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An obstruction to the existence of certain dynamics in surface diffeomorphisms

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Abstract. Let M be a two-dimensional, compact manifold and $g: M \to M$ be a diffeomorphism with a hyperbolic chain recurrent set. We find restrictions on the reduced zeta function $p(t)$ of any zero-dimensional basic set of g. If deg $(p(t))$ is odd, then $p(1) = 0$ (in $\mathbb{Z}/2\mathbb{Z}$). Since there are infinitely many subshifts of finite type whose reduced zeta functions do not satisfy these restrictions, there are infinitely many subshifts which cannot be basic sets for any diffeomorphism of any surface.

In the theory of dynamical systems, we are often interested in giving a qualitative description of the kinds of dynamics which can occur on a given manifold. Given an arbitrary diffeomorphism g of a manifold M, its recurrent behaviour is captured in its chain recurrent set $R(g)$. If the restriction $g|R(g)$ has a hyperbolic structure, then $R(g)$ can be decomposed into invariant, topologically transitive pieces

$$R(g) = R_1 \amalg R_2 \amalg \cdots \amalg R_n \quad \text{(disjoint union)}$$

[**9**] [**4**]. Each of these pieces is called a basic set. So, if $g|R(g)$ is hyperbolic, then giving a qualitative description of the dynamics of g is equivalent to giving a qualitative description of the action of g on each of its basic sets.

A key tool in the study of g restricted to a basic set R_i is symbolic dynamics and, in particular, subshifts of finite type. For example, if the basic set R_i is zero-dimensional, then $g|R_i$ is topologically conjugate to a subshift of finite type [**2**]. We consider the problem of realizing subshifts of finite type as basic sets.

Any subshift of finite type can be realized as a basic set of some Smale diffeomorphism $f: M \to M$ (i.e. not only does $f|R(f)$ have a hyperbolic structure but all basic sets are zero-dimensional) if the dimension of M is at least three [**11**]. By contrast, we prove that there are obstructions if M is two-dimensional, and in fact we exhibit an infinite class of subshifts, each of which cannot occur as a basic set on any surface.

Given a subshift of finite type $\sigma: S \to S$, its reduced homology zeta function $p(t)$ equals the mod 2 reduction of det $(I - At)$, where A is any matrix representing σ [**5**].

THEOREM. *Suppose M is a two-dimensional, compact manifold and $g: M \to M$ is a diffeomorphism with a hyperbolic chain recurrent set. If the subshift $\sigma|S$ is topologically conjugate to $g|R_i$ for some basic set R_i, then either*

(i) *$p(t)$ has even degree; or,*

(ii) *$p(1)=0$ (in $\mathbb{Z}/2\mathbb{Z}$; in other words, it has an even number of terms).*

Since there are infinitely many subshifts whose reduced zeta functions do not satisfy these conditions, we see that the two-dimensional situation contrasts sharply with the higher-dimensional situation.

1. *A special case of the theorem*

In this section we exclusively consider diffeomorphisms $f: X \to X$, where X is a two-dimensional manifold, which satisfy the conditions:

(1.1) $f|R(f)$ has a hyperbolic structure;

(1.2) $R(f) =$ (periodic sinks) $\amalg R \amalg$ (periodic sources); and

(1.3) $f|R$ is topologically conjugate to a subshift of finite type $\sigma: S \to S$.

We study $f|R$ using an identity relating the mod 2 reduction of the Artin–Mazur zeta function of f and the reduced homology zeta function of f.

The Artin–Mazur zeta function of f restricted to a basic set R_i is defined to be

$$\zeta_{f|R_i}(t) = \exp\left(\sum_{j=1}^{\infty} \frac{N_j}{j} t^j\right),$$

where N_j equals the number of fixed points of $f^j|R_i$ [**1**]. Since $f|R_i$ has a hyperbolic structure, then $\zeta_{f|R_i}(t)$ is a rational function with integer coefficients [**12**], [**7**], [**8**].

The reduced homology zeta function $z(f)$ can be calculated either locally or globally. Globally,

$$z(f) = \prod_{k=0}^{\dim X} \det(I + f_k t)^{(-1)^{k+1}},$$

where f_k is the map on $H_k(X; \mathbb{Z}/2\mathbb{Z})$ induced by f [**5**]. Locally, $z(f)$ is calculated using the formulae [**5**]

$$z_i(f) = (\zeta_{f|R_i}(t))^{(-1)^{u_i}} \quad \text{(reduced mod 2)},$$

where $u_i =$ unstable dimension of $f|R_i$, and

$$z(f) = \prod_{\substack{\text{all basic} \\ \text{sets } R_i}} z_i(f).$$

For example, if R_i is a periodic source or sink of period p_i, then

$$\zeta_{f|R_i}(t) = \frac{1}{(1-t^{p_i})} \quad \text{and} \quad z_i(f) = \frac{1}{(1+t^{p_i})}.$$

Since $f|R$ is topologically conjugate to a subshift $\sigma: S \to S$ with matrix representation A, then

$$\zeta_{f|R}(t) = \frac{1}{\det(I - At)}$$

[3]. Consequently, $f|R$ will contribute the mod 2 reduction $p(t) \in (\mathbb{Z}/2\mathbb{Z})[t]$ of the integral polynomial det $(I - At)$ to the reduced zeta function $z(f)$.

LEMMA 1. *Suppose* $f: X \to X$ *is a diffeomorphism of a compact surface which satisfies* (1.1)–(1.3). *If* $p(t)$ *is the* mod 2 *reduction of the polynomial* $[\zeta_{f|R}(t)]^{-1}$ *then either*

(i) $p(t)$ *has even degree*; *or*

(ii) $p(1) = 0$ (*in* $\mathbb{Z}/2\mathbb{Z}$).

Remark. In lemma 1 we do not assume that X is connected.

Proof. By discarding components which do not intersect R, we can assume that X consists of components which are cyclically permuted (since $f|R$ has a dense orbit). Thus

$$X = X_1 \cup \cdots \cup X_k,$$

where each X_i is a component of X and

$$f(X_i) = X_{i+1} \mod k.$$

Computing $z(f)$ from homology, we have

$$z(f) = \frac{\det(I + f_1 t)}{(1 + t^k)^2} = \frac{\det(I + f_1 t)}{(1 + t^d)^{2r}},$$

where $k = rd$, d is odd, r is a power of 2, the two factors $(1 + t^k)$ come from zero- and two-dimensional homology, and $f_1: H_1(X; \mathbb{Z}/2\mathbb{Z}) \to H_1(X; \mathbb{Z}/2\mathbb{Z})$.

We can also compute $z(f)$ from the basic sets and obtain

$$z(f) = \frac{p(t)}{\prod (1 + t^{p_i})},$$

where the product has one factor for each source or sink orbit and p_i is its period. Since components of X are permuted, we must have for each i, $p_i = kd_i$ for some integer d_i. Thus

$$p_i = (rd)d_i = rm_i,$$

where $m_i = dd_i$, and

$$z(f) = \frac{p(t)}{\prod (1 + t^{m_i})^r}.$$

Equating the two expressions for $z(f)$ we obtain

$$(1 + t^d)^{2r} p(t) = \det(I + f_1 t)[\prod (1 + t^{m_i})^r].$$

There must have been at least one source and one sink for f, so there must be at least two factors of the form $(1 + t^{m_i})^r$ in the right-hand side of this equation. If there are more than two factors, then $(1 + t)$ is a factor of the right-hand side with multiplicity at least $3r$. Hence, $(1 + t)$ must factor $p(t)$ and $p(1) = 0$.

Consequently, we need only consider the situation where there are exactly two factors of the form $(1 + t^{m_i})^r$. We then have

$$(1 + t^d)^{2r} p(t) = [\det(I + f_1 t)](1 + t^{m_1})^r (1 + t^{m_2})^r. \tag{1}$$

There are two cases.

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Case 1. X is non-orientable.

Then 1 is always a root of $\det(I+f_1t)$ because the universal coefficient theorem implies that 1 is always an eigenvalue of

$$f_1: H_1(X;\mathbb{Z}/2\mathbb{Z}) \to H_1(X;\mathbb{Z}/2\mathbb{Z}).$$

Therefore, the right-hand side of equation (1) has at least $2r+1$ factors of $(1+t)$, so $(1+t)$ must factor $p(t)$. Then $p(1)=0$.

Case 2. X is orientable.

(Note that this is the only situation where $p(1)$ may not equal zero independent of its degree.) Suppose the degree of $p(t)$ is odd. The degree of $\det(I+f_1t)$ is even because X is orientable. Since the degree of the left-hand side of equation (1) is odd, we see that $(m_1+m_2)r$ is odd. So $r=1$. Consequently, either $p_1=m_1$ or $p_2=m_2$ is even. Suppose $p_1=2q_1$, then

$$1+t^{p_1}=(1+t^{q_1})^2.$$

Equation (1) becomes

$$(1+t^d)^2p(t)=[\det(I+f_1t)](1+t^{q_1})^2(1+t^{p_1}).$$

Because the right-hand side of this equation has at least three factors of $(1+t)$ we see that $p(1)=0$. □

2. *A reduction in the number of basic sets*

In this section we prove a geometric lemma which allows us to apply lemma 1 more generally. Actually this lemma can be helpful whenever one is concerned with the dynamics of one specific basic set of a diffeomorphism of a surface.

Definition. Suppose that $g: M\to M$ is a diffeomorphism of a closed, connected manifold M. An increasing sequence (with respect to inclusion) of compact manifolds

$$\varnothing = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

all of which have boundary except M_0 and M_n, is a *filtration* for g if

$$g(M_j) \subset \operatorname{int}(M_j)$$

for $j=1,2,\cdots,n$.

Given a filtration for g, the chain recurrent set $R(g)$ of g can be decomposed into disjoint, invariant pieces R_j by

$$R_j = R(g)\cap(M_j - M_{j-1}).$$

Also, given a filtration

$$\varnothing = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

for g, there corresponds a filtration

$$\varnothing = N_0 \subset N_1 \subset \cdots \subset N_n = M$$

for g^{-1} defined by

$$N_j = M - \operatorname{int} M_{(n-j)}.$$

We now prove a lemma involving the modification of filtrations when the manifold M is two-dimensional.

LEMMA 2. *Suppose $g: M \to M$ is a diffeomorphism of a closed, connected, two-dimensional manifold M and*

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is a filtration for g. Given an integer i between 1 and n, there exists a diffeomorphism f of a closed surface X such that

(1) *$R(f)$ has an invariant, disjoint decomposition of the form (hyperbolic sinks) $\amalg R \amalg$ (hyperbolic sources).*

(2) *$g|R_i$ is topologically conjugate to $f|R$. Moreover, if $g|R_i$ is hyperbolic with unstable dimension u, then $f|R$ is hyperbolic with unstable dimension u.*

Remark. The manifold X may be disconnected.

Proof. Start with $g|M_i$. Form a new closed manifold Y by attaching disks to the boundary circles of M_i. We shall extend $g: M_i \to M_i$ to a diffeomorphism $h: Y \to Y$.

The manifold $\overline{Y - M_i}$ is homeomorphic to a disjoint union of disks $\amalg_{l=1}^{k} D_l$. First, we prove that the Euler characteristic

$$\chi(\overline{Y - g(M_i)})$$

equals k.

$$\chi(Y) = \chi(\overline{Y - M_i}) + \chi(M_i).$$

So

$$\begin{aligned} k = \chi(\overline{Y - M_i}) &= \chi(Y) - \chi(M_i) \\ &= \chi(Y) - \chi(g(M_i)) \\ &= \chi(\overline{Y - g(M_i)}). \end{aligned}$$

To conclude that $\overline{Y - g(M_i)}$ is homeomorphic to $\amalg_{l=1}^{k} D_l$ we need only show that $\overline{Y - g(M_i)}$ has no more than k components. Each component of $\overline{Y - g(M_i)}$ contains at least one boundary circle of $g(M_i)$, and each boundary circle is contained in no more than one component. Since there are k such boundary circles, $\overline{Y - g(M_i)}$ has at most k components.

The extension $h: Y \to Y$ of $g: M_i \to M_i$ is determined by $g|\partial M_i$. The disk $h(D_l)$ is the disk in $\overline{Y - g(M_i)}$ whose boundary circle is $g(\partial D_l)$. The extension h can be defined so that $R(h) \cap (Y - M_i)$ is a union of hyperbolic periodic sources. Moreover,

$$R(h) \cap M_i = R(g) \cap M_i.$$

To conclude the proof of the lemma, apply the same argument to the inverse filtration on the inverse h^{-1}. □

3. *The proof of the theorem*

According to the hypothesis of the theorem, $g|R_i$ is topologically conjugate to a subshift of finite type. Using C^∞ Lyapunov functions, it is possible to construct a filtration for g

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

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such that $R(g) \cap (M_j - M_{j-1}) = R_j$ for $j = 1, 2, 3, \cdots, n$ [**10**], [**13**], [**6**]. Apply lemma 2 to this filtration. The resulting diffeomorphism satisfies the assumptions made in § 1. Hence, lemma 1 applies. This proves the theorem.

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REFERENCES

[**1**] M. Artin & B. Mazur. On periodic points. *Ann. Math.* **81** (1965), 82–99.
[**2**] R. Bowen. Entropy versus homology for certain diffeomorphisms. *Topology* **13** (1974), 61–67.
[**3**] R. Bowen & O. Lanford. Zeta functions of restrictions of the shift transformation. Proceedings of the Symposia in Pure Mathematics. *Amer. Math. Soc.* **14** (1970), 43–49.
[**4**] J. Franke & J. Selgrade. Hyperbolicity and chain recurrence. *J. Diff. Equations* **26** (1977), 27–36.
[**5**] J. Franks. A reduced zeta function for diffeomorphisms. *Amer. J. Math.* **100** (1978), 217–243.
[**6**] J. Franks. Homology and dynamical systems. The Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics. *Amer. Math. Soc.* (to appear).
[**7**] J. Guckenheimer. Axiom A and no cycles imply $\zeta_f(t)$ rational. *Bull. Amer. Math. Soc.* **76** (1970), 592–594.
[**8**] A. Manning. Axiom A diffeomorphisms have rational zeta functions. *Bull. London Math. Soc.* **3** (1971), 215–220.
[**9**] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817.
[**10**] S. Smale. The Ω-stability theorem. Proceedings of the Symposia in Pure Mathematics. *Amer. Math. Soc.* **14** (1970), 289–297.
[**11**] R. F. Williams. Classification of subshifts of finite type. *Ann. Math.* **98** (1973), 120–153.
[**12**] R. F. Williams. The zeta function of an attractor. *Proceedings of the Conference on the Topology of Manifolds* (Michigan State, 1967) (ed. J. G. Hocking), pp. 155–161. Prindle, Weber and Schmidt, 1968.
[**13**] F. W. Wilson Jr. Smoothing derivatives of functions and applications. *Trans. Amer. Math. Soc.* **139** (1969), 413–428.