2) = 1020^2, 943^2, 468^2.$$

The corresponding node labelled (425, 576, −1001) is one of those adjacent to the node labelled (3, 5, −8) given in Section 12. Its other neighbours are labelled (−20, 17, 3), (44, −65, 21), (165, −221, 56). I have a few other examples with $r_i = \pm 1$ or $r_i = \pm$ squares, but these do not seem to show any systematic pattern.

The solutions

$$120^2 + (90^2, 119^2, 209^2) = 150^2, 169^2, 241^2,$$

$$119^2 + 8(90^2, 119^2, 209^2) = 281^2, 357^2, 603^2,$$

$$627^2 - 8(90^2, 119^2, 209^2) = 573^2, 529^2, 209^2,$$

are believed to be the simplest that do not belong to any of the rooted components of the graph. The corresponding node labelled (90, 119, -209) has neighbours labelled (7, 12, -19), (-17, 6, 11), (7, -40, 33), (17, 40, -57). Of these, (-17, 6, 11) has a neighbour labelled (11, 80, -91), independently of the three nodes bearing this label in the principal component.

A numerical curiosity is the occurrence of a node labelled (9, 16, -25), corresponding to the solutions

$$195^2 + 1771(3^4, 4^4, 5^4) = 426^2, 701^2, 1070^2,$$

$$460^2 + 1001(3^4, 4^4, 5^4) = 541^2, 684^2, 915^2,$$

$$462^2 - 299(3^4, 4^4, 5^4) = 435^2, 370^2, 163^2.$$

This node is adjacent to nodes labelled (13, 20, -33), (39, 115, -154), (15, -92, 77), (-65, 23, 42). The first of these is adjacent to the node labelled (4, 7, -11) in the path for $N = 11$, so these nodes lie in the principal component of the graph. A search for solutions involving other Pythagorean triples in this way has been unsuccessful, but this search was limited by the very rapid increase of the size of the integers involved in the calculation.

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