1

Special Relativity Theory

1.1 Basics

Problem 1.1 a) In Figure 1.1, a spacetime diagram for an observer $\mathcal{O}$ with an inertial frame is shown along with another observer $\mathcal{O}'$ with another inertial frame. The 4-vectors $\vec{A}$, $\vec{B}$, $\vec{U}$, and $\vec{V}$ are drawn. Which of the following statements are true?

1. In $\mathcal{O}$'s inertial frame, the scalar product between $\vec{A}$ and $\vec{B}$ is zero.
2. In $\mathcal{O}'$'s inertial frame, the scalar product between $\vec{A}$ and $\vec{B}$ is zero.
3. The scalar product between $\vec{A}$ and $\vec{B}$ is always nonzero.
4. In $\mathcal{O}$'s inertial frame, the scalar product between $\vec{U}$ and $\vec{V}$ is zero.
5. In $\mathcal{O}'$'s inertial frame, the scalar product between $\vec{U}$ and $\vec{V}$ is zero.
6. The scalar product between $\vec{U}$ and $\vec{V}$ is always nonzero.

b) Which of the 4-vectors in a) could be proportional to a 4-velocity? Explain why.

Problem 1.2 Show that

a) every 4-vector (i.e., vector in Minkowski space) that is orthogonal to a timelike 4-vector is spacelike.

b) the sum of two future directed time-like 4-vectors is another future directed timelike 4-vector.

c) every space-like 4-vector can be written as the difference between two future-directed lightlike 4-vectors.

d) the inner product of two future-directed timelike 4-vectors is positive.

Problem 1.3 In a particular inertial frame, two observers have the 3-velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. Find an expression for the gamma factor of observer 2 in the rest frame of observer 1 in terms of these velocities.

Problem 1.4 a) Can a rest frame be chosen for a photon? Explain why!

1. always
2. sometimes
3. never

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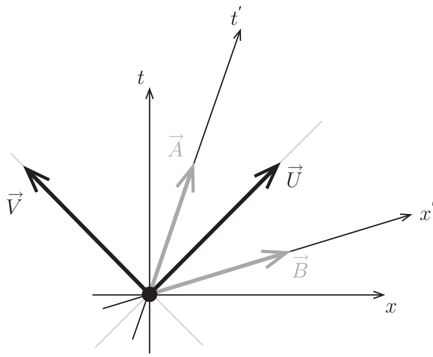

Figure 1.1 Spacetime diagram for observers $\mathcal{O}$ and $\mathcal{O}'$.

b) Can a rest frame be chosen for the center of momentum for a system of two photons? Explain why!

1. always
2. sometimes
3. never

1.2 Length Contraction, Time Dilation, and Spacetime Diagrams

Problem 1.5 a) State, explain, and derive the formula for length contraction in special relativity.

b) State, explain, and derive the formula for time dilation in special relativity.

Problem 1.6 A rod with length of 1 m is inclined 45° in the xy-plane with respect to the x-axis. An observer with the speed $\sqrt{2/3}\,c$ approaches the rod in the positive direction along the x-axis. How long does the observer measure the rod to be and at which angle does (s)he observe it to be inclined relative to its x-axis?

Problem 1.7 When the primary cosmic rays hit the atmosphere, muons are created at an altitude between 10 km and 20 km. A muon in the laboratory lives on average the time $\tau_0 = 2.2 \cdot 10^{-6}$ s before it decays into an electron (or a positron) and two neutrinos. Even though a muon can only move $\tau_0 c \approx 660$ m under the time τ_0, a large fraction of the muons will reach the surface of the Earth. How can this be explained? Make a numerical computation for a muon that moves with velocity $0.999c$.

Problem 1.8 An express train passes a station with velocity v. A measurement of the length of the train can be performed in the following different ways:

a) A "continuum" of linesmen is ordered to align along the track. The two men that see the front or the end of the train pass in front of them when their watches show 12:30 make a mark where they stand. The distance L_a between the marks is measured.

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b) One conductor goes to the front of the train and another goes to the end. When the watches of the conductors show 12:15, they quickly drive a nail into the track. The linesmen measure the distance L_b between the nails.

c) The stationmaster inspects the receding train through a pair of binoculars. Through the binoculars the stationmaster sees the front of the train to be at the semaphore A at the same time as its end is at the railway point B. The linesmen measure the distance L_c between A and B.

d) The stationmaster uses a radar to measure the length of the train. The arrival times of the radar pulses reflected from the front and end of the receding train are t_1 and t_2, respectively. The distance $L_d = (t_1 - t_2)c/2$ is a measure of the length of the train.

Express L_a, L_b, L_c, and L_d in terms of L_0, the rest length of the train.

Problem 1.9 A hitchhiker in the Milky Way sits waiting on a small asteroid when a formidably long express space cruiser passes very close to the asteroid. Just as the rear end is opposite to the hitchhiker, (s)he sees lanterns in the front and in the rear end of the cruiser go on simultaneously. Actually, the rear watchman also saw them go on, but according to his hydrogen maser wristwatch he measured a small time difference of $4 \cdot 10^{-9}$ s between the lightening of the forward and rear lanterns. From the type indication on the cruiser – X2000 – our hitchhiker realized that its length was $2 \cdot 10^3$ m. Had they known what you know, they could have calculated the speed of the cruiser. What was it, according to Einstein's special theory of relativity?

Problem 1.10 Two lamps, which are separated by the distance ℓ in an inertial coordinate system K, are switched on simultaneously (in K). In another inertial coordinate system K', an observer measures the distance between the lamps to be ℓ' and observes the lamps go on with the time difference τ. Express ℓ in terms of ℓ' and τ. Assuming that the inertial coordinate system K' is moving along the axis connecting the two lamps, also find the expression for the relative velocity v between the two inertial coordinate systems.

Problem 1.11 A rod of length ℓ lies in the xz-plane of a coordinate system. If the angle between the rod and the x-axis is θ, calculate the length of the rod as seen by an observer moving with velocity v along the x-axis.

Problem 1.12 Two events A and B with coordinates x_A and x_B are simultaneous for an observer K with rest frame S. Another observer, K', moving with velocity $-u$ along the x-axis of S measures these events to not be simultaneous, but such that B is earlier than A by the amount $\Delta t'$. What is the distance L between the events A and B expressed in the frame of K if it is L' in the rest frame of K?

Problem 1.13 An observer S with rest frame K observes two events x_α and x_β. The α event takes place at the origin and the β event 2 years later at a distance of 10 light years (ly) forward along the x^1-axis. Another observer S' with rest frame K' moves with velocity v along the x^1-axis of K, passing S at the origin. The observer S' instead observes the β event 1 year later than the α event.

a) How far away does S' find the β event?

b) What is the relative velocity between S and S'?

Problem 1.14 The ratio $R(\mu/e)$ of muon neutrinos to electron neutrinos measured at ground level from the cosmic radiation is $R(\mu/e) = 2$ at low energies. These neutrinos come from the decay of pions, created by the primary cosmic radiation, which consists mostly of protons. The relevant reaction chain can be written in simplified form as follows

$$\pi \longrightarrow \mu + \nu_\mu,$$
$$\mu \longrightarrow e + \nu_\mu + \nu_e.$$

As we can see there are two muon neutrinos ν_μ produced for every electron neutrino ν_e. When the energy of the muon neutrinos, and therefore the muons, is high enough this ratio goes up, since the muons hit the Earth before they decay, and no electron neutrinos are produced. In the muon's rest frame, the muon lifetime is $\tau_0 = 2.2$ μs. The speed of light is $3 \cdot 10^8$ m/s. What is the smallest energy of the muons that hit the ground before they decay substantially if they are produced at an altitude of 10 km above ground? The rest mass of the muon is 106 MeV.

Problem 1.15 A circular accelerator has a radius of 50 m. How many turns can a muon take on average in this ring before it decays if its energy is kept constant at 1 GeV? The average lifetime of the muon in its rest frame is 2.2 μs and the muon mass is 106 MeV.

Problem 1.16 Consider a triangle at rest in the inertial system K with sides of length $a = 3\ell$, $b = 4\ell$, and $c = 5\ell$ in K.

a) Compute the lengths of the sides and the area of this triangle as measured in an inertial frame K' moving with constant velocity v parallel to the a-side of the triangle.

b) Same as in a), but now the observer K' moves parallel to the c-side of the triangle.

Problem 1.17 Consider a pole of proper length L moving along the x-axis in the negative direction with a constant velocity so that the pole is parallel to the x-axis (see Figure 1.2). At a fixed time, an observer at rest at the spatial origin sees (the optical effect is referred to) the front of the pole at an angle $\pi/3$ with the x-axis, a mark on the pole at an angle $\pi/4$, and the end of the pole at an angle $\pi/6$. What is the quotient r between the distance from the front of the pole to the mark and the full length of the pole?

Problem 1.18 Two spaceships, which are initially at rest in some common rest frame, are connected by a straight tensionless string. At time $t = 0$ in this frame, both spaceships start to accelerate in the same direction, in the direction of the string, such that their separation is constant in the initial rest frame. Both spaceships agree to stop accelerating once a predetermined time t_0 has passed in the initial rest frame.

a) Does the string break, i.e., does the distance between the two spaceships increase in the new rest frame of the spaceships?

b) If the distance between the spaceships is originally 40 km, $t_0 = 30$ s, and the spaceships have constant acceleration of 1/50 c/s in the initial rest frame, what is the

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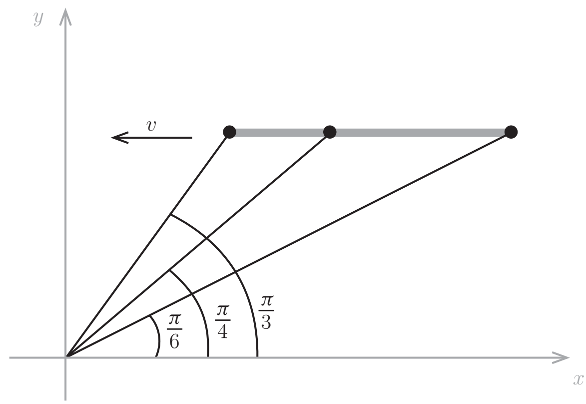

Figure 1.2 The pole is moving in the negative direction of the x-axis with a constant velocity v. The z-direction is neglected.

distance between the two spaceships in the frame of the leading spaceship after the engines are turned off?

Problem 1.19 Professor A. Einstein is traveling in a train on a rainy night. He is situated in the exact middle of the train, and suddenly lightning strikes right next to him. The train has reflectors in the rear and front, and since the reflections from rear and front reach him at the same time, he falls into slumber convinced that the reflections happened at the same time and that the speed of light is the same in both directions. What he did not see was that Professor W. Wolf was standing on the ground, also next to the lightning strike, observing the events. Draw spacetime diagrams showing how the light signals travel in each of the professors' rest frames. Use these to answer (including motivation) the following

a) Does Professor Wolf see the light reflections reaching Professor Einstein at the same time?

b) Would Professor Wolf agree with Professor Einstein that the light signals were reflected at the same time?

c) Do the reflections reach Professor Wolf at the same time?

Problem 1.20 Two rockets with rest lengths L and $2L$, respectively, move with constant velocities on an interstellar highway. Since the velocities are different, the rockets will pass each other. Call the event when the front of the faster rocket reaches the slower rocket A and the event when the end of the faster rocket reaches the front of the slower rocket B (see Figure 1.3). In each rocket there is an observer. Draw one or more spacetime diagrams describing the events, and use it/them to deduce which observer will consider time between A and B to be larger (the observer in the short rocket or the observer in the long rocket).

Problem 1.21 Muons created by cosmic rays hitting the atmosphere have a lifetime of $2.2 \cdot 10^{-6}$ s. If the muons are created at a height of 10 km, the time to reach the surface of the Earth (measured in the rest frame of the Earth) is at least 10 km/$c \simeq 3 \cdot 10^{-5}$ s, yet a large fraction of the muons can be measured at sea level.

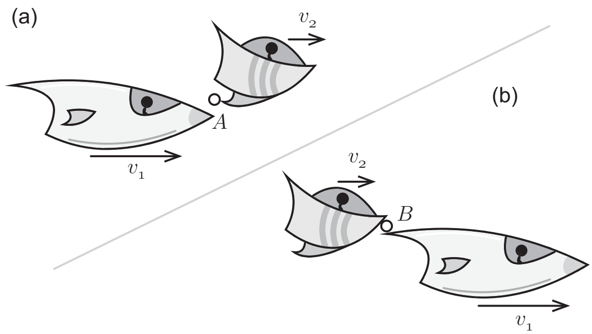

Figure 1.3 Two rockets with rest-lengths L and $2L$, respectively. Part (a) of the figure shows the event A, whereas part (b) shows the event B.

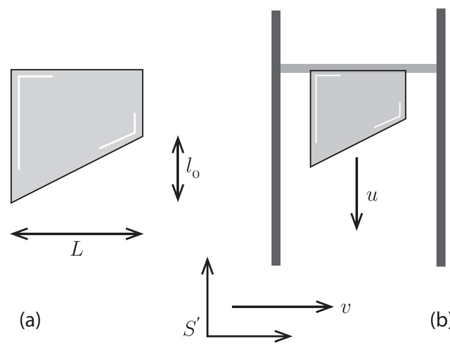

Figure 1.4 Part (a) of the figure shows the dimensions of a guillotine blade in its rest frame, whereas part (b) shows the event of the Scarlet Pimpernel riding by on his horse at velocity v (S' is the rest frame of the Scarlet Pimpernel and his horse) and the guillotine blade falls at velocity u in its own rest frame.

Explain qualitatively why this occurs by describing the situation using spacetime diagrams.

Problem 1.22 During the French Revolution, guillotines with a slanted blade were used to decapitate nobility. The guillotine blade at rest has the dimensions shown in Figure 1.4. Eager to save the nobility, the Scarlet Pimpernel rides by on his horse at velocity v. How fast does he have to ride in order for the guillotine blade to be horizontal in his rest frame S' if it falls at velocity u in the guillotine rest frame?

Problem 1.23 In an inertial frame S, two lights located on the positive x-axis are moving in the negative x-direction at speed v. An observer placed in the origin of S looks at the light signals coming from the lights. What is the distance between the *seen* positions of the lights if their separation in their common rest frame is ℓ_0?
Note: The problem is asking for the separation *as seen* by the observer, not the actual distance between the lights at a given time.

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1.3 Lorentz Transformations and Geometry of Minkowski Space

Problem 1.24 Verify directly from the form of the Lorentz transformation representing a boost in the x-direction that any object traveling at speed c in an inertial frame S travels at speed c in the boosted frame.

Problem 1.25 A train passes a station just after sunset. The length of the train is L. In the front and in the rear, it has two lanterns. The lanterns are turned on simultaneously in the train's rest frame. A stationman observes the train pass with velocity v. Does the stationman see the lanterns go on simultaneously? If not, what is the time difference between the turning on of the two lanterns for the stationman, expressed in terms of L and v?

Problem 1.26 An observer O on a train of length L and velocity v relative to the ground is standing at a distance xL $(0 \leq x \leq 1)$ from the front A of the train. When the light from the lamps at A and B, at the rear, reach him/her simultaneously, (s)he can calculate at which times $t_1(A)$ and $t_2(B)$ they turned on. Another observer O' on the ground can also determine these two times t'_1 and t'_2 in his rest frame, where the light reaches him as O just passes him. If (s)he then finds that $t'_1 = t'_2$, it turns out that the velocity v of the train can be expressed as a rather simple function of x. Find this function and show that if $v = 0$, then $x = 1/2$.

Problem 1.27 A particle of mass m and energy E falls from zenith to the Earth along the z-axis in the rest frame of observer K. Another observer, K', moves with velocity v along the positive x-axis of K and will observe the particle to approach K' with an angle θ relative to the z'-axis.

a) Calculate the angle θ expressed in terms of the velocity u of the particle and the velocity v of K'.

b) Based on the result of a) give a description of how the starry sky would look like for a space cruiser moving with high speed in our galaxy.

Problem 1.28 Consider a particle with 4-velocity $V = \gamma(v')(c, v', 0, 0)$. By making a Lorentz transformation with velocity $-v$ along the x^1-axis, show that you can obtain the formula for relativistic addition of velocities, by expressing the velocity v'' of the particle in the new system in terms of the velocity v' in the old system and the velocity v of the motion of the observer.

Problem 1.29 Consider an equilateral triangle with sides of length ℓ, which is at rest in the inertial coordinate system K. Assume that one of the sides in the triangle is parallel to the x^1-axis of K. In an inertial coordinate system K' moving relative to K with velocity v along the positive x^1-axis of K, an observer measures the lengths of the sides and angles in the triangle. What expressions in ℓ and v for the lengths and angles does the observer find?

Problem 1.30 An observer K' is moving with constant speed v along the positive x^1-axis of an observer K. A thin rod is parallel to the x'^1-axis and moving in the direction of the positive x'^2-axis with relative velocity u. Show that according to the observer K the rod forms an angle ϕ with the x^1-axis, with

$$\tan\phi = -\frac{uv/c^2}{\sqrt{1-v^2/c^2}}. \tag{1.1}$$

Problem 1.31 A cylinder is rotating around its axis with angular velocity ω (rad/s) in an inertial system. A straight line is drawn along the length of the cylinder. Show that the observer in an inertial system, which moves with velocity v parallel to the direction of the cylinder axis, will measure the line as twisted around the cylinder. Determine the twist angle per unit length.

Problem 1.32 A fast train (velocity v) is passing a station during the night. As the train passes the station, all compartment lights are turned on simultaneously with respect to the rest frame of the train. Relative to an observer standing at the station, the lights seem to be turned on at various times. Compute the velocity u of the line separating the illuminated and unilluminated parts of the train in the station rest frame.

Problem 1.33 A planet is moving along a circular orbit (radius R and angular velocity ω) around a star. A space ship is passing by the star, orthogonal with respect to the plane of motion of the planet, with velocity v. Compute the orbit of the planet in the rest frame coordinates of the space ship.

Problem 1.34 An observer B is moving with constant velocity v along the positive x^1-axis in the rest frame K of an observer A. An observer C is moving with constant velocity v' along the positive x'^2-axis in the rest frame K' of the observer B. Compute the absolute value of the relative velocity of C with respect to A. What is the time interval Δt between two events E_1 and E_2 that occur at the same spatial point with time difference $\Delta t''$ in the rest frame K'' of observer C.
Hint: It is sufficient to compute the time coordinate x''^0 of C as a function of the coordinates x^μ of A.

Problem 1.35 Let x be a lightlike vector in Minkowski space. Show that

$$u = N\begin{pmatrix} x^0 + x^3 \\ x^1 + ix^2 \end{pmatrix}, \tag{1.2}$$

where N is a real normalization factor, u is a spinor that satisfies $X \propto uu^*$, where X is a complex 2×2 matrix, so that $\det X = \det(uu^*) = 0$. Normalize this spinor by the requirement that $\operatorname{tr} X = 2x^0$.

A Lorentz transformation along the 3-axis is given by

$$a(v) = \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix}, \tag{1.3}$$

where $\tanh\theta = v/c$. Show explicitly that this transformation satisfies

$$a(v)u = u(L(a(v))x), \tag{1.4}$$

where $L(a(v))x$ is the Lorentz-transformed vector and u is the normalized spinor.

Problem 1.36 Use Einstein's postulate to derive the expressions for a Lorentz boost in the x-direction.

Problem 1.37 In an inertial frame S, rockets A and B traveling with velocities v and $-v$, respectively, pass each other at time $t = 0$ at the spatial origin. A time t_0 later, light signals are sent from the origin toward each of the spaceships. Compute the time difference between the spaceships receiving the light signals in the rest frame of one of the rockets.

1.4 Relativistic Velocities and Proper Quantities

Problem 1.38 a) Explain the concept of "relativity of simultaneity." Illustrate it in a spacetime diagram.

b) The worldline of a massive particle in Minkowski space is described by the following equations in some inertial frame $(x^\mu) = (ct, x, y, z)$,

$$x(t) = \frac{3}{2}at^2, \quad y(t) = 2at^2, \quad z(t) = 0, \tag{1.5}$$

where a is constant and $0 \leq t \leq t_0$ for some value of t_0. Compute the particle's 4-velocity and 4-acceleration components. What values of t_0 are possible and why? Compute the proper time along this worldline from $t = 0$ to $t = t_0$.

Problem 1.39 A rod moves with velocity v along the positive x-axis in an inertial frame S. An observer at rest in S measures the length of the rod to be L. Another observer moves with the velocity $-v$ along the x-axis. What length, expressed as a function of L and v, will this observer measure for the rod? The measurement is done as usual with the endpoints being measured simultaneously for each observer in their respective frames.

Problem 1.40 The worldline of a particle is described by the coordinates $x^\mu(t)$ in the system S. An observer at rest in the system S', with velocity u along the positive x^2-axis relative to S, measures the velocity of the particle at time t'. Express his result as a function of the velocity of the particle in S and u.

Problem 1.41 A spaceship is moving away from Earth. The effect of the engines is regulated so that the the passengers feel the constant acceleration g. Calculate the distance between the Earth and the spaceship (measured in the rest frame of the Earth) as a function of

a) the time on Earth.

b) the time on the spaceship.

The commander of the spaceship is 40 years of age at the beginning of the voyage. How old is (s)he when the spaceship reaches the Andromeda Galaxy, which lies about 2 500 000 light years away from Earth?
Hint: 1 year $\approx \pi \cdot 10^7$ s and $g \approx 10\,\text{m/s}^2$.

Problem 1.42 A rocket (with rest mass m_0) starts from rest at the origin of a coordinate system K. Its velocity along the positive x-axis is increased by shooting

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matter from the rocket with constant velocity w relative to the instantaneous rest frame of the rocket in the negative x-direction. Compute the remaining mass m of the rocket as a function of its velocity v with respect to the origin of K.

Problem 1.43 You and your friend are in intergalactic space (assume the Minkowski metric). You leave simultaneously from a space station, with equal speeds v, in orthogonal directions. Neglect acceleration. After a time T has passed in your inertial frame, you want to send a message to your friend using a light signal. In which direction (in your rest frame) should you send it?

Problem 1.44 a) In an inertial frame S, an object travels with 3-velocity u. A different inertial frame S' is moving in the negative x-direction relative to S with relative speed v'. Write down the 4-velocities of the object and S' in S.

b) Show that the gamma factor of an object in any inertial frame is given by the inner product of the 4-velocity of the object and the 4-velocity of an object at rest in the frame.

c) Express the gamma factor of the object in a) in the frame S' using the result from b).

Problem 1.45 Show that the 4-velocity $V^\mu = dx^\mu/d\tau$ and 4-acceleration $A^\mu = dV^\mu/d\tau$ of an object are always perpendicular, where τ is the proper time of the object such that $V^2 = 1$.

Problem 1.46 You are traveling in your spaceship in flat intergalactic space such that special relativity can be used. You are on your way to a space station when you suddenly discover an enemy spaceship on your radar. You immediately send out a light signal for help to the space station. When you send out your signal, the distance in your coordinate system to the space station is 1 light day. You have a relative speed of $c/4$ toward the space station. When the space station receives your signal, they send out a rescue spaceship with a speed $3c/4$ relative the space station. How long does it take before you get help (as measured by your own clock)? First, how long does it take your light signal to reach the space station, and second, how long does it take for the rescue spaceship to reach you?

Problem 1.47 On an interstellar highway there is a speed limit of u relative to a reference frame S. A member of the intergalactic police force is at rest in this system when a spaceship passes at constant velocity $v > u$ (see Figure 1.5). Eager to do the job properly, the police officer starts the pursuit, accelerating with constant proper acceleration a. The pursuit ends when the police officer catches up with the criminal.

a) How long does the pursuit take according to the criminal?

b) How long does the pursuit take according to the police officer?

c) What is the relative velocity between the police officer and the criminal at the end of the pursuit?

Problem 1.48 An astronaut on an accelerated spaceship uses a coordinate system (T, X, Y, Z) related to an inertial system (t, x, y, z) as follows (we set $c = 1$)

$$t = X \sinh(aT), \quad x = X \cosh(aT), \quad y = Y, \quad z = Z. \tag{1.6}$$

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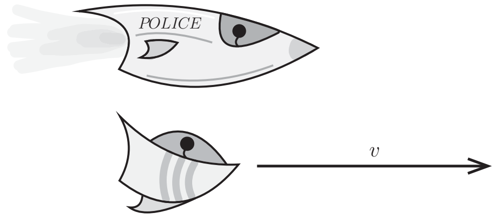

Figure 1.5 A space ship passing the intergalactic police at a relative constant velocity v.

a) Compute the metric tensor in the astronaut's coordinate system. (The metric in the inertial system is, of course, $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$.)

b) Let $(k^\mu) = (\omega, \omega\cos(\theta), 0, \omega\sin(\theta))$ be the 4-wavevector of a photon emitted at time $t = t_0$ at the position $x = x_0$, $y = z = 0$ in the abovementioned inertial system. Compute the components of this 4-wavevector in the astronaut's coordinate system.

c) Compute the duration of a trip of the spaceship on the astronaut's watch (i.e., the proper time) if the trajectory of his spaceship on this trip is, in the astronaut's coordinate system, $X(T) = X_0 =$ constant, $Y(T) = vT$ for some constant $v > 0$, $Z(T) = 0$, and $0 \le T \le T_0$.

Problem 1.49 A rocket A is accelerating with constant proper acceleration α such that its worldline is given by $t^2 - x^2 = -1/\alpha^2$ in the reference frame S. Assume that S' is a different reference frame related to S by a Lorentz transformation in standard configuration with velocity v. What is the coordinate acceleration a' in the system S' at time $t' = 0$?

Problem 1.50 Two observers A and B are initially colocated at rest in the inertial system S. At a given time in S, observer B starts accelerating with a proper acceleration α. A time t_0 later (as measured by A), a light signal is sent from A toward B. Find an expression for the proper timed elapsed for observer B when B receives the signal. Discuss the limiting cases.

Problem 1.51 Particles in a circular accelerator are accelerated by an electromagnetic field in such a way that they are kept in a circular orbit with constant velocity. What are the corresponding 4-acceleration and proper acceleration of the particle and what is the eigentime required for the particles to complete one orbit? Introduce any quantities required to solve the problem.

Problem 1.52 An object of internal energy M moving with 4-velocity V is being acted upon by a force $F = fU$, where the known 4-vector U fulfills $U^2 = 1$ and f is a scalar. How fast is the internal energy of the object increasing (with respect to its proper time) and what is the proper acceleration of the object?

Problem 1.53 Consider a 4-force $F^\mu = (0, \boldsymbol{f})^\mu$ acting on an object of rest energy m with 3-velocity $\boldsymbol{v}$. Compute the rate of change in the rest energy $dm/d\tau$ and the

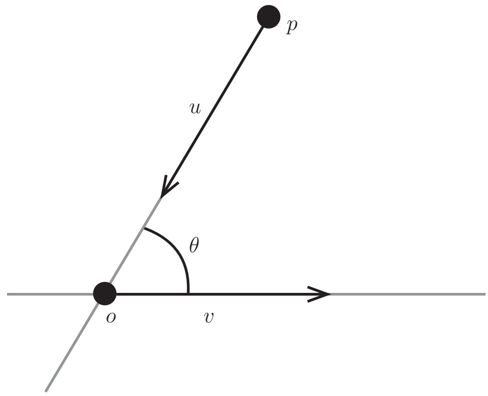

Figure 1.6 An observer o moving at velocity v and a particle p moving toward the observer o at velocity u in a direction that makes an angle θ with the direction of the velocity v.

proper acceleration α, where τ is the proper time of the object worldline. Discuss the requirements on $\boldsymbol{f}$ for the 4-force to be a pure force.

Problem 1.54 Given an object with acceleration $\boldsymbol{a}_0$ in its instantaneous rest frame S, find an expression for the acceleration in the inertial frame S', which is moving in the x-direction with velocity v relative to S. What is the maximal and minimal acceleration in S' depending on the direction of the acceleration based on your expression?

Problem 1.55 A particle has 4-momentum $P = (E, \boldsymbol{p})$ in an inertial frame S. An observer is moving by with velocity v in the x-direction. Compute the total energy this observer will measure for the particle and the velocity of the particle in the x'-direction of the observer's rest frame.

Problem 1.56 An object originally at rest with mass $m(0) = m_0$ is affected by a constant 4-force $F^\mu = f(1, \mathbf{1})^\mu$. Find the object's mass and the time elapsed in the initial rest frame as a function of the object's proper time τ.

Problem 1.57 An observer o is moving at velocity v in the x-direction in an inertial frame S. In the same inertial frame, a particle p hits the observer while traveling at a speed u in a direction that makes an angle θ with the negative x-direction, see Figure 1.6. What is the speed and angle that the observer will measure for the particle? Verify that your result is consistent with both the ultrarelativistic and nonrelativistic limits, i.e., $u \to 1$ and $u, v \ll 1$, respectively.

Problem 1.58 In an inertial frame S an object starting at rest at $t = 0$ is moving with constant *coordinate acceleration* a, i.e., $x = at^2/2$. Determine the proper time for the object to reach the speed v_0 in S and the proper acceleration of the object as a function of the time t in S.

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1.5 Relativistic Optics

Problem 1.59 In 1851, Fizeau measured the speed of light in running water. His result can be summarized in the formula

$$u = u_0 + kv, \tag{1.7}$$

where u is the speed of light in water, that runs with velocity v. The speed of light in water at rest is u_0 and the *drag coefficient* k is given by

$$k = 1 - \frac{1}{n^2}, \tag{1.8}$$

where $n = c/u_0$ is the refractive index of water. Explain Fizeau's result!

Problem 1.60 See Problem 1.59. Is Fizeau's result still valid if the water runs perpendicular to the motion of light? If not, what is the correction?

Problem 1.61 In 1965, Maarten Schmidt at the Mount Palomar Observatory could identify the strongly redshift Lyman α line in the spectrum of the quasi-stellar radio source 3C 9. Normally, this line has the wavelength 1 215 Å. Schmidt instead found the value 3 600 Å for this line in this radio source. It is possible to explain the redshift in terms of the Doppler effect. This would imply that 3C 9 moves with an enormous speed relative to our galaxy. Determine a lower bound for the speed of 3C 9.

Problem 1.62 A plane electromagnetic wave moving along the x^1-axis has the form

$$E(x) = E_0 \sin\left[2\pi\left(\frac{x^1}{\lambda} - \nu t\right)\right]. \tag{1.9}$$

Introduce the angular frequency $\omega = 2\pi\nu$ and show that the argument of the wave can be written in the form $-x_\mu k^\mu$, where $k = \left(\frac{\omega}{c}, \frac{\omega}{c}, 0, 0\right)$ is the 4-wave vector of the light wave traveling along the positive x^1-axis. Show that this vector is lightlike and deduce the formula for the Doppler shift by calculating the change in angular frequency ω under a Lorentz transformation along the x^1-axis. What does the formula for the Doppler shift look like expressed in terms of the rapidity θ?

Problem 1.63 A *gamma ray burst* (GRB) observed in a cluster of faraway galaxies is time dilated and therefore has a total duration about twice as long as GRBs in nearby galaxies. According to the Hubble law, the recession speed is proportional to the distance to the GRB. Calculate the Doppler redshift $z = \Delta\lambda/\lambda_0$ of a typical spectral line from the distant GRB, where λ and λ_0 are the observed and emitted wavelengths, respectively.
Hint: All GRBs can be considered to have the same duration when measured in their respective rest frames.

Problem 1.64 A person watches two objects with constant velocities on a collision course, i.e., they approach each other on a straight line.

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a) Assuming that both objects' velocities have the same absolute value $c/2$ in the person's frame of reference, compute the absolute value of the velocity with which a person traveling with the first object sees the other object approaching.

b) Assume that the first object sends a light pulse from a ruby laser, which produces visible light with a wavelength $\lambda_0 = 694.3$ nm, toward the second object. Compute the wavelength of this light pulse as seen by an observer on the second object.

Problem 1.65 A light source is moving at speed v and at an angle θ relative to the separation between the source and a stationary observer.

a) Consider a light pulse with frequency ω_0 in the rest frame of the source and determine the frequency ω measured by the observer.

b) Compute the angle θ for which $\omega = \omega_0$.

Problem 1.66 A large disk rotates at uniform angular speed Ω in an inertial frame S. Two observers, O_1 and O_2, ride on the disk at radial distances r_1 and r_2, respectively, from the center (not necessarily on the same radial line). They carry clocks, C_1 and C_2, which they adjust so that the clocks keep time with clocks in S, i.e., the clocks speed up their natural rates by the Lorentz factors

$$\gamma_1 = \frac{1}{\sqrt{1 - r_1^2\Omega^2/c^2}}, \quad \gamma_2 = \frac{1}{\sqrt{1 - r_2^2\Omega^2/c^2}}, \tag{1.10}$$

respectively. By the stationary nature of the situation, C_2 cannot appear to gain or lose relative to C_1. Deduce that, when O_2 sends a light signal to O_1, this signal is affected by a Doppler shift $\omega_2/\omega_1 = \gamma_2/\gamma_1$.
Note that, in particular, there is no relative Doppler shift between any two observers equidistant from the center.

Problem 1.67 A light source is moving with speed v through an optical medium with refractive index n. Derive an expression for the ratio between the frequency in the frame of the medium and the frequency in the frame of the source as a function of v, n, and the angle θ between the movement direction of the source and the propagation direction of the light (in the frame of the medium).

Problem 1.68 In an inertial frame S, a mirror is oriented perpendicular to the x-axis and moving with velocity v in the x-direction, see Figure 1.7. A light pulse with frequency ω approaches the mirror at an angle θ_{in} in S. What is the scattering angle θ_{out} and what frequency does the outgoing light have? Explain what happens when $v < -\cos\theta_{\text{in}}$?

Problem 1.69 In an inertial frame S, a light pulse is being directed at an optical medium with refractive index n which is moving with velocity v orthogonal to its surface, see Figure 1.8. In the rest frame of the medium, the light pulse is refracted according to Snell's law. An observer in S makes the interesting observation that the light pulse is still traveling in the same direction after entering the medium. Compute the index of refraction for the medium in terms of the velocity v and the angle θ' between the initial direction of the light pulse and the direction of motion for S in the rest frame of the medium.

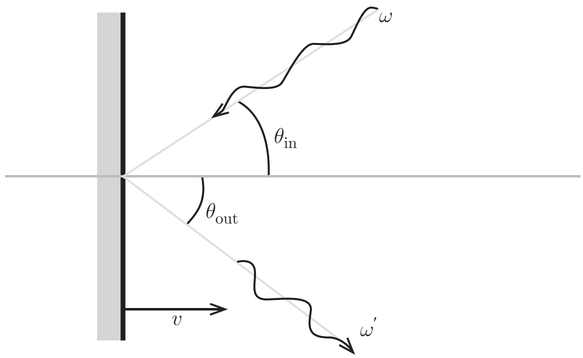

Figure 1.7 A mirror perpendicular to the x-axis moving with velocity v in the x-direction. The ingoing light has frequency ω and makes an angle θ_{in} with the x-axis, whereas the outgoing light has frequency ω' and makes an angle θ_{out} with the x-axis.

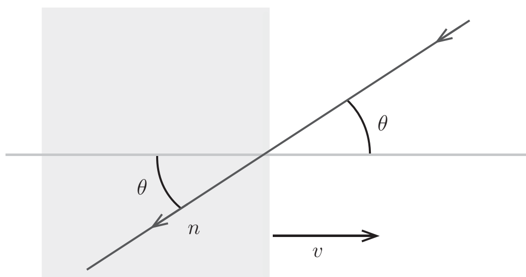

Figure 1.8 An optical medium with refractive index n moving with velocity v orthogonal to its surface. In the inertial frame S, an observer makes the observation that a light pulse is traveling in the same direction (at an angle θ relative to the velocity v) before and after entering the medium.

Problem 1.70 A sine wave propagating in a medium can be described by the function $\sin(N \cdot x)$ in the medium rest frame. Here, $(N^\mu) = (\omega, k)$, where ω is the angular frequency and k the wave number. Assuming the wave velocity in the medium is u, the relationship between k and ω is $ku = \omega$. A source with internal frequency ω_0 is moving through the medium with velocity v. Compute the Doppler shifted frequency ω in the medium rest frame when the waves are traveling in the direction of motion. Also discuss the special cases $v = u$ and $v = -u$ and make sure that your solution reduces to the classical Doppler formula when $v \ll 1$.

Problem 1.71 Using the same setup as in Problem 1.50 and assuming that A sends the light signal using a frequency ω, compute the frequency observed by B. In addition, if B carries a mirror and reflects the signal back at A, find the frequency observed by A for the reflected signal.

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1.6 Relativistic Mechanics

Problem 1.72 The rest energy of an electron is about 0.51 MeV, i.e., the energy a charged particle, with charge equal to the electron charge, would receive when falling down a potential difference of 0.51 MV. Assuming that the electron is accelerated through a linear accelerator (starting from rest) with a potential difference of 10^6 V. Compute the final velocity of the electron.

Problem 1.73 a) An electron e^- (with mass m_e) collides with a positron e^+ (i.e., the antiparticle of the electron with the same mass m_e as the electron). Show that they cannot annihilate into a single photon γ (a photon has zero mass), i.e., the process $e^- + e^+ \longrightarrow \gamma$ is impossible due to conservation of energy and momentum.

b) Also show that an electron cannot spontaneously emit a photon.

c) Can an electron colliding with a positron annihilate into two photons? Justify your answer.

Problem 1.74 An elementary particle with mass M decays into two particles a and b with masses m_a and m_b, respectively. Calculate the momentum of particle a in the rest frame of particle b.

Problem 1.75 A particle A with mass m_A decays into two particles B and C with masses m_B and m_C, respectively. Assume that particle A has speed v_A before the decay and that particle B is at rest after the decay, i.e., $\boldsymbol{p}_B = \boldsymbol{0}$. Express the speed v_A in the masses m_A, m_B, and m_C.

Problem 1.76 Two particles, 1 and 2, with masses m_1 and m_2, respectively, collide and form a new particle with mass M. Calculate the mass M and the velocity $\boldsymbol{v}$ of this new particle in the rest frame of particle 2 as a function of the velocity $\boldsymbol{v}_1$ of particle 1 in the rest frame of particle 2 and the masses m_1 and m_2.

Problem 1.77 a) Two particles with rest masses m_1 and m_2, respectively, move along the x-axis in the inertial frame of some observer at uniform velocities u_1 and u_2, respectively. They collide and form a single particle with rest mass m moving at uniform velocity u. Assuming that $c = 1$, prove that

$$m^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma(u_1)\gamma(u_2)(1 - u_1u_2), \tag{1.11}$$

and also find u.

b) Show that the above expression can be written as

$$m^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma(v), \tag{1.12}$$

when v is the velocity of particle 2 as measured in the rest frame of particle 1.

c) Consider two different situations and in both of the situations the relative velocity v as defined above is the same, and thus, the rest mass m is the same in both situations, but in one $u_1 = 0$ and in the other $m_1\gamma(u_1)u_1 = -m_2\gamma(u_2)u_2$. What is the difference in total energy for the two situations in the frame of the observer?

Problem 1.78 A pion with mass m_π and energy E_π moves along the x-axis. It decays into a muon with mass m_μ and a neutrino with approximately zero mass. Calculate the energy E_μ of the muon when it moves at a right angle relative to the x-axis in terms of the velocity of the incoming pion and the masses.

Problem 1.79 A pion with mass m_π decays into an electron with mass m and an antineutrino with mass m_ν. Calculate the velocity of the antineutrino in the rest frame of the electron as a function of the masses of the particles and determine the limiting value of this velocity as the mass of the antineutrino goes to zero.

Problem 1.80 In June 1998, the Super-Kamiokande Collaboration in Japan reported that it had found evidence for massive neutrinos. Super-Kamiokande measures so-called atmospheric neutrinos, which are produced in hadronic showers resulting from collisions of cosmic rays with nuclei in the upper atmosphere. Two of the dominating processes in the production of atmospheric neutrinos are

$$\pi^+ \longrightarrow \mu^+ + \nu_\mu,$$

where π^+ is a pion, μ^+ is an antimuon, and ν_μ is a muon neutrino, followed by

$$\mu^+ \longrightarrow e^+ + \bar{\nu}_\mu + \nu_e,$$

where e^+ is a positron, $\bar{\nu}_\mu$ is an antimuon neutrino, and ν_e is an electron neutrino.

a) Calculate the kinetic energy of the antimuon, T_{μ^+}, and the absolute value of the 3-momentum of the muon neutrino, p_{ν_μ}, when the pion decays at rest according to the first decay. Despite the small mass of the muon neutrino, neglect it! The mass of the pion is m_π and the mass of the antimuon is m_μ.

b) How far will one of the antimuons, which are produced in the first decay, travel (on average) in the pion rest frame before it decays according to the second decay? The mean lifetime of an antimuon at rest is τ_μ.

Problem 1.81 The pions in the sky that are decaying into muons as in Problem 1.14 are produced in collisions between protons in the primary cosmic rays and nitrogen or oxygen in the air. When a pion with energy of 2 GeV is produced, what energy does the muon have if it continues in the same direction as the pion? The expression can be simplified due to the high energy of the pion. What is the resulting expression? What is the muon energy? The pion has a rest mass of 140 MeV, and the neutrino mass can be neglected.

Problem 1.82 A beam of protons that are accelerated to a very high energy hits a beryllium target and produces a shower of particles. Two detectors are placed in a plane behind the target symmetrically around the proton beam axis. Each detector makes an angle of 45° with this axis and detects $\mu^+\mu^-$-pairs, one type of particle in each detector. When the momentum of each muon is 2.2 GeV, one sees an enhancement in the muon rate. This is interpreted as the production of a resonance R of mass M_R that decays into the muons. What is the mass M_R of this resonance? The muon mass is 106 MeV.

Problem 1.83 A particle with mass M and 4-momentum $p = (E, \mathbf{p})$ moves toward a detector when it suddenly decays and emits a photon in the direction of motion.

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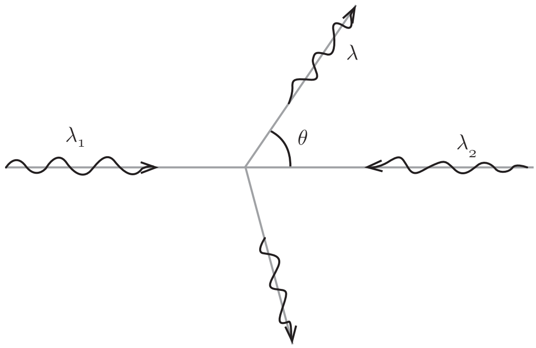

Figure 1.9 Scattering of two photons $\gamma + \gamma \longrightarrow \gamma + \gamma$.

The energy registered by the detector is ω. Determine what energy the photon had in the rest frame of the decaying particle.

Problem 1.84 An electron moves with constant velocity toward a positron at rest and they annihilate into two photons. The photons go out with angles ϕ and $-\phi$ relative to the direction of the incoming electron.

a) Calculate the angle as a function of the total energy of the electron.

b) Show that in the nonrelativistic limit the angle is given by $\cos\phi = v/(2c)$.

Problem 1.85 Two photons with wavelengths λ_1 and λ_2, respectively, are scattered against each other according to Figure 1.9. Calculate the wavelength of the photon with scattering angle θ, i.e., express λ as a function of λ_1, λ_2, and θ.

Hint: $p = \frac{h}{\lambda}$, where h is Planck's constant.

Problem 1.86 A K-meson with mass M decays at rest into two charged pions with the same mass m and a photon according to the reaction formula

$$K^0(P) \longrightarrow \pi^+(p_1) + \pi^-(p_2) + \gamma(k).$$

The momenta of the particles are given in parentheses after each particle symbol. Calculate the speed v of the pions in center-of-mass frame where (where $\mathbf{p}_1 + \mathbf{p}_2 = 0$) as a function of the masses of the particles and the photon energy $k^0 = \omega$ in the rest frame of the decaying particle.

Problem 1.87 A Σ^0 particle with speed $c/3$ in the direction toward a gamma detector suddenly decays into a Λ particle and a photon. The photon continues toward the detector.

a) What energy does the Σ^0 particle have in the system in which the detector is at rest?

b) What energy does the photon have in the rest system of the Σ^0 particle?

c) What energy will be registered in the detector?

The mass of the Λ is $m_\Lambda \approx 1\,115.7$ MeV and that of Σ^0 is $m_{\Sigma^0} \approx 1\,192.6$ MeV.

Problem 1.88 In elastic scattering of two particles onto each other, the same type of particles are present before and after the collision. Thus, in $e + p \longrightarrow e + p$ elastic

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scattering of electrons on protons with corresponding 4-momenta p_e, p_p, p'_e, and p'_p, one can form an invariant called t, defined as $t = (p_e - p'_e)^2$.

a) Show that, in the center-of-mass system defined by the total 3-momentum being $\mathbf{0}$, the quantity $-t$ equals the square of the change of the 3-momentum, i.e., $-t = (\mathbf{p}_e - \mathbf{p}'_e)^2$ and express this quantity in terms of the scattering angle θ between the incoming and outgoing electrons and the modulus of the 3-momentum $|\mathbf{p}_e|$ of the incoming electron.

b) Calculate the kinetic energy, T'_p, of the outgoing proton in the laboratory system, where the incoming proton is at rest before the collision, in terms of the variable t.

Problem 1.89 What is the kinetic energy T of the pion required to create the resonance $\Delta(1232)$ in the reaction

$$\pi + p \longrightarrow \pi + \Delta,$$

where π is a pion and p is a proton? The proton is at rest before the collision. The result should be expressed in terms of the masses of the particles involved.

Problem 1.90 The scattering probabilities for the reactions $\pi + d \longrightarrow p + p$ and for the reversed reaction $p + p \longrightarrow \pi + d$ are related due to so-called time reversal invariance. However, they must be compared at the same center-of-mass energy. Calculate the relation between the kinetic energy T_π of the pion (π), in the frame where the deuteron (d) is at rest before the collision in the first reaction, and the kinetic energy T_p of one of the protons (p) in reversed reaction, when the other proton is at rest, respecting the above condition on the center-of-mass energy.

Problem 1.91 Consider the reaction $\pi^+ + n \longrightarrow K^+ + \Lambda$ in the rest frame of n. The masses of the particles are m_{π^+}, m_n, m_{K^+}, and m_Λ, respectively. What is the kinetic energy T of the π^+ when the K^+ has total energy E and moves off at an angle of 90° to the direction of the incident π^+? (T should be expressed in m_{π^+}, m_n, m_{K^+}, m_Λ, and E.)

Problem 1.92 The mass of the meson π^0 can be measured by the reaction

$$p + \pi^- \longrightarrow \pi^0 + n,$$

where p is a proton, π^- is a negative pion, and n is a neutron. The uncharged π^0 meson decays very quickly into two photons and cannot be easily measured. However, the velocity of the final neutron can be measured and is found to be $v_n = (0.89418 \pm 0.00017)$ cm/ns. Derive the formula that expresses the mass of the π^0 meson as a function of the masses of the proton, the π^-, the neutron, and the velocity v_n, assuming that the reaction takes place at rest for the incoming particles. Simplify the result by showing that the velocity is small, so that we need to retain only lowest nontrivial order in v_n/c.

Problem 1.93 A thermal neutron is absorbed by a proton at rest and a deuteron is formed together with a photon. This exothermic reaction is formally

$$p + n \longrightarrow d + \gamma.$$

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The binding energy B of the deuteron is about 2.23 MeV. Calculate, relativistically, the energy of the emitted photon as a function of the masses of the particles and the binding energy B.

Problem 1.94 A hydrogen atom H, consisting of an electron and a proton with binding energy $B = 13.6$ eV, can disintegrate into its two constituent particles by being hit by a photon. The reaction is

$$\gamma + \mathrm{H} \longrightarrow p + e.$$

Calculate, relativistically, the smallest photon energy in the rest frame of H required for this process to occur expressed in terms of B and the hydrogen mass m_{H}.

Problem 1.95 Similarly to the cosmic microwave background (CMB) of photons with a temperature of $T_{\mathrm{CMB}} \sim 2.7$ K, there should be a cosmic neutrino background (CNB) with a temperature of $T_{\mathrm{CNB}} \sim 1.9$ K. At these temperatures, their kinetic energy is very tiny. Suppose a very high-energy antineutrino would hit such a neutrino and annihilate it. A result of this collision could be the production of a Z^0 boson which decays hadronically. The reaction is formally

$$\bar{\nu} + \nu \longrightarrow Z^0.$$

What is the threshold energy for the antineutrino for this to occur? In particular, consider the two limits

a) The CNB neutrinos have a mass of $m_\nu = 0.15$ eV.

b) The CNB neutrinos have very small masses ($m_\nu/(k_B T) \to 0$).

Hint: In a gas of particles at temperature T, the mean kinetic energy of the particles is given by $E_k = 3k_B T/2$, where $k_B \simeq 8.6 \cdot 10^{-5}$ eV/K is Boltzmann's constant. The Z^0 mass is $m_{Z^0} \simeq 91$ GeV.

Problem 1.96 Consider elastic scattering of photons on electrons

$$\gamma(k) + e^-(p) \longrightarrow \gamma(k') + e^-(p'),$$

where k and p are the incoming photon and electron 4-momenta and k' and p' the corresponding outgoing 4-momenta.

a) In the laboratory system, the incoming electron is at rest and the outgoing photon is scattered at an angle θ with respect to the direction of the incoming photon. Use invariants to derive the so-called "Compton formula," i.e., the difference between the outgoing and incoming photon wavelengths, as a function of θ, in units $c = 1$ and $\hbar = 1$.

b) Derive the angular frequency (energy) of the outgoing photon in the center-of-mass system in terms of the incoming photon angular frequency (energy) in the laboratory system.

Problem 1.97 In Compton scattering $\gamma + e \longrightarrow e + \gamma$, photons of a fixed energy ω are scattered against electrons, which can be considered at rest in the laboratory frame. Compute the kinetic energy of the outgoing *electron* as a function of the scattering angle θ of the outgoing *photon*.

Problem 1.98 Inverse Compton scattering occurs when low-energy photons collide with high-energy electrons. Assuming that the photon and electron are originally moving in the same direction, find an expression for the photon energy after the collision as a function of the initial photon energy, the velocity and mass of the electron, and the scattering angle θ of the photon.

Problem 1.99 An antimuon μ^+ decays into a positron e^+ and two neutrinos ν_e and $\bar{\nu}_\mu$. The reaction is

$$\mu^+ \longrightarrow e^+ + \nu_e + \bar{\nu}_\mu.$$

Give an expression for the largest possible total energy of the electron neutrino ν_e in the rest frame of the antimuon. You may assume that the neutrino masses are negligible compared to lepton masses.

Problem 1.100 A ρ-meson with mass $m_\rho \simeq 770\ \mathrm{MeV}/c^2$ sometimes decays into a pair of muons (μ^- and μ^+) with mass $m_{\mu^-} = m_{\mu^+} \simeq 106\ \mathrm{MeV}/c^2$ and a photon, γ. What is the maximal *kinetic* energy that the μ^+ can have in this decay in the rest frame of the ρ-meson?

Problem 1.101 There is a possibility that neutrinos are their own antiparticles. If this is true, then the so-called neutrinoless double beta decay

$$^{76}\mathrm{Ge} \longrightarrow {}^{76}\mathrm{Se} + e^- + e^-,$$

is allowed. Derive expressions for the maximal and minimal possible values of the sum of the kinetic energy of the electrons in the rest frame of ^{76}Ge. Express your answer in terms of the particle masses.

Problem 1.102 At the LHC (Large Hadron Collider), two photons are measured with 4-momenta

$$p_1 = \omega_1(1,1,0,0) \quad \text{and} \quad p_2 = \omega_2(1, \cos\theta, \sin\theta, 0), \tag{1.13}$$

respectively. Assuming that the photon pair results from the decay of a new particle ϕ such that $\phi \longrightarrow \gamma\gamma$, what is the mass of the new particle?

Problem 1.103 In an accelerator, protons are accelerated until they reach a kinetic energy of 8 000 MeV and are then made to collide with protons at rest. If the sum of the kinetic energies of two colliding protons (measured in the center-of-mass system) is larger than the rest energy of a proton-antiproton pair, then such a pair can be formed according to the reaction formula

$$p + p \longrightarrow p + p + p + \bar{p},$$

where p is a proton and $\bar{p}$ is an antiproton.

Is the energy 8 000 MeV sufficient for the reaction to go? The rest mass of the proton is 938 MeV.

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Problem 1.104 Protons at rest are bombarded with π-mesons. How large kinetic energy do the mesons need to have for the reaction

$$\pi^- + p \longrightarrow \pi^+ + \pi^- + n,$$

to take place? The rest mass of the particles are $m_{\pi^-} = m_{\pi^+} \approx 140$ MeV, $m_p \approx 938$ MeV, and $m_n \approx 940$ MeV.

Problem 1.105 In the CELSIUS ring at the The Svedberg Laboratory in Uppsala, Sweden, one would like to study the reaction

$$p + d \longrightarrow p + p + n + \eta.$$

The available kinetic energy of the protons is $T_p = 700$ MeV and the deuterons (d) can be considered to be at rest. The rest masses of the particles are $m_p \approx m_n$, $m_d \approx m_p + m_n$, $m_n = 940$ MeV, and $m_\eta = 550$ MeV.

a) Is the reaction possible?

b) If the kinetic energy of the protons in the beam is increased to $T_p = 1\,350$ MeV, what is the maximum kinetic energy that the η can get in the system in which the nucleons are at rest after the reaction, expressed in terms of the rest masses and the kinetic energies?

Problem 1.106 In neutrino detection, the quasi-elastic ($\nu_\mu + X \longrightarrow \mu + Y$, where X and Y are different nuclei) and 1π ($\nu_\mu + X \longrightarrow \mu + Y + \pi^0$) processes are relevant at relatively low energies. Compute the ratio between the neutrino threshold energies for these processes in the rest frame of the nucleus X. Express your answer in terms of the different particle masses (the neutrino may be considered massless for the purposes of this problem).

Problem 1.107 Consider the particle collision $e^- + e^- \longrightarrow e^- + e^- + e^- + e^+$. Compute the necessary total energy of one of the initial electrons in the rest frame of the other for this process to occur. Also, compute the ratio between this energy and the total required energy in the center-of-momentum frame.

Problem 1.108 We can produce neutral kaons in a proton collision through the reaction $p + p \longrightarrow p + p + K^0$. Find an expression for the threshold kinetic energy of the protons of this reaction when

a) One proton is stationary in the lab frame (find the threshold kinetic energy of the *other* proton).

b) Both protons have the same kinetic energy (quote the *total* kinetic energy of both protons).

Problem 1.109 A particle χ hits a stationary proton p and undergoes inelastic scattering to a new state χ^* while keeping the proton intact. Determine the threshold kinetic energy of χ for this scattering to occur if $m_{\chi^*} = m_\chi + \delta > m_\chi$. Discuss your result in the limit when $\delta \ll m_\chi$.

Problem 1.110 Neutrinos are emitted from core collapse supernovae. If a core collapse supernova occurs at a distance L from Earth and each neutrino has a total energy E, how much more time would pass (in the rest frame of the Earth) until the neutrinos reach us if they have a small mass $m > 0$ compared to if they were massless ($m = 0$)? Give an exact answer as well as a reasonable approximation for when $m \ll E$.

Problem 1.111 An elementary particle of charge e (e is the elementary charge) is accelerated from rest in a 100 m long straight insulated vacuum cylinder (a linear accelerator) with a constant electric field of 10^4 V/m across the endpoints.

a) What kinetic energy will the particle obtain after the acceleration?

b) How long time does it take for particle to pass through the tube if it starts from rest? *Hint:* Use the energy as an integration variable.

1.7 Electromagnetism

Problem 1.112 Show by explicit calculation, using the chain rule for derivation and the properties of the Lorentz transformations, that

$$\Box A^{\mu}(x) = 0, \tag{1.14}$$

is invariant under Lorentz transformations, i.e., if $A^{\mu}(x)$ is a solution to Eq. (1.14), then $A'^{\mu}(x')$ is a solution to the same equation in the primed variables $x' = \Lambda x$, where Λ is a Lorentz transformation.

Problem 1.113 Show that the gauge transformation $A_{\mu} \mapsto A'_{\mu} = A_{\mu} + \partial_{\mu}\psi$, where ψ is an arbitrary scalar field, does not affect the field tensor $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

Problem 1.114 An inertial coordinate system K' is moving relative to another inertial coordinate system K with constant velocity v along the positive x^1-axis of K.

a) Assume that a stick of length ℓ is at rest in K such that $\Delta\mathbf{x} = (\ell, 0, 0)$. Calculate $\Delta\mathbf{x}'$ in K'.

b) Assume that there is a constant electric field $\mathbf{E} = (0, 0, E)$ in K (no magnetic field, i.e., $\mathbf{B} = \mathbf{0}$ in K). Calculate $\mathbf{E}'$ and $\mathbf{B}'$ in K'.

Problem 1.115 An observer at rest in a frame K experiences only an electric field $\mathbf{E}$. Another observer in another frame K', moving with velocity v along the positive x-axis, will observe a magnetic field $\mathbf{B}'$. Calculate this magnetic field for small velocities (linear terms in v) and show that this field is perpendicular to both the electric field $\mathbf{E}'$ and the velocity of K relative to K'.

Problem 1.116 Let K, K', and K'' be as in Problem 1.34. Assume that there is a constant electric field $\mathbf{E} = (0, 1, 0)$ (in some given physical units) in the coordinate system K. We assume that the magnetic field $\mathbf{B}$ vanishes in K. Compute the components of both the electric and magnetic fields in the coordinate systems K' and K''.

Problem 1.117 Compute the electric and magnetic field components due to a point charge q moving with velocity v along the positive x-axis.

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Problem 1.118 A particle of mass m and electric charge q is moving in a constant electric field $\boldsymbol{E}$. Use the Lorentz force law to calculate the velocity of the particle as a function of the displacement r from the origin along the direction of motion. The particle starts off at rest.

Problem 1.119 A current I is flowing through a straight uncharged conductor. Determine the electromagnetic field in an inertial system K' that moves parallel to the conductor with velocity v

a) by transforming the electromagnetic field tensor from the rest frame K of the conductor to K',

b) by transforming the current-density 4-vector from K to K', and then, knowing the charge of the conductor and its current relative to K determine the field in K'.

Problem 1.120 Maxwell's equations can be expressed by means of the electromagnetic 4-vector potential A. When $\partial_\mu A^\mu = 0$ (i.e., the Lorenz gauge), they take on a simple form. What is this form? Assuming that Maxwell's equations are in this simple form, and furthermore, $J = 0$ (i.e., current free), show for a plane wave, $A^\mu = \varepsilon^\mu e^{ik\cdot x}$, where ε is the polarization vector, that

$$\boldsymbol{E} \cdot \boldsymbol{k} = \boldsymbol{B} \cdot \boldsymbol{k} = 0, \tag{1.15}$$

i.e., the electric and magnetic fields are perpendicular to the direction of motion.

Problem 1.121 Calculate the Lorentz invariants $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon_{\mu\nu\omega\lambda}F^{\mu\nu}F^{\omega\lambda}$ for a free electromagnetic plane wave $A^\mu(x) = \epsilon^\mu e^{ik\cdot x}$, where ϵ is the polarization vector. Give a physical interpretation of your result.

Problem 1.122 a) Prove that the scalar product $\boldsymbol{E} \cdot \boldsymbol{B}$ between the electric and magnetic field vectors is invariant under Lorentz transformations.

b) Show that if the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal for one observer, they are orthogonal for any observer.

c) Show that $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal for free plane waves with $A^\mu(x) = \varepsilon^\mu e^{ik\cdot x}$, where ε is the polarization vector.

d) Show for the plane waves that $\boldsymbol{E} \times \boldsymbol{B} = A\boldsymbol{k}$, where $\boldsymbol{k}$ is the wave vector and A is a nonvanishing expression.

Problem 1.123 An electron with mass m_0 is moving in a homogeneous magnetic field $\boldsymbol{B} = (0,0,B)$ and no electric field. Calculate its trajectory if it has velocity $\boldsymbol{u} = (u,0,0)$ at time $t = 0$.

Problem 1.124 In an inertial coordinate system K, there is a constant electric field $\mathbf{E} = (cB,0,0)$ and a constant magnetic field $\mathbf{B} = (0,B,0)$. In another inertial system K', the same fields are measured to be $\mathbf{E}' = (0,2cB,cB)$ and the x-component $B'_x = 0$. Compute B'_y and B'_z.

Problem 1.125 Observer A measures the electric and magnetic field strengths to be $\mathbf{E} = (\alpha,-\alpha,0)$ and $\mathbf{B} = (0,0,2\alpha/c)$, respectively, where $\alpha \neq 0$. Another observer, observer B, makes the same measurements and finds $\mathbf{E}' = (0,0,2\alpha)$ and $\mathbf{B}' = (B'_x,\alpha/c,B'_z)$. Determine B'_x and B'_z.

Problem 1.126 Observer A measures the electric and magnetic field strengths to be $\mathbf{E} = (0, \beta, -\beta)$ and $\mathbf{B} = (2\beta/c, 0, 0)$, respectively, where $\beta \neq 0$. Another observer, observer B, makes the same measurements and finds $\mathbf{E}' = (2\beta, 0, 0)$ and $\mathbf{B}' = (B'_x, B'_y, \beta/c)$. Determine B'_x and B'_y.

Problem 1.127 Observer A measures the electric and magnetic field strengths to be $\mathbf{E} = (\alpha, 0, 0)$ and $\mathbf{B} = (\alpha/c, 0, 2\alpha/c)$, respectively, where $\alpha \neq 0$. Another observer, observer B, makes the same measurements and finds $\mathbf{E}' = (E'_x, \alpha, 0)$ and $\mathbf{B}' = (\alpha/c, B'_y, \alpha/c)$. Express E'_x and B'_y in terms of α and c. Finally, a third observer, observer C, is moving relative to observer B with constant velocity v along the positive x-axis of observer B. Find the electric and magnetic field strengths, $\mathbf{E}''$ and $\mathbf{B}''$, as observer C measures them.

Problem 1.128 Assume that a muon originally travels vertically down toward the ground from an altitude of 10 km. There is a magnetic field coming from the Earth of $B = 50\ \mu$T affecting the motion of the muon. To make a simple model we take the magnetic field to be constant all the way from 10 km altitude to ground level. Suppose the field lines go from south to north and we are in Japan on the northern hemisphere. How far in length and in which direction is the deviation from the point where the muon would hit the ground without magnetic field, compared to where it hits the ground due to the deviation induced by the magnetic field of the Earth, if it has the energy of 2 GeV and is negatively charged?
Hint: The combination cB, where c is the speed of light, has the value $cB = 300$ V/m, for $B = 1\ \mu$T. The trajectory of a charged particle in a homogeneous magnetic field is a circle, it is sufficient to compute the radius of the circle and then use geometric arguments.

Problem 1.129 An observer in the system S has observed an electromagnetic field tensor $F^{\mu\nu}$ with nonvanishing $\boldsymbol{E}$- and $\boldsymbol{B}$-fields. Performing a Lorentz transformation with velocity u along the positive x_1-axis to another system S' he finds that the $\boldsymbol{B}$-field is absent, i.e., all its components are equal to 0. What is the electric field in this system expressed in u and the components of the electric field in S?

Problem 1.130 In an inertial frame S there is a constant time-independent magnetic field $\boldsymbol{B}$ and no electric field ($\boldsymbol{E} = \mathbf{0}$). Consider another inertial frame S', which moves with velocity v along the positive x^1-axis of S.

a) What are the $\boldsymbol{E}'$ and $\boldsymbol{B}'$ fields in the system S' expressed in the original $\boldsymbol{B}$-field and the velocity v?

b) Verify that the Lorentz invariants are indeed invariant under this transformation.

Problem 1.131 An electron in a linear particle accelerator of length $L = 3$ km (e.g., SLAC in California, USA) is accelerated through an electric potential U.

a) Compute the trajectory $x(t)$ of this electron for $0 < |x(t)| < L$ if its motion starts at time $t = 0$ at rest at one end of the accelerator.

b) Compute the time it takes for this electron to pass through the whole accelerator.

c) Compute the time dependence of the energy of this electron in the accelerator.

Problem 1.132 a) Find the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ generated by a particle with charge q moving with constant velocity v parallel with the x-axis in an

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inertial system S, using that the electric and magnetic potentials in the particle's rest frame are

$$\phi(t', \boldsymbol{x}') = \frac{q}{4\pi|\boldsymbol{x}'|}, \quad \boldsymbol{A}(t', \boldsymbol{x}') = \boldsymbol{0}, \tag{1.16}$$

we use the notation $\boldsymbol{A} = (A^1, A^2, A^3)$, and similarly for $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{x}$.

b) Explain why it is possible to check your result in a) by computing $\boldsymbol{E} \cdot \boldsymbol{B}$ and $\boldsymbol{E}^2 - \boldsymbol{B}^2$ in both inertial systems. Perform these checks!

Problem 1.133 Bubble chambers were frequently used in the 1960s in particle collision experiments. In a bubble chamber, there is a strong constant magnetic field, which bends the motion of charged particles. The charged particles give rise to bubbles, which make the trajectories of the charged particles visible.

a) In the lab frame of the bubble chamber, there is a strong magnetic field in the z-direction and no electric field. Use the Lorentz force law to show that the trajectory of a charge particle can be parametrized in the lab frame as

$$x = R\cos\omega\tau, \quad y = -R\sin\omega\tau, \tag{1.17}$$

and determine ω. Show that for a charged particle, you can obtain the 3-momentum from knowing the radius of the trajectory and the strength of the magnetic field (any energy losses can be neglect)

$$p_i p^i = q^2 R^2 B_i B^i. \tag{1.18}$$

b) In a bubble chamber, one can only see the traces of charged particles in terms of bubbles. Consider the following process

$$\Sigma^- \longrightarrow \pi^- + X^0,$$

where Σ^- and π^- are known charged particles. Here X^0 is an unknown uncharged particle, which we cannot see, since it does not give rise to bubbles. For the other two particles, we know their rest masses and their trajectory radii R_Σ and R_π (therefore, we also know their 3-momenta). From this information, derive an expression for the rest mass of the unknown particle expressed in terms of the rest masses M_Σ of Σ^- and M_π of π^-, and their respective 3-momenta, as well as the angle θ between the recorded trajectories of the charged particles close to the collision.

Problem 1.134 Starting from the plane wave solution to Maxwell's equations

$$A^\mu = \varepsilon^\mu \sin(k \cdot x), \tag{1.19}$$

show that the electric and magnetic fields are orthogonal and have the same magnitude without referring to a particular gauge condition.

Problem 1.135 Assume that the electromagnetic field in an inertial frame S satisfies $|\boldsymbol{E}| = |\boldsymbol{B}|$ and that the angle between the electric and magnetic field is α. In another inertial frame, the fields are $\boldsymbol{E}'$ and $\boldsymbol{B}'$ with a corresponding angle α'. Show that

$$\cos\alpha' = \frac{\boldsymbol{E}^2}{\boldsymbol{E}'^2}\cos\alpha. \tag{1.20}$$

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Problem 1.136 The 4-potential of a stationary point charge Q in its rest frame is given by

$$A^{\mu} = \frac{Q}{4\pi r}(1, \mathbf{0})^{\mu}, \tag{1.21}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance to the particle. Compute the electromagnetic stress–energy tensor $T^{\mu\nu}$ in $(x, y, z) = (1, 0, 0)$ and the corresponding trace T^{μ}_{μ}.

Problem 1.137 Starting from Maxwell's equations and without assuming a particular gauge condition, show that the components of the electromagnetic field tensor $F^{\mu\nu}$ satisfy the sourced wave equation

$$\Box F^{\mu\nu} \equiv \partial_{\sigma}\partial^{\sigma} F^{\mu\nu} = S^{\mu\nu}, \tag{1.22}$$

and express the source tensor $S^{\mu\nu}$ in terms of the 4-current density J^{μ}.

Problem 1.138 The electromagnetic stress–energy tensor is given by

$$T^{\nu}_{\mu} = -\varepsilon_0 \left[F_{\mu\sigma} F^{\nu\sigma} - \frac{1}{4}\delta^{\nu}_{\mu}(F_{\rho\sigma} F^{\rho\sigma}) \right]. \tag{1.23}$$

Given the electromagnetic plane wave solution for the 4-potential

$$A^{\mu} = \varepsilon^{\mu} \sin(k \cdot x), \tag{1.24}$$

express T^{ν}_{μ} in terms of the 4-vector k. You also need to assure that the wave actually fulfills Maxwell's equations in the absence of a source term $\partial_{\mu} F^{\mu\nu} = 0$.
Hint: You may assume the Lorenz gauge condition $\partial_{\mu} A^{\mu} = 0$.

Problem 1.139 The 4-potential A_{μ} is not physical, but may be transformed according to $A_{\mu} \mapsto A_{\mu} + \partial_{\mu}\varphi$, where φ is a scalar field, without changing the physical observables. Show that the physical electromagnetic field tensor $F^{\mu\nu}$ is invariant under this transformation.

Problem 1.140 The electric field of an electric dipole with dipole moment $\boldsymbol{d} = d\boldsymbol{e}_z$ is given by

$$\boldsymbol{E} = \frac{d}{4\pi\varepsilon_0}\left(\frac{3xz}{r^5}\boldsymbol{e}_x + \frac{3yz}{r^5}\boldsymbol{e}_y + \frac{3z^2 - r^2}{r^5}\boldsymbol{e}_z \right), \tag{1.25}$$

in its rest frame S. Compute the value of the quantity $F_{\mu\nu}\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho}$ as a function of time and position in the frame S', which is moving in the positive x-direction with velocity v relative to S'.

Problem 1.141 A particle at rest acting as an electric monopole and a magnetic dipole has the electromagnetic fields

$$\boldsymbol{E} = \frac{q}{4\pi r^2}\boldsymbol{e}_r \quad \text{and} \quad \boldsymbol{B} = \frac{1}{4\pi r^3}\left[3(\boldsymbol{m} \cdot \boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{m}\right], \tag{1.26}$$

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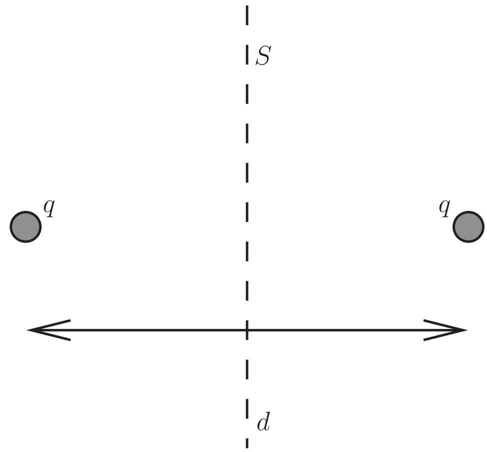

Figure 1.10 Two particles (each with charge q) and the plane S equidistant from both particles.

where $\boldsymbol{r}$ is the position vector relative to the particle, r its magnitude, q the charge of the particle, and $\boldsymbol{m}$ its magnetic dipole moment. Determine whether or not there exists a region of spacetime where the electric field is equal to zero in some inertial frame (although that frame may generally be different for different points in the region). If such a region exists, determine the shape of the region in the particle's rest frame.

Problem 1.142 Two particles with the same charge q are held fixed with a separation distance d (see Figure 1.10). Compute the stress–energy tensor of the static electric field between the charges and use your result to find the total 4-force between the electromagnetic fields on either side of the plane S that is equidistant from both charges.

Problem 1.143 Compute the Lorentz 4-force between two electrons moving in parallel with constant velocity $\boldsymbol{v}$ and a separation d orthogonal to the direction of motion.

1.8 Energy–Momentum Tensor

Problem 1.144 Determine the momentum density of a gas consisting of massless noninteracting particles in a frame which is moving with velocity $\boldsymbol{v}$ relative to the gas rest frame. Express your result in terms of $\boldsymbol{v}$ and the energy density of the gas *in the frame where the gas is moving*.

Problem 1.145 A star cruiser is moving through space with velocity v relative to the galaxy. Suddenly it encounters a gas cloud of dust particles. What is the 4-force from the dust cloud on the star cruiser at the moment it enters the cloud? You may assume that the star cruiser has a cross sectional area A relative to the direction of

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motion and that all of the dust particles encountered will be absorbed in the hull of the star cruiser. In addition to computing the 4-force, motivate and state whether it is pure, heatlike, or neither.

Problem 1.146 The energy–momentum tensor of a string with tension t is given by

$$(T^{\mu\nu}) = \begin{pmatrix} \rho_0 & 0 \\ 0 & -\sigma \end{pmatrix}, \tag{1.27}$$

where ρ_0 is the string density and $\sigma = t/A < \rho_0$ is the stress across the string cross section A.

a) Does a frame exist where the stress (T^{11}) is equal to zero?

b) Does a frame exist where the energy density is smaller than ρ_0?

Your answers should be accompanied by solid argumentation.

Problem 1.147 A pure photon gas such as the cosmic microwave background (CMB) can be described as a perfect fluid with pressure $p = \rho_0/3$ in its rest frame. In a frame moving with velocity $\boldsymbol{v}$ in relation to the rest frame of the CMB, compute the energy density, momentum density, and stress tensors. In addition, comment on whether the shear stress (off-diagonal elements of the stress tensor) in an arbitrary frame is zero or not.

Problem 1.148 In a perfect fluid with proper density ρ_0 and positive proper pressure p, find an expression for the energy density ρ in an arbitrary inertial frame S' and derive an upper bound on ρ/γ^2, where γ is the gamma factor between the fluid's rest frame and S'.

Problem 1.149 The energy density in the frame of an observer with 4-velocity V is given by $\rho = T_{\mu\nu}V^\mu V^\nu$. The *weak energy condition* is a condition requiring the energy density to be nonnegative for all observers, i.e., $\rho \geq 0$. For a perfect fluid, determine the condition on the equation of state parameter w in the relation $p = w\rho_0$ that the weak energy condition implies.

1.9 Lagrange's Formalism

Problem 1.150 The 4-momentum of a free particle of mass m is $p^\mu = mc\dot{x}^\mu$.

a) Show that the momentum is conserved (i.e., independent of time) by deriving the Euler–Lagrange variational equations for the Lagrangian $\mathscr{L} = p^2/(2m)$ in Minkowski space.

b) When the particle moves in an electromagnetic field, one can obtain the relevant equations of motion by using the substitution $p \mapsto p + qA/c$, where $A = A(x)$ is the electromagnetic potential and q is the charge of the particle. Show that, to lowest nontrivial order in q, the equations of motion for the particle give the equations of the Lorentz force.

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