

# Gaps of saddle connection directions for some branched covers of tori

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*Abstract.* We compute the gap distribution of directions of saddle connections for two classes of translation surfaces. One class will be the translation surfaces arising from gluing two identical tori along a slit. These yield the first explicit computations of gap distributions for non-lattice translation surfaces. We show that this distribution has support at zero and quadratic tail decay. We also construct examples of translation surfaces in any genus  $d > 1$  that have the same gap distribution as the gap distribution of two identical tori glued along a slit. The second class we consider are twice-marked tori and saddle connections between distinct marked points with a specific orientation. These results can be interpreted as the gap distribution of slopes of affine lattices. We obtain our results by translating the question of gap distributions to a dynamical question of return times to a transversal under the horocycle flow on an appropriate moduli space.

Key words: gap distribution, translation surface, affine lattice, horocycle flow, Poincaré section

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## 1. Introduction

We are interested in the distribution of directions of saddle connections on translation surfaces. A *translation surface* is a pair  $(X, \omega)$ , where  $X$  is a compact, connected Riemann surface of genus  $g$  and  $\omega$  is a non-zero holomorphic 1-form on  $X$ . The singular flat metric induced by  $\omega$  has isolated singularities at the zeros of  $\omega$ , with a zero of order  $k$  corresponding to a point with cone angle  $2\pi(k + 1)$ .

A *saddle connection* is a geodesic segment in the flat metric connecting two (not necessarily distinct) singular points, with no singular points in its interior. See Figure 1 for an example of a saddle connection on a translation surface. Integrating  $\omega$  over  $\gamma$  yields the *holonomy vector* of  $\gamma$ , which keeps track of how far and in what direction  $\gamma$  travels.

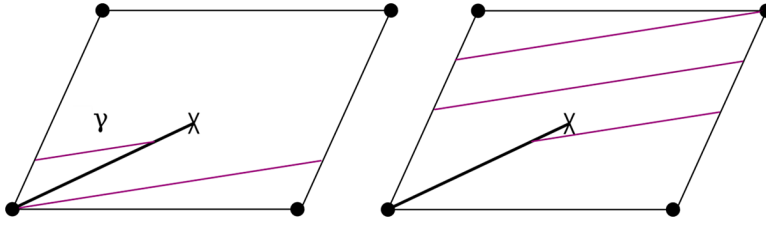


FIGURE 1. A saddle connection  $\gamma$  on a translation surface. This translation surface is an example of a doubled slit torus.

We write

$$v_\gamma := \int_\gamma \omega \in \mathbb{C}.$$

The set of all holonomy vectors,

$$\Lambda_\omega = \{v_\gamma : \gamma \text{ is an oriented saddle connection on } (X, \omega)\} \subset \mathbb{C}$$

is a discrete subset of the plane, see e.g. [16].

The counting and distribution of saddle connections and their holonomy vectors is a very well-studied problem and we review the relevant history in §1.1. Our motivating question is understanding how random the set of saddle connection directions are for some classes of translation surfaces. We study this by considering the *gap distribution* of slopes of holonomy vectors which we now describe.

The *slopes* of saddle connections of a translation surface  $(X, \omega)$  in the first quadrant and underneath the diagonal are given by

$$\mathcal{S}(\Lambda_\omega) := \left\{ \text{slope}(w) = \frac{w_2}{w_1} \mid w = w_1 + iw_2 \in \Lambda_\omega, 0 < w_1, 0 \leq w_2 \leq w_1 \right\}.$$

We will see in §2.2 why we need the restriction to the first quadrant and underneath the first diagonal. In fact, by applying a rotation matrix, we can move saddle connections outside of the first quadrant and underneath the diagonal into it so that this restriction is a mild one.

Let  $\mathcal{S}^R(\Lambda_\omega)$  be the set of slopes of saddle connections in  $\omega$  that are in the first quadrant, underneath the diagonal, and in the ball of radius  $R$  about 0 with respect to the max norm on  $\mathbb{R}^2$ ,

$$\mathcal{S}^R(\Lambda_\omega) = \left\{ \text{slope}(w) = \frac{w_2}{w_1} \mid w \in \Lambda_\omega, w_1 \in (0, R], w_2 \in [0, w_1] \right\}.$$

Let  $N(\omega, R) := |\mathcal{S}^R(\Lambda_\omega)|$  and write the elements of  $\mathcal{S}^R(\Lambda_\omega)$  in increasing order:

$$\mathcal{S}^R(\Lambda_\omega) = \{s_0 = 0 < s_1 < \dots < s_{N(\omega,R)}\}.$$

By the seminal work of Eskin and Masur [13], for almost every translation surface, there are positive constants  $c_1 = c_1(\omega) \leq c_2 = c_2(\omega)$  so that

$$c_1 R^2 \leq N(\omega, R) \leq c_2 R^2.$$

Later on, Eskin, Marklof, and Morris [12] obtained similar asymptotics for branched covers of lattice surfaces.

We are interested in slope gaps and since  $N(\omega, R)$  grows quadratically, then it is natural to consider the set of *renormalized slope gaps*,

$$\mathcal{G}^R(\Lambda_\omega) = \{R^2 \cdot (s_i - s_{i-1}) \mid i = 1, \dots, N(\omega, R)\}. \quad (1)$$

The *gap distribution* of saddle connection directions on a translation surface  $(X, \omega)$  is given by the weak-\* limit as  $R \rightarrow \infty$  of the probability measures

$$\eta_R(\omega) = \frac{1}{N(\omega, R)} \sum_{x \in \mathcal{G}^R(\Lambda_\omega)} \delta_x \quad (2)$$

if it exists.

Evaluating the weak-\* limit of these measures on the indicator function of an interval  $I$ , we obtain

$$\lim_{R \rightarrow \infty} \frac{|\mathcal{G}^R(\Lambda_\omega) \cap I|}{N(\omega, R)} \quad (3)$$

the proportion of gaps in an interval  $I$ .

1.1. *Motivations for gap distributions.* The counting and distribution properties of saddle connections and their directions have been studied extensively. Masur [18] showed that the *angles of saddle connections* are dense on the circle for any translation surface. Vorobets [30] and Eskin and Masur [13] showed that the angles equidistribute on the circle for a generic translation surface. More recently, Dozier [9, Theorem 1.3] showed that for every translation surface, the distribution of angles converge to a measure that is in the same measure class as the Lebesgue measure on the circle. Precise descriptions of their results are in the references above, but they suggest that angles (and thus slopes) seem to behave ‘randomly’ at first glance. Thus, the gap distribution of translation surfaces can be thought of as a second test of randomness that yields insight to the fine-scale statistics of holonomy vectors.

Probability theory provides some context for what one would expect from ‘truly random’ gaps. In the case of a sequence of independent identically distributed uniform random variables on  $[0, 1]$ , the gaps converge to an exponential distribution. Following [2], we call gap distributions that deviate from an exponential distribution *exotic*.

1.2. *Main results.* We compute the gap distribution for the set of translation surfaces that arise from gluing two identical tori along a slit. See Figure 1. We call such a surface a *doubled slit torus*. Every doubled slit torus has genus two and two cone-type singularities of angle  $4\pi$ . Denote the class of doubled slit tori by  $\mathcal{E}$ . We prove that the gap distribution of doubled slit tori exists and is given by an explicit density function. This provides the first explicit computation of a gap distribution for a non-lattice surface.

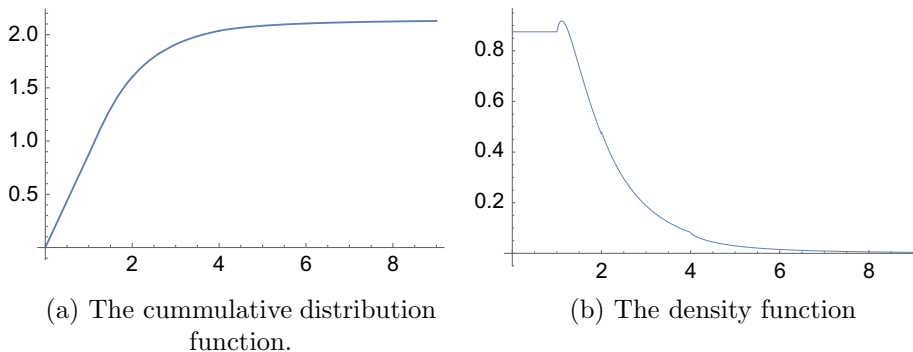


FIGURE 2. Proportion of gaps for doubled slit tori.

**THEOREM 1.1.** *There is a limiting probability density function  $f : [0, \infty) \rightarrow [0, \infty)$ , with*

$$\lim_{R \rightarrow \infty} \frac{|\mathcal{G}^R(\Lambda_\omega) \cap I|}{N(\omega, R)} = \int_I f(x) dx$$

*for almost every doubled slit torus  $\omega$  with respect to an  $SL_2(\mathbb{R})$ -invariant probability measure on  $\mathcal{E}$  that is in the same measure class as the Lebesgue measure.*

*The density function is continuous, piecewise differentiable with four points of non-differentiability, and each piece is expressed in terms of elementary functions.*

The proof of Theorem 1.1 can be found in §4 and an explicit description of the cumulative distribution function can be found in Appendix A.1.3. As a corollary, we also show that the gap distribution has a quadratic tail so that the gap distribution of doubled slit tori is exotic (Theorem 4.4(3)). We prove other results related to the gap distribution, but we wait until §4 Theorem 4.4 before more precisely stating them.

Perhaps surprisingly, Theorem 1.1 will follow from the following dynamical result (relevant definitions will be given in §2).

**THEOREM 1.2.** *The set  $\mathcal{W}$  consisting of doubled slit tori with a horizontal saddle connection of length less than or equal to 1 forms a transversal for the space of doubled slit tori  $\mathcal{E}$  under horocycle flow. There is a four-dimensional parameterization of  $\mathcal{W}$  (see Proposition 3.10) and the return time map in these coordinates is given explicitly (see Theorem 3.11).*

In fact, the connection between the slope gap questions and return times on moduli space is relatively well understood, see Athreya [2] or §2.3 where we recall this connection. The main contribution of this theorem is the explicit return time map (Theorem 3.11) that requires new techniques arising from the high dimensionality of the parameter space. Another key ingredient in proving Theorem 1.2 is the study of unipotent flows on the space of affine lattices. By an affine lattice we simply mean a set of the form  $g\mathbb{Z}^2 + v$ , where  $g \in SL_2(\mathbb{R})$  and  $v \in \mathbb{R}^2$ . We prove the following result for the space of affine lattices.

**THEOREM 1.3.** *The set  $\Omega$  consisting of affine lattices with a horizontal vector of length less than or equal to 1 forms a transversal for the space of affine lattices  $AX_2$  under*

horocycle flow. There is a four-dimensional parameterization of  $\Omega$  (see Proposition 3.5) and the return time map in these coordinates is given explicitly (see Theorem 3.6).

In addition to using the above theorem to prove Theorem 1.2, we also use it to prove a slope gap question for affine lattices that we phrase in §2.5.

1.3. *Why doubled slit tori are important.* The space  $\mathcal{E}$  of doubled slit tori has a long history of being studied to provide insight into general questions on the dynamics and geometry of translation surfaces. That the space and the surfaces within it have very concrete descriptions make it an excellent testing ground for conjectures.

For example, Masur [20] showed that the space  $\mathcal{E}$  provided the first examples of higher genus surfaces with a minimal but non-uniquely ergodic directional flow. Cheung [6] used these surfaces to show that an upper bound on Hausdorff dimension of the directions which are non-ergodic was sharp. A dichotomy on the Hausdorff dimension of such directions for doubled slit tori was later given by Cheung *et al* [7].

The space  $\mathcal{E}$  is also interesting from the point of view of the  $\mathrm{SL}_2(\mathbb{R})$  action on translation surfaces. In the remarkable paper [21], McMullen completely classifies orbit closures (under  $\mathrm{SL}_2(\mathbb{R})$ ) of genus 2 translation surfaces. One consequence of his work is that there are translation surfaces that have non-closed  $\mathrm{SL}_2(\mathbb{R})$  orbits but are not dense. These intermediate orbit closures consist of an infinite family called the eigenform loci. The space  $\mathcal{E}$  is a part of this family and often called the *Eigenform locus of discriminant four* in the literature.

1.4. *Some history of gap distributions.* This paper makes use of unipotent flows on the space of affine lattices to prove a gap distribution result for doubled slit tori. Elkies and McMullen [10] famously used unipotent flows on the space of affine lattices to compute the distribution of gaps of the sequence  $(\sqrt{n})_{n \in \mathbb{N}}$  on the unit circle. Similarly, Marklof and Strömbergsson [17] used dynamics on the space of affine lattices to study the gap distribution for the angles of visible affine lattice points.

Through personal communication with Athreya, we learned that the paper of Marklof and Strömbergsson provided much inspiration for the use of homogenous dynamics in the context of gap distributions of translation surfaces. Athreya and Chaika [3] showed that the gap distribution for generic surfaces exists and has support at zero. Gap distributions of lattice surfaces were considered shortly thereafter. Athreya *et al* [4] computed the explicit gap distribution for a genus 2 lattice surface (the *Golden L*) and Uyanik and Work [29] for the genus 2 surface arising from the regular octagon. In some sense, these papers were generalizations of Athreya and Cheung [5], which used homogenous dynamics to compute gap distribution of Farey fractions (which can be interpreted as slopes of saddle connections of the torus with a marked point). There is also the related work by Heersink [15] who considered the limiting gap distribution of various subsets of Farey fractions. In all the cases above, the parameter space is three-dimensional whereas the parameter space for doubled slit tori and affine lattice is five-dimensional.

The present paper and [4, 5, 29] are examples of the general strategy developed by Athreya [2]. This strategy involves turning the question of gap distributions of slopes of

saddle connections to a dynamical question of return times to a transversal under horocycle flow on an appropriate moduli space. We recall this strategy in detail in §2. Examples of explicit cross-sections for horocycle flow can also be found in [28, 31].

1.5. *Organization of the paper.* In §2, we outline the general strategy from [2], which reduces the gap distribution question to a dynamical question. We also introduce twice-marked tori and construct transversals needed for our paper. In §3, we give a parameterization of the transversals for twice-marked tori (Proposition 3.5) and doubled slit tori (Proposition 3.10). We also compute the first return time under horocycle flow for each (Theorem 3.6 and Theorem 3.11). In §4, we consider measures on our transversals, classify measures for the transversal on twice-marked tori (Theorem 4.2), and give precise descriptions of our results (Theorem 4.4). In Corollary 4.6, we construct examples of translation surfaces in any genus  $d > 1$  that have the same gap distribution as doubled slit tori. In the appendix, we prove our estimates for the various gap distributions we consider.

## 2. Setting the stage

Here, we will outline the general strategy from [2], introduce the relevant spaces, and define our transversals.

2.1. *On translation surfaces.* We review the basics of translation surfaces needed for this article. For a detailed treatment of these topics, we refer the reader to [19, 20, 32].

Recall that a translation surface is a pair  $(X, \omega)$ , where  $X$  is a compact, connected Riemann surface of genus  $g$  and  $\omega$  is a non-zero holomorphic 1-form on  $X$ . We can also, equivalently, view a translation surface as a union of finitely many polygons  $P_1 \cup P_2 \cup \dots \cup P_n$  in the Euclidean plane with gluings of parallel sides by translations such that for each edge, there exists a parallel edge of the same length and these pairs are glued together by a Euclidean translation. In this case, the total angle about each vertex of the polygons is necessarily an integer multiple of  $2\pi$ . The vertices where the total angle  $\theta$  is greater than  $2\pi$  are called *cone points* and correspond to the zeros of the 1-form. In fact, a zero of order  $k$  is a cone point of total angle  $2\pi(k + 1)$ .

By the Riemann–Roch theorem, the sum of order of the zeros is  $2g - 2$ , where  $g$  denotes the genus of  $X$ . Thus, the space of genus  $g$  translation surfaces can be stratified by integer partitions of  $2g - 2$ . If  $\underline{k} = (k_1, \dots, k_s)$  is an integer partition of  $2g - 2$ , we denote by  $\mathcal{H}(\underline{k})$  the moduli space of translation surfaces  $\omega$ , such that the multiplicities of the zeroes are given by  $k_1, \dots, k_s$ . As an example, any doubled slit torus belongs in the stratum  $\mathcal{H}(1, 1)$ . In the case that there are no zeros of the differential but we want to mark points, it is common to use a vector  $\underline{k}$  with all zeros. For example, the space of genus 1 translation surfaces have no cone points, but it is convenient to mark a single point and denote it as  $\mathcal{H}(0)$ .

There is a natural action by  $\mathrm{SL}_2(\mathbb{R})$  on the space of translation surfaces. This is most easily seen via the polygon definition: Given a translation surface  $(X, \omega)$  that is a finite union of polygons  $\{P_1, \dots, P_n\}$  and  $A \in \mathrm{SL}_2(\mathbb{R})$ , we define  $A \cdot (X, \omega)$  to be the translation surface obtained by the union of the polygons  $\{AP_1, \dots, AP_n\}$  with the same side gluings as for  $\omega$ . This action makes sense since the linear action of  $A$  preserves the

notion of parallelism. Notice that the action of  $SL_2(\mathbb{R})$  preserves the multiplicities and number of zeros so that it induces an action on each strata  $\mathcal{H}(k)$ . Furthermore,  $SL_2(\mathbb{R})$  acts equivariantly on the holonomy vectors of translation surfaces in that  $\Lambda_{g \cdot \omega} = g \cdot \Lambda_\omega$ . A *lattice surface* is a translation surface  $(X, \omega)$  whose stabilizer under the  $SL_2(\mathbb{R})$ -action is a lattice in  $SL_2(\mathbb{R})$ .

We will consider the action of the one-parameter family of matrices

$$\left\{ h_u := \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \in SL_2(\mathbb{R}) : u \in \mathbb{R} \right\}.$$

The action of these matrices on  $\mathcal{H}(k)$  is called the *horocycle flow*.

2.2. *Reducing to gaps in a vertical strip.* We now phrase a parallel gap question. We will see that this question will turn out to be equivalent to our original one. Let  $V$  denote the vertical strip  $V = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1, y > 0\}$ . For a translation surface  $(X, \omega)$ , we can consider the set of slopes of saddle connections contained in the vertical strip  $V$

$$\mathcal{S}_V(\Lambda_\omega) = \{0 = s_0 < s_1 < \dots < s_N < \dots\}.$$

Let  $\mathcal{S}_V^N$  denote the first  $N$  slopes. We can associate to these slopes the set of *gaps*

$$\mathcal{G}_V^N(\Lambda_\omega) = \{s_i - s_{i-1} \mid i = 1, \dots, N\}. \tag{4}$$

Notice that the slopes tend to infinity and so we do not need to normalize. Then the gap distribution for slopes in the vertical strip given by

$$\lim_{N \rightarrow \infty} \frac{|\mathcal{G}_V^N(\Lambda_\omega) \cap I|}{N} \tag{5}$$

for an interval  $I$ . We prove the following.

**THEOREM 2.1.** *There is a limiting probability density function  $f : [0, \infty) \rightarrow [0, \infty)$ , with*

$$\lim_{N \rightarrow \infty} \frac{|\mathcal{G}_V^N(\Lambda_\omega) \cap I|}{N} = \lim_{R \rightarrow \infty} \frac{|\mathcal{G}^R(\Lambda_\omega) \cap I|}{N(\omega, R)} = \int_I f(x) dx$$

for almost every doubled slit torus  $\omega$  with respect to an  $SL_2(\mathbb{R})$ -invariant probability measure on  $\mathcal{E}$  that is in the same measure class as the Lebesgue measure. The density function is continuous, piecewise differentiable with four points of non-differentiability, and each piece is expressed in terms of elementary functions.

The density function  $f$  is the same for both gap distribution questions. For the remainder of the paper, we focus on proving the above theorem since it implies Theorem 1.1. Indeed, the two gap distribution questions are related by applying the diagonal matrix

$$\gamma_R = \begin{pmatrix} R^{-1} & 0 \\ 0 & R \end{pmatrix}$$

which takes vectors in the first quadrant with (max norm) norm less than or equal to  $R$  and sends them to vectors in the vertical strip. Notice  $\gamma_R$  changes slopes of vectors in the first quadrant by a factor of  $1/R^2$  and so the renormalized gaps from equation 1 become the

unnormalized gaps from equation 4. Thus,

$$\lim_{R \rightarrow \infty} \frac{|\mathcal{G}^R(\Lambda_\omega) \cap I|}{N(\omega, R)} = \lim_{R \rightarrow \infty} \frac{|\mathcal{G}_V^{N(\omega, R)}(\gamma_R \cdot \Lambda_\omega) \cap I|}{N(\omega, R)}$$

and we have reduced proving Theorem 1.1 to proving Theorem 2.1, a gap distribution for the gaps coming from slopes of vectors in the vertical strip  $V$ .

2.3. *From gaps to transversals.* We show how to implement the strategy from [2], where they translated the question of gaps between slopes of vectors into return times under the horocycle flow on an appropriate moduli space to a specific transversal. This method was utilized in [4, 5, 29]. By *transversal* to horocycle flow, we mean a subset  $T$  so that almost every orbit under horocycle flow intersects  $T$  in a non-empty, countable, discrete set of times.

We consider the transversal on  $\mathcal{E}$  under horocycle flow given by the set of doubled slit tori that contain a non-zero horizontal saddle connection with horizontal component less than or equal to 1 and denote this by  $\mathcal{W}$ . More explicitly, this transversal is

$$\mathcal{W} = \{\omega \in \mathcal{E} : \Lambda_\omega \cap (0, 1] \neq \emptyset\},$$

where  $\Lambda_\omega$  denotes the set of holonomies of saddle connections. We will call a horizontal saddle connection with a horizontal component less than or equal to one a *short* saddle connection and an element in  $\mathcal{W}$  a *short* doubled slit torus. Occasionally the phrase ‘cross-section’ or ‘Poincaré section’ will be used instead of transversal, but these all mean the same thing in this paper. Note that  $\mathcal{W}$  is indeed a transversal to horocycle flow by Lemma 2.1 in [2].

Let us denote the return time map to  $\mathcal{W}$  under horocycle flow by  $\mathcal{R}$  and the return map by  $\mathcal{T}$ . Then,  $\mathcal{R} : \mathcal{W} \rightarrow (0, \infty)$  is given by

$$\mathcal{R}(\omega) = \min\{u > 0 : h_u(\omega) \in \mathcal{W}\}$$

and  $\mathcal{T} : \mathcal{W} \rightarrow \mathcal{W}$  is given by

$$\mathcal{T}(\omega) = h_{\mathcal{R}(\omega)}(\omega).$$

A simple, but key, observation is that the horocycle flow preserves slope differences of vectors  $\vec{v}$  since

$$\text{slope}(h_u \vec{v}) = \text{slope}(\vec{v}) - u.$$

Consequently, this holds for holonomy vectors of saddle connections. This observation links slopes with  $\mathcal{W}$  since if we apply horocycle flow to an element in  $\mathcal{W}$ , we see that the next vector to become short is the vector with short horizontal component and smallest slope, and the return time to the Poincaré section is exactly its slope. Similarly, the second vector to become short will be the vector with short horizontal component and second smallest slope and the return time will be its slope minus the return time of the first, which is a *slope difference*. Continuing in this fashion, we see that the  $i$ th-return time is the difference of the  $i$ th + 1 slope and the  $i$ th. This is formalized by

$$\mathcal{R}(\mathcal{T}^i(\omega)) = s_{i+1} - s_i.$$



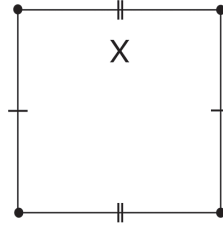


FIGURE 3. An example of a twice-marked torus. This is the standard torus of area 1 with marked points at the origin and at  $(\frac{1}{2}, \pi/3)$ .

Hence, we can relate our gap distribution to this induced dynamical system via

$$\frac{1}{N} |\mathcal{G}_V^N(\Lambda_\omega) \cap I| = \frac{1}{N} \sum_{i=0}^{N-1} \chi_{\mathcal{R}^{-1}(I)}(\mathcal{T}^i(\Lambda_\omega)), \tag{6}$$

where  $\mathcal{R}^{-1}(I)$  is the set of slope differences in an interval  $I$ . Thus, the gap distribution reduces to understanding return times of  $\mathcal{W}$ . The remainder of this paper will be concerned with computing the return times.

2.4. *Moduli space.* We introduce the moduli space of twice-marked tori and explain its connection to the space of doubled slit tori  $\mathcal{E}$ .

By a *twice-marked torus*, we simply mean a (unit area) torus with a preferred vertical direction and two marked points. We denote the set of all twice-marked tori by  $\mathcal{H}(0, 0)$ . Concretely, a twice-marked torus is given by the data of a torus  $\mathbb{C}/g\mathbb{Z}^2$ , the flat metric  $dz$  inherited from the complex plane, and two distinct marked points  $v_1 + \mathbb{Z}^2, v_2 + \mathbb{Z}^2$  where  $g \in \text{SL}_2(\mathbb{R})$ . We will always assume that the first marked point is at the origin and denote the second marked point by  $v$ . Denote such an element by  $\mathbb{T}_{g,v}^2$ . To each  $\mathbb{T}_{g,v}^2$ , we can associate the *affine lattice* given by the orbit of  $v$  under the lattice  $g\mathbb{Z}^2$ ,

$$g\mathbb{Z}^2 + v.$$

Denote the space of all affine lattices by  $AX_2$ . Given an affine lattice  $g\mathbb{Z}^2 + v$  and taking its quotient by  $g\mathbb{Z}^2$  yields a twice-marked torus  $\mathbb{T}_{g,v}^2$ . It is easy to see that these operations are inverses and so there is a bijection between the space of twice-marked tori  $\mathcal{H}(0, 0)$  and the space of affine lattices  $AX_2$ .

Recall the affine action of  $\text{SL}_2(\mathbb{R}) \times \mathbb{R}^2$  on  $\mathbb{R}^2$  given by  $(h, w) \cdot v = hv + w$ . Thus, we can consider the natural action of  $\text{SL}_2(\mathbb{R}) \times \mathbb{R}^2$  on  $AX_2$  defined by

$$(h, w) \cdot (g\mathbb{Z}^2 + v) = (hg)\mathbb{Z}^2 + (hv + w).$$

Note that  $\text{SL}_2(\mathbb{R}) \times \mathbb{R}^2$  acts transitively on  $AX_2$  and that the stabilizer of  $\mathbb{Z}^2$  is given by  $\text{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2$ . Thus, we may identify  $AX_2$  (and hence  $\mathcal{H}(0, 0)$ ) as the homogenous space

$$\mathcal{H}(0, 0) = AX_2 = \text{SL}_2(\mathbb{R}) \times \mathbb{R}^2 / \text{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2.$$

We denote an element in any of the above spaces (and hence all) by  $(g, v)$  and leave the coset implicitly defined. Denote by  $X_2$  the *space of lattices* which we can identify with the

homogeneous space

$$X_2 = \text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z}).$$

There is the projection map  $\mathcal{H}(0, 0) = AX_2 \rightarrow X_2$  given by  $(g, v) \mapsto g$ . This will be a torus bundle since the fiber of a point  $g\mathbb{Z}^2$  in  $X_2$  is a vector  $v$  that is only defined up to  $g\mathbb{Z}^2$  and hence we think of  $v$  as living in the torus  $\mathbb{R}^2/g\mathbb{Z}^2$ .

Recall, that  $\mathcal{E}$  is the family of translation surfaces obtained by gluing two identical tori along a slit. Every such surface projects to a twice-marked torus by restricting to one torus and forgetting the slit. By reversing this process, we can obtain a doubled slit torus from a twice-marked torus, although some care must be taken. Indeed, suppose we have an element in  $\mathcal{H}(0, 0)$  which depends on the data  $g \in \text{SL}_2(\mathbb{R})$  and  $v \in \mathbb{C}/g\mathbb{Z}^2$ . We can take two copies of it and then consider a slit between the marked points to construct an element  $\omega = \omega_{(g,v)}$  in  $\mathcal{E}$ . There are four oriented trajectories between the marked points that we can choose corresponding to the ‘corners’ of the fundamental domain of  $\mathbb{C}/g\mathbb{Z}^2$  spanned by the columns of  $g$ . Hence, each pair  $(g, v) \in \mathcal{H}(0, 0)$  gives rise to four elements in  $\mathcal{E}$ , which is to say there is a degree-four map

$$\Pi : \mathcal{E} \rightarrow \mathcal{H}(0, 0)$$

given by ‘forgetting the slit’. Any time  $\Pi(\omega) = (g, v)$ , we use the notation  $\omega = \omega_{(g,v)}$ . Since the degree of  $\Pi$  is four, then  $\mathcal{E}$  can be identified as four copies of  $\mathcal{H}(0, 0)$ , one copy for each oriented trajectory between the marked points.

Ultimately, we are interested in saddle connections of elements in  $\mathcal{E}$  and one way to find saddle connections of a doubled slit torus is to look for trajectories between marked points of the twice-marked torus. This explains our interest in  $\mathcal{H}(0, 0)$ .

2.5. *Gap distribution for twice-marked tori.* We also consider the gap distribution of twice-marked tori  $\mathcal{H}(0, 0)$ . However, instead of considering all trajectories between marked points, we will only be interested in those between *distinct* marked points with a specific orientation. The reason for this is because these trajectories define an affine lattice.

Let  $\Lambda_\omega^{d,+}$  denote the holonomies of trajectories that start at the origin and end at the other marked point. Let  $\mathcal{S}_V(\Lambda_\omega^{d,+})$  denote the set of slopes of holonomy vectors in the vertical strip  $V$ . Then as before, we can order them

$$\mathcal{S}_V(\Lambda_\omega^{d,+}) = \{0 = s_0 < s_1 < \dots < s_N < \dots\}.$$

Let  $\mathcal{S}_V^N$  denote the first  $N$  slopes. We can associate to these slopes the set of *gaps*

$$\mathcal{G}_V^N(\Lambda_\omega^d) = \{s_i - s_{i-1} \mid i = 1, \dots, N\}$$

and consider a gap distribution question for the slopes in the vertical strip  $V$

$$\lim_{N \rightarrow \infty} \frac{|\mathcal{G}_V^N(\Lambda_\omega^{d,+}) \cap I|}{N}$$

for an interval  $I$ . We prove the following.

THEOREM 2.2. *There is a limiting probability density function  $g : [0, \infty) \rightarrow [0, \infty)$ , with*

$$\lim_{N \rightarrow \infty} \frac{|\mathcal{G}_V^N(\Lambda_\omega^{d,+}) \cap I|}{N} = \int_I g(x) dx$$

for almost every twice-marked torus  $\omega \in \mathcal{H}(0, 0)$  with respect to an  $SL_2(\mathbb{R})$ -invariant probability measure on  $\mathcal{H}(0, 0)$  that is in the same measure class as the Lebesgue measure.

In fact, we prove that the slope gap distribution for twice-marked tori *always* exists, but is not always the same. This is done by proving a measure classification result for the transversal under the map induced by horocycle flow (see §4.1). The gap distribution density is the same for twice-marked tori that are in the support of the same measure.

Moreover, if  $\omega \in \mathcal{H}(0, 0)$  corresponds to  $(g, v) \in AX_2$ , then  $\Lambda_\omega^{d,+}$  is the affine lattice  $g\mathbb{Z}^2 + v$ . See §3.2.1 for a detailed discussion on this. Hence, the above theorem can be interpreted as a theorem about gaps of slopes of an affine lattice. The distribution of gaps of angles of affine lattices was studied in [17]. They show that the gap distribution agrees with the distribution of gaps of the sequence  $(\sqrt{n})_{n \in \mathbb{N}}$  on the unit circle found by Elkies and McMullen [10].

A consequence of Theorem 2.2 is that the gap distribution of affine lattices has a quadratic tail and hence is not exotic (Theorem 4.4(1)). As with the gap distribution of doubled slit tori, we prove much finer results, but we wait until §4 Theorem 4.4 before more precisely stating them. Our proof technique is the same as for doubled slit tori, namely, we translate the gap distribution question to one of return times to a transversal on the moduli space of twice-marked tori  $\mathcal{H}(0, 0)$ . The explicit computation of these return times addresses a question from [2, §6.2].

The transversal we consider is given by the set of twice-marked tori that have a horizontal saddle connection that starts at the origin and ends at the other marked point with a length less than or equal to 1. Denote this set by  $\Omega$ . Thus,

$$\Omega = \{\omega \in \mathcal{H}(0, 0) : \Lambda_\omega^{d,+} \cap (0, 1] \neq \emptyset\}.$$

Under the identification of  $\mathcal{H}(0, 0)$  and  $AX_2$ ,  $\Omega$  is the same as the set of affine lattices that contain a non-zero horizontal vector with horizontal component less than or equal to 1. That is,  $\Omega$  can be identified with

$$\{\Lambda \in AX_2 : \Lambda \cap (0, 1] \neq \emptyset\}.$$

We will abuse notation and refer to the above set as  $\Omega$ . The set  $\Omega \subset \mathcal{H}(0, 0)$  is a transversal for horocycle flow. We postpone proof of these statements until §3. We will call a horizontal vector with a horizontal component less than or equal to one a *short* vector and an element in  $\Omega$  a *short* affine lattice.

Henceforth, we will only think about elements  $(g, v)$  as affine lattices. Furthermore, unless explicitly stated, we will assume that for an element  $(g, v) \in \mathcal{H}(0, 0)$  or  $\omega_{(g,v)} \in \mathcal{E}$ , we have  $v \notin g\mathbb{Q}^2$ . This assumption only removes a set of measure zero elements from our spaces and, since we are concerned only with almost everywhere results, is a mild one.

3. *Parameterization of the transversal and return times*

We give coordinates to the transversals from the last section and compute the first return time under these coordinates.

3.1. *On twice-marked tori  $\mathcal{H}(0, 0)$ .* Here, we parameterize our Poincaré section  $\Omega$  and compute the first return time to  $\Omega$ . We also recall some results from [5] that will aid us.

3.1.1. *A Poincaré section for  $X_2$ .* Before parameterizing our section, we recall some results from [5] where they considered a Poincaré section for  $X_2$ , which they used to obtain statistical properties of Farey fractions. We fix the notation

$$p_{a,b} := \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

and

$$g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Let  $\Delta$  denote the set of lattices that contains a short vector (recall that *short* for us means length less than or equal to 1). It is shown in [5] that this set is

$$\Delta = \left\{ \Lambda_{a,b} := p_{a,b}\mathbb{Z}^2 = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mathbb{Z}^2 : 0 < a \leq 1, 1 - a < b \leq 1 \right\}.$$

There is a bijection between  $\Delta$  with the actual triangle

$$\{(a, b) : 0 < a \leq 1, 1 - a < b \leq 1\} \subset \mathbb{R}^2$$

via  $(a, b) \mapsto \Lambda_{a,b}$  and thus we give an element in  $\Lambda_{a,b} \in \Delta$  the coordinates  $(a, b)$ . Because of the bijection, we abuse notation and use  $\Delta$  for the subset of  $X_2$  and the actual triangle.

We now restate the main theorem [5, Theorem 1.1] with slightly different notation.

**THEOREM 3.1.** *The subset  $\Delta \subset X_2$  is a Poincaré section for the horocycle action on  $X_2$ . More precisely, every horocycle orbit  $\{h_s\Lambda\}_{s \in \mathbb{R}}$  (outside a codimension 1 set of lattices) intersects  $\Delta$  in a non-empty, countable, discrete set of times.*

*Moreover, the first return time map  $r : \Delta \rightarrow [0, \infty)$  is given by*

$$r(\Lambda_{a,b}) = r(a, b) = \min\{s > 0 : h_s\Lambda_{a,b} \in \Delta\} = (ab)^{-1}$$

*and the return map  $S : \Delta \rightarrow \Delta$  is*

$$S(\Lambda_{a,b}) := h_{r(a,b)}\Lambda_{a,b} := \Lambda_{S(a,b)}$$

*where  $S(a, b) = (b, -a + \lfloor (1 + a)/b \rfloor b)$ .*

They also identify the codimension 1 set as those lattices that contain a short vertical vector, which geometrically correspond to embedded closed horocycles that foliate the cusp. Dynamically, for any  $0 < a \leq 1$ , each horocycle is a periodic orbit under horocycle

flow  $h_s$  with period  $a^2$  and orbit

$$\text{Per}(a) := \left\{ h_s g_{\log(a)} \mathbb{Z}^2 = \begin{pmatrix} 0 & -a^{-1} \\ a & sa^{-1} \end{pmatrix} \mathbb{Z}^2 : s \in [0, a^2) \right\}.$$

Notice above, we implicitly used that

$$h_s g_{\log(a)} \mathbb{Z}^2 = h_s g_{\log(a)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbb{Z}^2 = \begin{pmatrix} 0 & -a^{-1} \\ a & sa^{-1} \end{pmatrix} \mathbb{Z}^2$$

so as to further drive the point that such elements are outside of the generic set of lattices. Of course, any  $a > 0$  gives rise to a periodic orbit  $\text{Per}(a)$ , but only the  $0 < a \leq 1$  give rise to points that never enter the transversal  $\Delta$ .

As a consequence of their theorem, we have that every  $g \in X_2$  is one of the two forms.

- (1)  $g = h_s p_{a,b}$ , where  $0 < a \leq 1$ ,  $1 - a < b \leq 1$ , and  $0 \leq s < (ab)^{-1}$ . These are the generic elements in  $X_2$ .
- (2)  $g = h_s g_{\log(a)}$ , where  $0 \leq a < 1$  and  $0 \leq s < a^2$ . These are the elements in the codimension 1 set of lattices with a short vertical vector.

3.1.2. *A Poincaré section for  $AX_2$ .* In this section, we show  $\Omega$  is indeed a transversal and give coordinates to  $\Omega$ . Observe that since the dimension of the space of affine lattices  $AX_2$  is five, the cross-section by horocycle flow will have dimension equal to four. Thus, we aim to find a four-dimensional parameterization.

We prove a lemma that says that any affine lattice  $(g, v) \in \Omega$  can be expressed as an affine lattice where the affine piece is horizontal.

LEMMA 3.2. *The set of short affine lattices is given by*

$$\Omega = \left\{ (g, v) \in AX_2 : v = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \text{ and } \alpha \in (0, 1] \right\}.$$

*Proof.* Suppose that  $(g, v) \in \Omega$ , that is that the affine lattice  $(g, v)$  contains a short horizontal vector. Recall that  $v$  is only defined up to elements in  $g\mathbb{Z}^2$ . Hence, if  $w \in g\mathbb{Z}^2$ , then as an affine lattice, we have

$$g\mathbb{Z}^2 + v = g\mathbb{Z}^2 + (v + w).$$

In particular, if we assume our affine lattice is short, then there is some  $w \in g\mathbb{Z}^2$  so that  $w + v$  is a short horizontal vector. In particular, it is of the form

$$w + v = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

for some  $\alpha \in (0, 1]$ . Thus,

$$g\mathbb{Z}^2 + v = g\mathbb{Z}^2 + v + w = g\mathbb{Z}^2 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}. \quad \square$$

Armed with the above lemma, we also provide proof of our claims from the previous section.

LEMMA 3.3. *Under the identification of  $\mathcal{H}(0, 0)$  and  $AX_2$ , we have that*

$$\{\omega \in \mathcal{H}(0, 0) : \Lambda_\omega^{d,+} \cap (0, 1] \neq \emptyset\}$$

*is the same as the set of affine lattices that contain a non-zero horizontal vector with horizontal component less than or equal to 1,*

$$\{\Lambda \in AX_2 : \Lambda \cap (0, 1] \neq \emptyset\}.$$

*Proof.* This follows from the identification of  $\mathcal{H}(0, 0)$  and  $AX_2$  and use of Proposition 3.9 that yields if  $\omega \in \mathcal{H}(0, 0)$  is the twice-marked torus formed by  $(g, v)$ , then  $\Lambda_\omega^{d,+} = g\mathbb{Z}^2 + v$ .

If  $\omega \in \Omega$  with short horizontal saddle connection  $\gamma$  with holonomy in  $\Lambda_\omega^{d,+}$ , then tiling  $\mathbb{C}$  by a fundamental domain of  $\omega$  produces an affine lattice with a short horizontal vector given by the holonomy of  $\gamma$ . Conversely, if we are given an affine lattice  $\Lambda$  with  $\Lambda \cap (0, 1] \neq \emptyset$ , then we can assume the affine part is producing the vector in  $(0, 1]$  by the previous lemma. That is, we can assume

$$\Lambda = g\mathbb{Z}^2 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

for  $\alpha \in (0, 1]$ . Taking the quotient by  $g\mathbb{Z}^2$  produces a twice-marked torus along with a saddle connection  $\gamma$  that has holonomy  $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ . □

LEMMA 3.4. *The set  $\Omega$  is a transversal for horocycle flow. That is, almost every orbit under horocycle flow intersects  $\Omega$  in a non-empty, countable, discrete set of times.*

*Proof.* This follows along the same lines of justification as Lemma 2.1 of Athreya [2] and the discussion in §2.3. Notice that generically an affine lattice  $\Lambda \in AX_2$  has infinitely many vectors contained in the vertical strip  $V = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1, y > 0\}$ . Then,  $h_u\Lambda \in \Omega$  exactly when  $u$  is the slope of the smallest vector contained  $\Lambda \cap V$ . Similarly,  $h_{u'}h_u\Lambda \in \Omega$  when  $u'$  is the slope of the smallest vector contained  $h_u(\Lambda) \cap V$ . This shows almost every orbit under horocycle flow intersects  $\Omega$  in a non-empty, countable set of times. These intersections have to happen at discrete times since  $\Lambda$  itself is discrete. □

We use this lemma along with the coordinates on the space of lattices developed in [5] to parameterize  $\Omega$ .

PROPOSITION 3.5. *The set of short affine lattices is given by the union of*

$$\left\{ \left( h_s p_{a,b}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right) : a \in (0, 1], b \in (1 - a, 1], s \in [0, (ab)^{-1}], \alpha \in (0, 1] \right\}$$

and

$$\left\{ \left( h_s g_{\log(a)}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right) : a \in (0, 1], s \in (0, a^2], \alpha \in (0, 1] \right\}.$$

*Proof.* First notice that any affine lattice from the union is a short affine lattice.

Now suppose  $(g, v) \in \Omega$ . By the last lemma, we can assume  $v$  is a short vector. Combining this with the coordinates developed in [5], which we recalled in the last section,

we can finish the proof of our parameterization. By the results of [5], a generic lattice  $g\mathbb{Z}^2 \in X_2$  (of a codimension 1 set) has the form

$$g\mathbb{Z}^2 = h_s p_{a,b}\mathbb{Z}^2 = \begin{pmatrix} a & b \\ -sa & a^{-1} - sb \end{pmatrix} \mathbb{Z}^2$$

with  $0 < a \leq 1$ ,  $1 - a < b \leq 1$ , and  $0 < s < (ab)^{-1}$ .

Thus, we can give such an element in  $(g, v) \in \Omega$  the coordinates  $(a, b, s, \alpha)$ , where

$$g\mathbb{Z}^2 = h_s p_{a,b}\mathbb{Z}^2 \text{ and } v = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

If  $g$  is in the codimension 1 set, then it is of the form

$$g\mathbb{Z}^2 = h_s g_{\log(a)}\mathbb{Z}^2 = \begin{pmatrix} 0 & -a^{-1} \\ a & sa^{-1} \end{pmatrix} \mathbb{Z}^2$$

with  $0 < a \leq 1$ , and  $0 < s < a^2$ , and we give it the coordinates  $(a, s, \alpha)$ . □

We note that the affine lattice  $(a, 0, s, \alpha)$  is *not* the same as the affine lattice  $(a, s, \alpha)$ .

3.1.3. *First return time to  $\Omega$ .* As mentioned above, our goal is to find the return map to our transversal, which will calculate the gap distribution for slopes in the vertical strip  $V$ . We outline the strategy of how to do this: If we had the point  $(g, v) \in \Omega$  that is an affine lattice with a short horizontal vector, then we have  $v = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$  for some  $0 < \alpha \leq 1$ . Choose any vector  $\begin{pmatrix} x \\ y \end{pmatrix} \in g\mathbb{Z}^2$ . Then the horocycle flow action on the vectors of the affine lattice  $(g, v)$  is given by

$$h_u \begin{pmatrix} x \\ y \end{pmatrix} + h_u \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} x + \alpha \\ y - u(x + \alpha) \end{pmatrix}.$$

Thus, if we want a short vector, then we need the horizontal  $x + \alpha$  between 0 and 1,  $y > 0$  (so that  $(x + \alpha, y)$  is in the vertical strip), and

$$u = \frac{y}{x + \alpha}$$

so that the vertical is zero. As there are several lattice points that will meet this criteria and since we are seeking the first return time, we will minimize  $u > 0$  over all lattice points in  $g\mathbb{Z}^2$  and the vertical strip  $V$ . Which is to say, we want to compute

$$u = \min_{(x,y)^T \in g\mathbb{Z}^2 \cap V} \frac{y}{x + \alpha}.$$

We can unpack a little more by observing that an element in a generic lattice has the form  $g = h_s p_{a,b}$ , where  $0 < a \leq 1$ ,  $1 - a < b \leq 1$ ,  $0 \leq s < (ab)^{-1}$ , and  $0 < \alpha \leq 1$ .

Thus, we are trying to find the minimum of the slopes of these vectors

$$R(a, b, s, \alpha) := \min_{m,n \in \mathbb{Z}} \frac{(a^{-1} - sb)n - sam}{ma + nb + \alpha}$$

subject to the constraints:

- (1)  $0 < ma + nb + \alpha \leq 1$  so that the horizontal is short;
- (2)  $(a^{-1} - sb)n - sam$  is positive so that we are sampling points in the vertical strip  $V$ , because this minimum is the return time. Our main result in this part is an explicit return time computation for generic affine lattices.

THEOREM 3.6. *Let  $j = \lfloor (1 + a - \alpha)/b \rfloor$ . If  $\alpha > a$ , then*

$$R(a, b, s, \alpha) = \begin{cases} \frac{sa}{\alpha - a} & \text{if } s < \frac{\alpha - a}{ab\alpha}, \\ \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} & \text{if } s > \frac{\alpha - a}{ab\alpha}. \end{cases}$$

If  $\alpha \leq a$ , then

$$R(a, b, s, \alpha) = \begin{cases} \frac{a^{-1} - sb}{b + \alpha} & \text{if } b + \alpha \leq 1, \\ \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} & \text{if } b + \alpha > 1. \end{cases}$$

*Proof.* The main idea is to notice that for any fixed  $n$ , there is a unique  $m = m_n$  for which the affine lattice point

$$\begin{pmatrix} ma + nb + \alpha \\ (a^{-1} - sb)n - sam \end{pmatrix}$$

may be a candidate to minimize slope. From there, we can single out a candidate vector that provides an upper bound to the minimum slope and then use that candidate along with the structure of the affine lattice to find the actual minimum.

Fix  $n \in \mathbb{Z}$ , if we have a candidate affine lattice point, then the horizontal must satisfy

$$0 < ma + nb + \alpha \leq 1 \iff \frac{-(nb + \alpha)}{a} < m \leq \frac{1 - (nb + \alpha)}{a}.$$

Since we require  $m \in \mathbb{Z}$ , we have that  $m$  must be one of the following:

$$\left\lfloor \frac{-(nb + \alpha)}{a} \right\rfloor, \left\lfloor \frac{-(nb + \alpha)}{a} \right\rfloor + 1, \left\lfloor \frac{-(nb + \alpha)}{a} \right\rfloor + 2, \dots, \left\lfloor \frac{1 - (nb + \alpha)}{a} \right\rfloor.$$

Now notice that to minimize the slope

$$\frac{(a^{-1} - sb)n - sam}{ma + nb + \alpha},$$

we should maximize  $m$ . Hence, we choose  $m_n = \lfloor (1 - (nb + \alpha))/a \rfloor$ . In particular, for every  $n \in \mathbb{Z}$ , there is a single  $m = m_n$  for which the associated affine lattice point may be a minimizer.

Using this fact, we will single out a candidate and use the structure of the lattice to find the true minimum.

We proceed by cases.

*Case 1:*  $\alpha < a$ . Consider the candidate minimizers when  $n = 1$ . Since their horizontal must be short, we deduce that

$$0 < ma + b + \alpha \leq 1 \iff \frac{-(b + \alpha)}{a} < m \leq \frac{1 - (b + \alpha)}{a}.$$



Since we are trying to minimize the slope  $(a^{-1} - sb - sam)/(ma + b + \alpha)$ , then we want to maximize  $m$  and so

$$m = \left\lfloor \frac{1 - (b + \alpha)}{a} \right\rfloor.$$

We claim that the only possible values for this floor are 0 and 1. Indeed, notice by our choice of coordinates we have

$$1 - a < b \leq 1 \iff -\alpha \leq 1 - (b + \alpha) < a - \alpha.$$

An obvious upper bound is  $a$  and due to our assumption that  $\alpha < a$ , we have that a lower bound is  $-a$ . Hence,

$$-a \leq 1 - (b + \alpha) < a \iff -1 \leq \frac{1 - (b + \alpha)}{a} < 1.$$

This proves the claim that  $m \in \{0, 1\}$ . Moreover, we see that

$$m = \left\lfloor \frac{1 - (b + \alpha)}{a} \right\rfloor = \begin{cases} 0 & \text{if } b + \alpha \leq 1, \\ -1 & \text{if } b + \alpha > 1. \end{cases}$$

Hence, in the case that  $\alpha < a$ , the candidate when  $n = 1$  is

$$\gamma_1 := \begin{cases} \begin{pmatrix} b + \alpha \\ a^{-1} - sb \end{pmatrix} & \text{if } b + \alpha \leq 1, \\ \begin{pmatrix} b + \alpha - a \\ a^{-1} - sb + sa \end{pmatrix} & \text{if } b + \alpha > 1. \end{cases}$$

Now, the minimum slope is bounded above by  $\text{slope}(\gamma_1)$  and that to get any other candidate vector, we must add/subtract multiples of the basis vectors of  $(\begin{smallmatrix} a & b \\ -sa & a^{-1}-sb \end{smallmatrix})\mathbb{Z}^2$ . Recall we found  $\gamma_1$  by fixing  $n = 1$ , this means that we cannot obtain any other candidates by adding/subtracting the first basis vector. First notice that

$$\gamma_1 - \begin{pmatrix} b \\ a^{-1} - sb \end{pmatrix}$$

has vertical zero so that it is not a candidate to minimize slope. Hence, to find other candidates to minimize slope, our only option to is to add the second basis vector  $(\begin{smallmatrix} b \\ a^{-1}-sb \end{smallmatrix})$  to  $\gamma_1$  to get the other candidates at different levels (as long as the horizontal is less than 1). For example, the level 2 candidate

$$\gamma_2 = \gamma_1 + \begin{pmatrix} b \\ a^{-1} - sb \end{pmatrix}.$$

More generally, all of our candidates are

$$\gamma_n = \gamma_1 + (n - 1) \begin{pmatrix} b \\ a^{-1} - sb \end{pmatrix},$$

where  $n \geq 1$  and so long as the horizontal is less than 1. Of course, once the horizontal becomes bigger than 1, we subtract the other basis vector, but since we are decreasing the horizontal, we are increasing the slope so that we do not need to look at these candidates.

Now notice that

$$\begin{cases} \text{if } b + \alpha \leq 1, \text{ adding } \begin{pmatrix} b \\ a^{-1} - sb \end{pmatrix} \text{ to } \gamma_1 \text{ increases slope,} \\ \text{if } b + \alpha > 1, \text{ adding } \begin{pmatrix} b \\ a^{-1} - sb \end{pmatrix} \text{ to } \gamma_1 \text{ decreases slope.} \end{cases}$$

We conclude that the return time is

$$\begin{cases} \text{slope } (\gamma_1) \text{ if } b + \alpha \leq 1, \\ \text{slope } (\gamma_j) \text{ if } b + \alpha > 1, \end{cases}$$

where  $j$  is the maximum number of times we can add  $\begin{pmatrix} b \\ a^{-1} - sb \end{pmatrix}$  to  $\gamma_1$  so that the horizontal of  $\gamma_j$  is smaller than 1. We compute this value of  $j$ : This is the largest number so that

$$\text{horizontal } (\gamma_j) = jb + \alpha - a \leq 1 < (j + 1)b + \alpha - a \iff j = \left\lfloor \frac{a + 1 - \alpha}{b} \right\rfloor.$$

Case 2:  $a < \alpha$ . This case is slightly easier since we have a candidate at  $n = 0$  given by

$$\gamma_0 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} - \begin{pmatrix} a \\ -sa \end{pmatrix} = \begin{pmatrix} \alpha - a \\ sa \end{pmatrix}.$$

This candidate is in the vertical strip  $V$  by our assumption  $a < \alpha$  and does not exist in Case 1. Notice that subtracting the second basis vector will yield a negative horizontal.

Now, as before, to obtain any other candidate, all we can do is add the second basis vector yielding the candidates

$$\gamma_n = \gamma_0 + n \begin{pmatrix} b \\ a^{-1} - sb \end{pmatrix},$$

where  $n \geq 0$ . Since we are considering a minimum, then slope  $(\gamma_0)$  is an upper bound. When we compare the slope of  $\gamma_0$  and

$$\gamma_1 = \gamma_0 + \begin{pmatrix} b \\ a^{-1} - sb \end{pmatrix},$$

we see that the slope of  $\gamma_1$  is

$$\begin{cases} \text{greater if } s < \frac{\alpha - a}{ab\alpha}, \\ \text{smaller if } s > \frac{\alpha - a}{ab\alpha}. \end{cases}$$

In the second situation, we continue to add the second basis vector as many times as possible so that the horizontal of the new vector is less than or equal to 1. We

deduce,

$$R(a, b, s, \alpha) = \begin{cases} \text{slope}(\gamma_0) = \frac{sa}{\alpha - a} & \text{if } s < \frac{\alpha - a}{ab\alpha}, \\ \text{slope}(\gamma_j) = \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} & \text{if } s > \frac{\alpha - a}{ab\alpha}, \end{cases}$$

where  $j$  is the maximum number of times we can add  $(\frac{b}{a^{-1}-sb})$  to  $\gamma_0$  so that the horizontal of  $\gamma_j$  is smaller than 1. We compute this value of  $j$ : This is the largest number so that

$$\text{horizontal}(\gamma_j) = jb + \alpha - a \leq 1 < (j + 1)b + \alpha - a \iff j = \left\lfloor \frac{1 + a - \alpha}{b} \right\rfloor. \quad \square$$

Notice the above theorem was for a generic affine lattice that is one such that the lattice part has the form  $g = h_s p_{a,b}$ . On the codimension 1 set of lattices, the proof is even easier.

**PROPOSITION 3.7.** *Suppose  $a \in (0, 1]$ . The return map for the codimension 1 set of lattices with a short vertical vector is given by*

$$R\left(h_s g_{\log(a)}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix}\right) = R(a, s, \alpha) = \frac{a}{\alpha}.$$

*Proof.* All of the points inside of a short vertical lattice  $(a, s, \alpha)$  are of the form

$$\begin{pmatrix} \alpha - a^{-1}n \\ sa^{-1}n + am \end{pmatrix}$$

for  $m, n \in \mathbb{Z}$ . Since we want a slope of such a point in the vertical strip, then we have the constraints  $0 < \alpha - a^{-1}n \leq 1$  and  $sa^{-1}n + am > 0$ . We have

$$0 < \alpha - a^{-1}n \leq 1 \iff a(\alpha - 1) \leq n < a\alpha.$$

Since  $a, \alpha < 1$ , then we must have  $n = 0$ . Hence, the slope of the points we are interested in reduces to minimizing

$$\frac{sa^{-1}n + am}{\alpha - a^{-1}n} = \frac{am}{\alpha}$$

subject to  $am + sa^{-1}n = am$  being positive. Since  $a$  is positive, then this is minimized when  $m = 1$ . □

*Remark 3.8.* We establish some notation suggested by the partition from the theorem. This notation will be used frequently in later sections.

$$\Omega = \Omega_1 \sqcup \Omega_2 \sqcup \Omega_3 \sqcup \Omega_4 \sqcup \Omega_{VL}$$

where

$$\begin{aligned} \Omega_1 &:= \left\{ (a, b, s, \alpha) \in \Omega : a < \alpha \text{ and } s < \frac{\alpha - a}{ab\alpha} \right\}, \\ \Omega_2 &:= \left\{ (a, b, s, \alpha) \in \Omega : a < \alpha \text{ and } \frac{\alpha - a}{ab\alpha} < s \right\}, \\ \Omega_3 &:= \{ (a, b, s, \alpha) \in \Omega : \alpha < a \text{ and } b < 1 - \alpha \}, \end{aligned}$$

$$\begin{aligned} \Omega_4 &:= \{(a, b, s, \alpha) \in \Omega : \alpha < a \text{ and } 1 - \alpha < b\}, \\ \Omega_{VL} &:= \{(a, s, \alpha) \in \Omega : a \in (0, 1], s \in (0, a^2], \text{ and } \alpha \in (0, 1]\}. \end{aligned}$$

Furthermore, we have that the return time map is given by

$$R(a, b, s, \alpha) = \begin{cases} \frac{sa}{\alpha - a} & \text{if } (a, b, s, \alpha) \in \Omega_1, \\ \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} & \text{if } (a, b, s, \alpha) \in \Omega_2, \\ \frac{a^{-1} - sb}{b + \alpha} & \text{if } (a, b, s, \alpha) \in \Omega_3, \\ \frac{j(a^{-1} - sb)}{\alpha - a + jb} & \text{if } (a, b, s, \alpha) \in \Omega_4, \\ \frac{a}{\alpha} & \text{if } (a, s, \alpha) \in \Omega_{VL}, \end{cases}$$

where  $j = \lfloor (1 + a - \alpha)/b \rfloor$ .

We remark that in the next sections, we will mostly be uninterested in the piece  $\Omega_{VL}$  since this is a lower dimensional subset of  $\Omega$ .

### 3.2. On doubled slit tori $\mathcal{E}$ .

3.2.1. *A Poincaré section for  $\mathcal{E}$ .* We describe a parameterization of  $\mathcal{W}$ . Before doing this, we describe more explicitly the saddle connections of a doubled slit torus.

The following description is well known, but does not seem to be written in the literature. A special case of the following proposition can be found in that paper by Cheung *et al* [7], where they consider the case  $(g, v) = (\text{Id}, (\frac{\alpha}{0}))$ , where  $\alpha$  is an irrational number in  $(0, 1)$ . Recall our standing assumption that  $v \notin g\mathbb{Q}^2$  and that  $\mathbb{Z}_{\text{prim}}^2$  denotes the set of primitive integers.

**PROPOSITION 3.9.** *Let  $\omega = \omega_{(g,v)} \in \mathcal{E}$  and suppose that the slope of  $v$  is not contained inside of the slopes of  $g\mathbb{Z}_{\text{prim}}^2$ . Then, the saddle connections set is given by*

$$\Lambda_{(g,v)} := \Lambda_{\omega_{(g,v)}} = g\mathbb{Z}_{\text{prim}}^2 \cup (g\mathbb{Z}^2 + v) \cup (-g\mathbb{Z}^2 - v).$$

*Proof.* This essentially amounts to observing that for an element  $\omega = \omega_{(g,v)} \in \mathcal{E}$ , the cone points are at the origin of the torus  $\mathbb{C}/g\mathbb{Z}^2$  or at the affine piece  $v \in \mathbb{C}/g\mathbb{Z}^2$ . Hence, any saddle connection begins at the origin or at the affine piece  $v$  and ends at the origin or the affine piece  $v$ .

Any saddle connection starting and ending at the origin necessarily has holonomy inside of  $g\mathbb{Z}_{\text{prim}}^2$ . The condition that the slope of  $v$  is not contained inside of the slopes of  $g\mathbb{Z}_{\text{prim}}^2$ , guarantees that the holonomies of saddle connections starting and ending at the origin are all of  $g\mathbb{Z}^2$ . Notice that this argument works for saddle connections starting and ending at the affine piece as well and so the set of all saddle connections that start and end at the origin (or at the affine piece) is  $g\mathbb{Z}_{\text{prim}}^2$ .

Next let us consider saddle connections that start at the origin and end at the affine piece  $v$ . In the notation of §2, this is the set  $\Lambda_\omega^{d,+}$ . The holonomy of such a saddle connection can be computed projecting the saddle connection to the nearest corner, taking the holonomy which will be in  $g\mathbb{Z}^2$ , and then adding the contribution of the affine piece. That is, the holonomy of these saddle connections is  $g\mathbb{Z}^2 + v$ . The fact that there is the contribution of the affine piece and the restriction that the slope of  $v$  is not contained inside of the slopes of  $g\mathbb{Z}_{\text{prim}}^2$  means that we obtain the full set  $g\mathbb{Z}^2 + v$ .

Using this argument on saddle connections that start at the affine piece and end at the origin shows they have holonomy in  $-g\mathbb{Z}^2 - v$ . □

Thus, saddle connections beginning and ending at themselves (loops) lie in  $g\mathbb{Z}_{\text{prim}}^2$ , while  $g\mathbb{Z}^2 + v$  represents those saddle connections beginning at one of the singularities and ending at the other one. The piece  $-g\mathbb{Z}^2 - v$  corresponds to saddle connections starting at the affine piece and ending at the origin. In the case that we remove our restriction on the slit  $v$ , then a lot can still be said but care is required since we can have affine lattice points blocking some saddle connections with holonomy in  $g\mathbb{Z}^2$ . We continue to focus on the generic case.

The explicit description of saddle connections of doubled slit tori allows us to describe  $\mathcal{W}$  more explicitly as well. Essentially, what we will deduce is that for a doubled slit torus  $\omega = \omega_{(g,v)}$  to have a short horizontal saddle connection (that is to be in  $\mathcal{W}$ ), then the short horizontal saddle connection must come from the primitive vectors  $g\mathbb{Z}_{\text{prim}}^2$  or from the affine lattice  $(g\mathbb{Z}^2 + v)$ .

Recall,  $\Delta = \{p_{a,b}\mathbb{Z}^2 : (a, b) \in \Delta\}$  and

$$\Omega = \left\{ h_s p_{a,b}\mathbb{Z}^2 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : (a, b) \in \Delta, s \in [0, (ab)^{-1}], \alpha \in (0, 1] \right\}.$$

We write  $\Omega$  more succinctly as

$$\Omega = \left\{ (g, v) = g\mathbb{Z}^2 + v : g \in \text{SL}_2(\mathbb{R}), v = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right\}.$$

Denote the projection map from  $AX_2 \rightarrow X_2$  by  $\pi$ . Then, define

$$\mathcal{W}_{\text{sl}} = \{\omega \in \mathcal{E} : \Pi(\omega) \in \pi^{-1}(\Delta)\}$$

and

$$\mathcal{W}_{\text{sa}} = \{\omega \in \mathcal{E} : \Pi(\omega) \in \Omega\}.$$

Of course both of these sets are preimages of  $\Pi$  but we use this notation to avoid notation overload. It is not hard to see that  $\mathcal{W}_{\text{sl}}$  consists of those doubled slit tori so that the image under  $\Pi$  is an affine lattice where the lattice part is short. Thus, it is a *short lattice* and that is from where the subscript ‘sl’ comes. Similarly,  $\mathcal{W}_{\text{sa}}$  consists of the doubled slit tori so that the image under  $\Pi$  is an affine lattice where the affine piece is short. Thus, it has a *short affine* vector and that is from where the subscript ‘sa’ comes. Unwrapping a bit more,

we see

$$\mathcal{W}_{sl} = \{\omega \in \mathcal{E} : \Pi(\omega) = (p_{a,b}, v)\} \text{ and } \mathcal{W}_{sa} = \left\{ \omega \in \mathcal{E} : \Pi(\omega) = \left( g, \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right) \right\}$$

for where  $(a, b) \in \Delta$  and  $\alpha \in (0, 1]$ .

This discussion proves that we have the following description of  $\mathcal{W}$ : if  $\omega$  is short ( $\omega \in \mathcal{W}$ ), then the short vector is either in the lattice part  $g\mathbb{Z}_{\text{prim}}^2$  or in the affine lattice part  $g\mathbb{Z}^2 + v$ . We state this as a proposition.

PROPOSITION 3.10. *The transversal  $\mathcal{W}$  is*

$$\mathcal{W} = \mathcal{W}_{sl} \cup \mathcal{W}_{sa}.$$

For doubled slit tori, we will remove lower dimensional subsets such as the affine lattices that have a periodic lattice.

3.2.2. *First return time to  $\mathcal{W}$ .* We compute the time it takes for an element in  $\mathcal{W}$  to return to  $\mathcal{W}$  under the horocycle flow. This return time will require the return map on  $AX_2$  computed in the last section.

Our strategy to do this is to consider the return time for an element  $\omega$  on the partition  $\mathcal{W}_{sl} \cup \mathcal{W}_{sa}$  of  $\mathcal{W}$ . Then,  $\Pi(\omega) = (g, v)$  can be decomposed into a lattice piece  $g$  and an affine vector piece  $v$ . We can check exactly how much time needs to pass under horocycle flow before returning  $\mathcal{W}$  using the results of [5] and the results on affine lattices. Then the minimum of the two times will be the return time to  $\Omega$ .

THEOREM 3.11. *Suppose  $\omega \in \mathcal{W}$  and let  $u$  denote the return time to  $\mathcal{W}$  under the horocycle flow. Then we have two cases, either  $\omega \in \mathcal{W}_{sa}$  or  $\omega \in \mathcal{W}_{sl}$ .*

(1) *Suppose  $\omega \in \mathcal{W}_{sl}$ . Then  $\Pi(\omega) = (p_{a,b}, v)$  for some  $a, b \in \Delta$ , where  $v = (v_1, v_2)^T$  and the return time to  $\mathcal{W}$  under horocycle flow is*

$$\mathcal{R}(\omega) = \begin{cases} v_2/v_1 & \text{if } b + v_1 \leq 1, \text{ and} \\ (ab)^{-1} & \text{if } b + v_1 > 1. \end{cases}$$

(2) *Suppose  $\omega \in \mathcal{W}_{sa}$ . Then  $\Pi(\omega) = (h_s p_{a,b}, (\alpha, 0)^T)$  for  $(a, b, s, \alpha) \in \Omega$  and the return time to  $\mathcal{W}$  under horocycle flow is*

$$\mathcal{R}(\omega) = \begin{cases} (ab)^{-1} - s & \text{if } \Pi(\omega) \in \Omega_2 \cup \Omega_4, \\ \frac{sa}{\alpha - a} & \text{if } \Pi(\omega) \in \Omega_1, \text{ and} \\ \frac{a^{-1} - sb}{b + \alpha} & \text{if } \Pi(\omega) \in \Omega_3, \end{cases}$$

where  $\Omega_i, i = 1, \dots, 4$ , form the atoms of the partition of  $\Omega$ .

We first prove a lemma that computes the time it would take to travel from  $\mathcal{W}_{sl}$  to  $\mathcal{W}_{sl}$  under horocycle flow.

LEMMA 3.12. Let  $\rho = \rho(a, b, v_1, v_2)$  denote the time it would take to travel from  $\mathcal{W}_{s1}$  to  $\mathcal{W}_{s1}$  under horocycle flow. Then

$$\rho(a, b, v_1, v_2) = \begin{cases} \frac{v_2}{v_1} = \text{slope}(v) & \text{if } b + v_1 \leq 1, \text{ and} \\ \frac{v_2 + ja^{-1}}{v_1 + jb - a} & \text{if } b + v_1 > 1. \end{cases}$$

*Proof.* Suppose that  $\omega \in \mathcal{W}_{s1}$ . Then  $\Pi(\omega) = (p_{a,b}, v)$  for some  $(a, b) \in \Delta$ . Then, the only way  $(p_{a,b}, v)$  will have a short horizontal affine vector under the horocycle flow will be when  $h_u v$  is short. Recall that  $v$  is actually only defined up to  $p_{a,b}\mathbb{Z}^2$ . However, when we write ‘ $v$ ’, we mean the representative inside of the standard fundamental domain of  $\mathbb{R}^2/p_{a,b}\mathbb{Z}^2$ .

Then,  $h_{\rho(a,b,v_1,v_2)}v$  is short when  $\rho(a, b, v_1, v_2) = \text{slope}(v) = v_2/v_1$ . Since  $v$  is actually only defined up to  $p_{a,b}\mathbb{Z}^2$ , we see the next value  $u$  so that  $h_u v$  is short is

$$\rho(a, b, v_1, v_2) = \min \frac{v_2 + a^{-1}n}{v_1 + am + bn},$$

where the minimum is taken over  $(m, n) \in \mathbb{Z}$  such that

- (1)  $(v_2 + a^{-1}n)/(v_1 + am + bn) > 0$ , since we want positive slope/return time, and
- (2)  $v_1 + am + bn \in (0, 1]$ , since we want a short vector.

The first condition says that  $v_2 + a^{-1}n > 0 \iff n > -v_2a > -1$ . Since  $n \in \mathbb{Z}$ , then  $n \geq 0$ .

Now fix  $n \geq 0$ . Then the second condition of  $v_1 + am + bn \in (0, 1]$  tells us that

$$\frac{-(nb + v_1)}{a} < m \leq \frac{1 - (nb + v_1)}{a}.$$

Since we are trying to minimize the slope, then we want to maximize  $m$ , so that we take

$$m = m_n = \left\lfloor \frac{1 - (nb + v_1)}{a} \right\rfloor.$$

Hence, for every  $n \geq 0$ , we have the candidate

$$\gamma_n = \begin{pmatrix} v_1 + am_n + bn \\ v_2 + a^{-1}n \end{pmatrix}.$$

Now observe that since we are looking at a minimum, over  $n$  of  $\gamma_n$ ,  $\gamma_1$  is an upper bound. We are interested in  $\gamma_1$ , because we can describe it explicitly. In particular,

$$m_1 = \begin{cases} 0 & \text{if } b + v_1 \leq 1, \\ -1 & \text{if } b + v_1 > 1, \end{cases}$$

so that

$$\gamma_1 = \begin{cases} \begin{pmatrix} v_1 + b \\ v_2 + a^{-1} \end{pmatrix} & \text{if } b + v_1 \leq 1, \text{ and} \\ \begin{pmatrix} v_1 + b - a \\ v_2 + a^{-1} \end{pmatrix} & \text{if } b + v_1 > 1. \end{cases}$$

We are interested in explicitly knowing  $\gamma_1$  because this allows us to compute all the other candidates  $\gamma_n$ . For example, to go from the candidate  $\gamma_1$  to the candidate at level 2, all we have to do is add the basis vector  $(b, a^{-1})^T$  of the lattice  $p_{a,b}\mathbb{Z}^2$ :

$$\gamma_2 = \gamma_1 + \begin{pmatrix} b \\ a^{-1} \end{pmatrix}.$$

Similarly,

$$\gamma_3 = \gamma_1 + 2 \begin{pmatrix} b \\ a^{-1} \end{pmatrix} \text{ and } \gamma_0 = \gamma_1 - \begin{pmatrix} b \\ a^{-1} \end{pmatrix}.$$

Of course, if we add the basis vector  $(b, a^{-1})^T$  too many times, the horizontal will become larger than 1 and hence we will cease to have a candidate. This is amended by subtracting the other basis vector  $(a, 0)^T$ , but this makes the slope larger since we are decreasing the horizontal. In other words, the only candidates  $\gamma_n$  in which we are interested are those to which we can add the basis vector  $(b, a^{-1})^T$  so that they still have a horizontal less than or equal to 1. In particular, this is a finite list. We can then obtain the candidate with minimum slope by observing how adding the basis vector  $(b, a^{-1})^T$  changes the slope compared with  $\gamma_1$ . Now notice that

$$\begin{cases} \text{if } b + v_1 < 1, \text{ adding } \begin{pmatrix} b \\ a^{-1} \end{pmatrix} \text{ increases slope,} \\ \text{if } b + v_1 > 1, \text{ adding } \begin{pmatrix} b \\ a^{-1} \end{pmatrix} \text{ decreases slope.} \end{cases}$$

We conclude the following.

- (1) If  $b + v_1 < 1$ , then the minimum of the slopes is  $\min\{\text{slope}(\gamma_0), \text{slope}(\gamma_1)\}$ . A direct calculation shows that the minimum of these two is  $\text{slope}(\gamma_0) = v_2/v_1$ .
- (2) If  $b + v_1 > 1$ , then the minimum of the slopes is  $\text{slope}(\gamma_j)$ , where  $j$  is the maximum number of times we can add  $(\frac{b}{a^{-1}})$  to  $\gamma_1$  so that the horizontal of  $\gamma_j$  is smaller than 1. We compute this value of  $j$ : this is the largest number so that

$$\text{horizontal}(\gamma_j) = jb + v_1 - a \leq 1 < (j + 1)b + v_1 - a \iff j = \left\lfloor \frac{a + 1 - v_1}{b} \right\rfloor.$$

Hence, for an element  $\omega \in \mathcal{W}_{sl}$ , with  $\Pi(\omega) = (p_{a,b}, v)$ , to return to  $\mathcal{W}_{sa}$  under the horocycle flow, it needs to travel  $\rho$  units of time, where

$$\rho(a, b, v_1, v_2) = \begin{cases} \text{slope}(\gamma_0) = \text{slope}(v) = v_2/v_1 & \text{if } b + v_1 \leq 1, \text{ and} \\ \text{slope}(\gamma_j) = \frac{v_2 + ja^{-1}}{v_1 + jb - a} & \text{if } b + v_1 > 1. \end{cases} \quad \square$$

Now we give the proof of the return time to  $\mathcal{W}$  under horocycle flow.



*Proof of Theorem 3.11.*

*Step 1: Computing time from  $\mathcal{W}_{sa}/\mathcal{W}_{sl}$  to  $\mathcal{W}_{sa}/\mathcal{W}_{sl}$ .* We implement the first part of the strategy outlined at the beginning of this section and compute how much time is needed for an element in  $\mathcal{W}$  to return to either  $\mathcal{W}_{sl}$  or  $\mathcal{W}_{sa}$ .

(1) *Time from  $\mathcal{W}_{sa}$  to  $\mathcal{W}_{sa}$ .* Suppose that  $\omega \in \mathcal{W}_{sa}$  so that  $\Pi(\omega) = (h_s p_{a,b}, (\alpha, 0)^T)$  for  $(a, b, s, \alpha) \in \Omega$ . Consequently, the return time to  $\mathcal{W}_{sa} \subset \mathcal{W}$  is exactly the return time from the previous section on affine lattices. Namely, the return time is  $R(a, b, s, \alpha)$ .

(2) *Time from  $\mathcal{W}_{sa}$  to  $\mathcal{W}_{sl}$ .* Suppose that  $\omega \in \mathcal{W}_{sa}$  so that  $\Pi(\omega) = (h_s p_{a,b}, (\alpha, 0)^T)$  for  $(a, b, s, \alpha) \in \Omega$ . Consequently, under the horocycle flow, the next time we get a short vector in the lattice part is when the lattice part contains a short vector. Hence, the return time to  $\mathcal{W}_{sl} \subset \mathcal{W}$  is  $(ab)^{-1} - s$  from the results of [5].

(3) *Time from  $\mathcal{W}_{sl}$  to  $\mathcal{W}_{sl}$ .* Suppose that  $\omega \in \mathcal{W}_{sl}$  so that  $\Pi(\omega) = (p_{a,b}, v)$  for some  $(a, b) \in \Delta$ . Thus, under the horocycle flow, the next time we get a short vector in the lattice part is the return time obtained from [5]. Hence, the return time to  $\mathcal{W}_{sl} \subset \mathcal{W}$  is  $(ab)^{-1}$  from the results of [5].

(4) *Time from  $\mathcal{W}_{sl}$  to  $\mathcal{W}_{sa}$ .* The only difficult return time  $\rho(a, b, v_1, v_2)$  to compute is for an element  $\omega \in \mathcal{W}_{sl}$  to return to  $\mathcal{W}_{sa}$ . This was computed in the previous lemma to be

$$\rho(a, b, v_1, v_2) = \begin{cases} \frac{v_2}{v_1} & \text{if } b + v_1 \leq 1, \text{ and} \\ \frac{v_2 + ja^{-1}}{v_1 + jb - a} & \text{if } b + v_1 > 1. \end{cases}$$

We collect the times collected above now to deduce that for  $\omega \in \mathcal{W}$ , the return time  $u$  to  $\mathcal{W}$  under the horocycle flow is given by the following two cases.

*Case 1:  $\omega \in \mathcal{W}_{sl}$ .* For  $\omega \in \mathcal{W}_{sl}$ , the return time is

$$u = \min\{(ab)^{-1}, \rho(a, b, v_1, v_2)\}$$

where  $\rho$  is the minimum time it takes for an affine lattice  $(p_{a,b}, v)$  to have a short affine vector in the affine piece.

*Case 2:  $\omega \in \mathcal{W}_{sa}$ .* For  $\omega \in \mathcal{W}_{sa}$ , the return time is

$$u = \min\{(ab)^{-1} - s, R(a, b, s, \alpha)\}$$

where  $R$  is the return map for affine lattices.

*Step 2: Computing the minimum.* We now implement the second step of the strategy and compute the minimums of the times calculated from step 1. We have two cases given by the partition of  $\mathcal{W}$ .

*Case 1:  $\omega \in \mathcal{W}_{sl}$ .* For  $\omega \in \mathcal{W}_{sl}$  the return time is

$$u = \min\{(ab)^{-1}, \rho(a, b, v_1, v_2)\}$$

where we have  $\Pi(\omega) = (p_{a,b}, v)$  for some  $(a, b) \in \Delta$  and  $v = (v_1, v_2)^T \in p_{a,b}\mathbb{Z}^2$ . The return time is

$$u = \begin{cases} \min\{(ab)^{-1}, v_2/v_1\} & \text{if } b + v_1 \leq 1, \text{ and} \\ \min \left\{ (ab)^{-1}, \frac{v_2 + ja^{-1}}{v_1 + jb - a} \right\} & \text{if } b + v_1 > 1. \end{cases}$$

By our choice of  $v$  as the representative lying in the standard fundamental domain of  $\mathbb{R}^2/p_{a,b}\mathbb{Z}^2$  spanned by the columns of  $p_{a,b}$ , we have

$$u = \begin{cases} \min\{(ab)^{-1}, v_2/v_1\} = v_2/v_1 & \text{if } b + v_1 \leq 1, \text{ and} \\ \min \left\{ (ab)^{-1}, \frac{v_2 + ja^{-1}}{v_1 + jb - a} \right\} = (ab)^{-1} & \text{if } b + v_1 > 1. \end{cases}$$

Case 2:  $\omega \in \mathcal{W}_{sa}$ . For  $\omega \in \mathcal{W}_{sa}$ , the return time is

$$u = \min\{(ab)^{-1} - s, R(a, b, s, \alpha)\}$$

where  $R$  is the return map for affine lattices and  $(a, b, s, \alpha) \in \Omega$ . Recall the map is defined piecewise as

$$R(a, b, s, \alpha) = \begin{cases} \frac{sa}{\alpha - a} & \text{if } (a, b, s, \alpha) \in \Omega_1, \\ \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} & \text{if } (a, b, s, \alpha) \in \Omega_2, \\ \frac{a^{-1} - sb}{b + \alpha} & \text{if } (a, b, s, \alpha) \in \Omega_3, \\ \frac{j(a^{-1} - sb)}{\alpha - a + jb} & \text{if } (a, b, s, \alpha) \in \Omega_4. \end{cases}$$

where  $j = \lfloor (1 + a - \alpha)/b \rfloor$ . We now compute the return time

$$u = \min\{(ab)^{-1} - s, R(a, b, s, \alpha)\}.$$

(1) If  $(a, b, s, \alpha) \in \Omega_1$ , then

$$u = \frac{sa}{\alpha - a}$$

because

$$\frac{sa}{\alpha - a} < (ab)^{-1} - s \iff s < \frac{\alpha - a}{ab\alpha}$$

and this inequality is one of the defining inequalities of  $\Omega_1$ .

- (2) If  $(a, b, s, \alpha) \in \Omega_2$ , then

$$u = (ab)^{-1} - s$$

because

$$(ab)^{-1} - s < \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} \iff s > \frac{\alpha - a}{ab\alpha}$$

and this inequality is one of the defining inequalities of  $\Omega_2$ .

- (3) If  $(a, b, s, \alpha) \in \Omega_3$ , then

$$u = \frac{a^{-1} - sb}{b + \alpha}$$

because

$$\frac{a^{-1} - sb}{b + \alpha} < (ab)^{-1} - s \iff s < \frac{1}{ab}$$

and this inequality is one of the defining inequalities of  $\Omega$ .

- (4) If  $(a, b, s, \alpha) \in \Omega_4$ , then

$$u = (ab)^{-1} - s$$

because

$$(ab)^{-1} - s < \frac{j(a^{-1} - sb)}{\alpha - a + jb} \iff s < \frac{1}{ab}$$

and this inequality is one of the defining inequalities of  $\Omega$ . □

#### 4. Invariant measures and gap distributions

Here, we prove the existence of a density function for the slope gap question for twice-marked tori and for doubled slit tori. We will classify measures on the transversal  $\Omega$  and show various properties of the cumulative distribution function of the gaps such as support at zero and the decay rate. We will also consider an invariant measure on the transversal  $\mathcal{W}$ .

4.1. *Invariant measures on  $\Omega$ .* In this section, we review some Ratner theory and use it to classify invariant measures on  $\Omega$ .

4.1.1. *Ratner theory for  $AX_2$ .* In the early 1990s, Marina Ratner published a series of influential works [22–24] that gave a mostly complete understanding of ergodic probability measures for unipotent dynamical systems on homogenous spaces. We briefly review her work in the context of horocycle flow on  $AX_2$ . Our exposition borrows ideas from [1, 11, 27].

Consider the vertical translation  $V_\beta : AX_2 \rightarrow AX_2$

$$V_\beta(g, v) = \left( g, v + \begin{pmatrix} 0 \\ \beta \end{pmatrix} \right)$$

for some  $\beta \in \mathbb{R}$ . Notice that this flow always commutes with  $h_s$ . Hence, if  $\mu$  is an ergodic probability measure invariant under horocycle flow, then so is the measure obtained by precomposing with  $V_\beta$ . This makes it interesting to consider the vertical translation if one is interested in measures invariant under the horocycle flow.

By disintegrating, any measure  $d\mu(g, v)$  on  $AX_2$  can be decomposed into the measure

$$d\mu(g, v) = d\lambda_g(v) d\nu(g),$$

where  $d\nu(g)$  is a measure on the lattice part and  $d\lambda_g(v)$  is a measure on the fiber. Notice that the vertical translation only acts on the fiber piece of an affine lattice  $(g, v)$ . (That is, it acts on the torus  $\mathbb{R}^2/g\mathbb{Z}^2$ .) We can use previous results of [14] (in the cocompact case) and [8] (in the general lattice case) to understand the possibilities for  $\nu(g)$ . These results show that either  $\nu$  is a Haar measure on  $X_2$  or supports a periodic measure. Given these two possible measures on the lattice part, we can then try to understand how vertical translation acts on the fiber  $\mathbb{R}^2/g\mathbb{Z}^2$  to gain a complete understanding of the possibilities for  $\mu$ . This is carried out in [11]. Before being more precise, we make some definitions.

The *denominator* of an affine lattice  $(g, v)$  is the minimal  $d > 0$  so that  $v \in d^{-1}g\mathbb{Z}^2$  whenever this is defined. We denote the denominator by  $d(g, v)$ . Let  $X[q]$  denote the set of affine lattices with denominator  $q > 0$ . That is,

$$X[q] = \{(g, v) \in AX_2 : d(g, v) = q\}.$$

In the literature, this is also referred to as the set of *torsion* affine lattices since the vector  $v$  for any  $(g, v) \in X[q]$  is a torsion point under the group structure of  $\mathbb{C}/g\mathbb{Z}^2$ . The set of torsion affine lattices forms an  $SL_2(\mathbb{R})$ -invariant subset of  $AX_2$  of dimension 3. In particular, it is a homogenous space and can be identified as the quotient of  $SL_2(\mathbb{R})$  by  $\Gamma(q)$ , where  $\Gamma(q)$  is the congruence subgroup

$$\Gamma(q) = \{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv Id \pmod{q}\}.$$

Explicitly, this identification is given by the map

$$g(\mathbb{Z}^2 + w) \mapsto g\Gamma(q),$$

where  $w$  is any vector with denominator  $q$ .

To get an idea of the properties of these types of affine lattices, notice that  $X[1]$  is just the space of lattices. Also notice that  $(Id, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix})$  is  $X[2]$ , but not in  $X[4]$  by our minimality condition on the denominator.

Thinking about torsion affine lattices as twice-marked tori, we see that torsion twice-marked tori are lattices surfaces. This explains our assumption that restricts to affine lattices with  $v \notin g\mathbb{Q}^2$ . In fact, we have  $(g, v)$  is a lattice surface if and only if  $v \in g\mathbb{Q}^2$ , see the work of Schmoll [25] for proof of this.

We now state Ratner’s measure classification for horocycle flow on  $AX_2$ .

**THEOREM 4.1.** *If  $\mu$  is an  $h_u$ -invariant ergodic probability measure, then  $\mu$  is one of the following:*

- (1)  $\mu = m_{AX_2}$ , the Haar measure on  $AX_2$ ;
- (2)  $\mu$  is a torsion measure, that is, there is  $q > 0$  so that the support of  $\mu$  is contained in the set of torsion affine lattices  $X[q]$ ;
- (3) there is  $a > 0$  and  $0 \leq \alpha < a^{-1}$  so that the support of  $\mu$  is

$$\text{Per}(a, \alpha) := \left\{ h_s V_\beta \left( g_{\log(a)}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right) = \left( h_s g_{\log(a)}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) : s \in [0, a^2), \beta \in [0, a) \right\};$$

- (4)  $\mu$  is supported on a  $h_u$ -periodic orbit.

The first two measure above are those whose lattice measure is a Haar measure on  $X_2$ , while the remaining two are those for which the lattice measure is supported on a  $h_u$ -periodic orbit.

Lastly, we observe that any measure in the list above can be precomposed by a vertical translation  $V_\beta$  to produce more ergodic probability measures. We will mainly concern ourselves with the above list.

**4.1.2. Invariant measures for  $\Omega$ .** To classify ergodic probability measures invariant under the return map on the transversal  $\Omega$ , we use Ratner’s theorem, which classifies ergodic probability measures for unipotent dynamical systems on homogenous spaces. Since we have a transversal, then there is a bridge between measures on  $(AX_2, h_s)$  and measures on  $(\Omega, T)$ . While this bridge is well known by now, the interested reader can find details in [5].

As a corollary to Ratner’s theorem, we can classify  $T$ -invariant measures on the transversal  $\Omega$  by disintegration.

**THEOREM 4.2.** *If  $\nu$  is a  $T$ -invariant ergodic probability measure, then:*

- (1)  $\nu$  is induced by the Haar measure on  $AX_2$ , that is,  $d\nu = 2 ds db da d\alpha$ ;
- (2)  $\nu$  is induced by a torsion measure on  $AX_2$  and is given by the two-dimensional Lebesgue measure on a finite number of triangles;
- (3)  $\nu$  is supported on

$$\Omega \cap \text{Per}(a, \alpha) = \left\{ \left( h_s g_{\log(a)}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right) : s \in [0, a^2) \right\}$$

for some  $a > 0$  and  $0 \leq \alpha < \min\{a^{-1}, 1\}$ ;

- (4)  $\nu$  is supported on a  $T$ -periodic orbit.

*Proof.* By our discussion about disintegration in the last section, it is clear that the disintegration of the Haar measure on  $AX_2$  is given by the product of the two-dimensional Lebesgue measure on the torus  $\mathbb{R}^2/g\mathbb{Z}^2$  and the three-dimensional Haar measure on  $X_2$ . This measure is  $SL_2(\mathbb{R})$ -invariant and hence horocycle invariant. On  $\Omega$ , we must consider only those affine lattices with a short horizontal vector so that the disintegration of the

Haar measure on  $\Omega$  will be

$$dm_\Omega = 2 ds db da d\alpha.$$

Now suppose that  $\nu$  is a  $T$ -invariant ergodic probability measure on  $\Omega$ , different than  $m_\Omega$ . Consider  $d\nu du$  on  $AX_2$ . By Ratner’s theorem, we know the support of this measure on  $AX_2$  can only be three-, two-, or one-dimensional.

In case it is three-dimensional, then it must be supported on  $X[q]$  for some  $q > 0$ . In particular, this means that we are in the territory of lattice surfaces and the work of Uyanik and Work [29] applies. Their work shows that the transversal of short horizontal saddle connections  $\Omega \cap X[q]$  is given by  $k$  triangles, where  $k$  is the number of cusps of  $X[q]$  thought of as  $SL_2(\mathbb{R})/\Gamma(q)$ . In particular, the  $T$ -invariant ergodic probability measure is given by the Lebesgue measure on these triangles. We refer the interested reader to their work for more details.

In case  $d\nu du$  is two-dimensional, then it is supported on  $Per(a, \alpha)$  for some  $a > 0$  and  $0 \leq \alpha < \min\{a^{-1}, 1\}$ , that is, those affine lattices for which the lattice part is periodic and the affine piece has an arbitrary vertical component. Hence,  $d\nu$  is supported on

$$\Omega \cap Per(a, \alpha) = \left\{ \left( h_s g_{\log(a)}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right) : s \in [0, a^2] \right\}.$$

Lastly, if  $d\nu du$  is one-dimensional, then on  $\Omega$  it becomes zero-dimensional. This simply means that it is supported on a periodic orbit for  $T$ . □

We continue to refer to the measure  $m_\Omega$  as the Haar measure on  $\Omega$  and we use  $m_q$  to denote a torsion measure on the transversal. We obtain the following.

**THEOREM 4.3.** *The measure  $dm_\Omega = 2 ds db da d\alpha$  is a unique  $T$ -invariant ergodic probability measure on  $\Omega$  that is absolutely continuous with respect to the Lebesgue measure. Hence, for almost every  $m_\Omega$  point  $(a, b, s, \alpha)$  and every  $f \in L^1(\Omega)$ ,*

$$\frac{1}{N} \sum_{i=0}^{N-1} f(T^i(a, b, s, \alpha)) \rightarrow \int_\Omega f dm_\Omega.$$

In coordinates  $(a, b, s, \alpha)$ , integrating with respect to  $dm_\Omega$  takes the form

$$\int_\Omega f dm_\Omega = \int_0^1 \int_0^1 \int_{1-a}^1 \int_0^{(ab)^{-1}} f(a, b, s, \alpha) ds db da d\alpha.$$

**4.2. An invariant measure on  $\mathcal{W}$ .** Recall that  $\mathcal{H}(0, 0) = AX_2$  is a homogenous space so that it comes equipped with a Haar measure  $m_{AX_2}$ . Since  $\mathcal{W}$  is a four-fold cover of  $AX_2$ , we can endow it with a natural measure that is four copies of  $m_{AX_2}$ . Denote this measure by  $m_{\mathcal{E}}$ . This measure is  $SL_2(\mathbb{R})$ -invariant and finite. In particular, this measure is also invariant under the horocycle flow.

There is a correspondence of measures invariant under a flow and measures invariant under the first return map of that flow to a transversal. Specializing to our situation, there is an induced measure on our transversal  $\mathcal{W}$ , which we denote by  $m_{\mathcal{W}}$ , that is invariant under the first return map.

It can be written as the sum of a Lebesgue measure on  $W_{sl}$  and  $W_{sa}$ , which we denote as  $m_{sa}$  and  $m_{sl}$ . Explicitly, on  $W_{sa}$ , we have  $\Pi(\omega) = \Lambda_{(a,b,s,\alpha)} \in \Omega$  and so for  $f \in L^1(W_{sa})$ , we have

$$\int_{W_{sa}} f \, dm_{sa} = 4 \int_{\Omega} f(a, b, s, \alpha) \, ds \, db \, da \, d\alpha,$$

where the 4 reflects the fact that the  $\Pi : \mathcal{E} \rightarrow \mathcal{H}(0, 0) = AX_2$  is a four-to-one map. Similarly, on  $W_{sl}$ , we have  $\Pi(\omega) = (p_{a,b}, v)$ , where  $(a, b) \in \Delta$  and  $v \in \mathbb{R}^2/p_{a,b}\mathbb{Z}^2$ , and so for  $f \in L^1(W_{sl})$ , we have

$$\int_{W_{sl}} f \, dm_{sl} = 4 \iint_{(a,b) \in \Delta} \int_{v \in \mathbb{R}^2/p_{a,b}\mathbb{Z}^2} f(a, b, v) \, dv \, db \, da.$$

Thus, the flat measure on  $\mathcal{W}$  is given by the sum,

$$\int_{\mathcal{W}} f \, dm_{\mathcal{W}} = \int_{W_{sa}} f \, dm_{sa} + \int_{W_{sl}} f \, dm_{sl}.$$

4.3. *Gap distribution results.* In this section, we prove the existence of the slope gap limit for twice-marked tori and doubled slit tori and state various properties about their distribution and density functions.

*Proof of Theorem 2.2.* We prove that the slope gap limit in Theorem 2.2 of twice-marked tori exists. Recall from (6), we have

$$\frac{1}{N} |\mathcal{G}_V^N(\Lambda) \cap I| = \frac{1}{N} \sum_{i=0}^{N-1} \chi_{R^{-1}(I)}(T^i(\Lambda)).$$

Now suppose that  $\Lambda$  is  $\mu$ -generic for some ergodic probability measure  $\mu$ . We classified all such measures in Theorem 4.3. Then by definition of a  $\mu$ -generic point, we have that

$$\frac{1}{N} \sum_{i=0}^{N-1} \chi_{R^{-1}(I)}(T^i(\Lambda)) \rightarrow \mu(R^{-1}(I)).$$

Since the cumulative distribution function is given by the total area bounded by the return time hyperbolas, it is piecewise real analytic. In particular, it is continuous and differentiable. The probability density associated with the slope gap is simply the derivative of the cumulative distribution function. For example, for  $m_{\Omega}$ -generic points, we have the probability density is given by

$$g(t) := \frac{d}{dt} m_{\Omega}(R^{-1}(0, t)).$$

Hence, the proportion of gaps in an interval  $I$  of an  $m_{\Omega}$ -generic affine lattice  $\Lambda \in \Omega$  is given by

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\mathcal{G}_V^N(\Lambda) \cap I| = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{R^{-1}(I)}(T^i(\Lambda)) = m_{\Omega}(R^{-1}(I)) = \int_I g(t) \, dt.$$

Similarly, for  $m_q$ -generic points, we have the probability density is given by

$$g_q(t) := \frac{d}{dt}m_q(\mathcal{R}^{-1}(0, t))$$

and the proportion of gaps given by

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\mathcal{G}_V^N(\Lambda) \cap I| = \int_I g_q(t) dt. \quad \square$$

*Proof of Theorem 2.1.* A similar argument works to prove that the slope gap limit (5) for doubled slit tori exists for any  $m_{\mathcal{W}}$ -generic point of  $\mathcal{W}$ . Indeed,

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\mathcal{G}_V^N(\Lambda_\omega) \cap I| = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{\mathcal{R}^{-1}(I)}(\mathcal{T}^i(\Lambda_\omega)) = m_{\mathcal{W}}(\mathcal{R}^{-1}(I)).$$

This proves Theorem 1.1. The probability density associated with the slope gap is simply the derivative of the cumulative distribution function. We denote this density by  $f$ . That is,

$$f(t) := \frac{d}{dt}m_{\mathcal{W}}(\mathcal{R}^{-1}(0, t)),$$

where  $\mathcal{R}$  is the first return map to  $\mathcal{W}$ . Hence, the proportion of gaps of saddle connections in an interval  $I$  of an  $m_{\mathcal{W}}$ -generic translation surface  $\omega \in \mathcal{W}$  is given by

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\mathcal{G}_V^N(\Lambda_\omega) \cap I| = \int_I f(t) dt. \quad \square$$

We state more precisely the conclusions of the above theorems now as well. Recall that  $F \sim G$  means that for large enough  $t$ , the ratio is a positive constant and  $F \ll G$  means there is a positive constant  $c$  so that for large enough  $t$ ,  $|F(t)| \leq c|G(t)|$ .

**THEOREM 4.4.**

- (1) For any affine lattice in  $\Omega$  that is generic for  $m_\Omega$ , we have that the proportion of gaps larger than  $t$ ,

$$\int_t^\infty g(t) dt$$

satisfies

$$t^{-2} \ll \int_t^\infty g(t) dt \ll t^{-1}.$$

Here  $g(t)$  is the density function for affine lattices from the proof of Theorem 2.2.

- (2) There is support at zero for any affine lattice in  $\Omega$  that is generic for  $m_\Omega$ . That is, for any  $\varepsilon > 0$ ,

$$\int_0^\varepsilon g(t) dt > 0.$$

- (3) For any doubled slit torus in  $\mathcal{W}$  that is generic with respect to  $m_{\mathcal{W}}$ , we have that the proportion of gaps of saddle connections larger than  $t$  decays quadratically. More



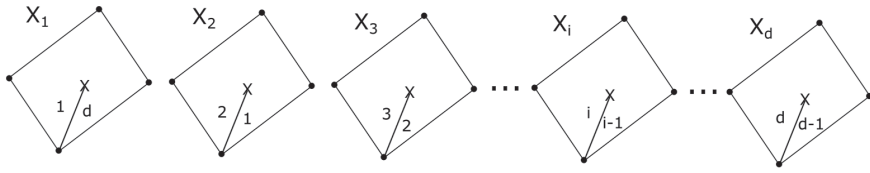


FIGURE 4. An example of a  $d$ -symmetric torus cover.

precisely,

$$\int_t^\infty f(t) dt \sim t^{-2}.$$

Here,  $f(t)$  is the density function for doubled slit tori from the proof of Theorem 1.1.

- (4) There is support at zero for any doubled slit torus in  $\mathcal{W}$  that is generic with respect to  $m_{\mathcal{W}}$ . That is, for any  $\varepsilon > 0$ ,

$$\int_0^\varepsilon f(t) dt > 0.$$

*Remark 4.5.* While the statements in part (1) of Theorem 4.4 are for gaps of affine lattices that are already in the transversal  $\Omega$  and generic for  $m_\Omega$ , we can obtain information about the gaps of an affine lattice  $\Lambda$  that is generic for  $m_{AX_2}$ . Let  $\Lambda' \in \Omega$  be such that  $h_u \Lambda = \Lambda'$  and  $u$  is the smallest time for which this occurs. Then,

$$\lim_{R \rightarrow \infty} \frac{|\mathcal{G}_V^R(\Lambda) \cap (t, \infty)|}{N(R)} = \lim_{R \rightarrow \infty} \frac{|\mathcal{G}_V^R(\Lambda') \cap (t, \infty)|}{N(R)} = \int_t^\infty g(t) dt.$$

Similar arguments can be used for the rest of Theorem 4.4 to turn statements about the transversal to statements about the ambient space.

**4.4.  $d$ -Symmetric torus covers.** We can apply our results of doubled slit tori to a more general class of translation surfaces. Let  $d > 1$  and consider  $d$ -copies of a torus  $X$  that we label  $X_1, \dots, X_d$ . Now glue the copies along a slit and identify opposite sides of the slits according to the permutation  $(1, 2, \dots, d)$ . Following [26], we call a surface constructed in this way a  $d$ -symmetric torus cover. Notice that a doubled slit torus simply corresponds to this construction with  $d = 2$ .

Any  $d$ -symmetric torus cover has two cone points of angle  $2d\pi$  and has genus  $d$ . Let  $\mathcal{E}^d$  denote the class  $d$ -symmetric torus covers. We have  $\mathcal{E}^d \subset \mathcal{H}(d - 1, d - 1)$ . See Figure 4 for an example of a  $d$ -symmetric torus cover.

We remark that we can do a similar construction with any cycle  $\sigma \in S^d$  (cycle so that the resulting surface is connected). By relabeling the  $d$ -copies, there is no loss in generality in assuming  $(1, 2, \dots, d)$ .

The results for doubled slit tori allow us to compute the gap distribution of a typical element in  $\mathcal{E}^d$ . In particular, this yields an explicit gap distribution for a translation surface in any genus  $d > 1$ .

COROLLARY 4.6. *For  $d > 1$ , the gap distribution of a  $d$ -symmetric torus cover can be computed using the density function for doubled slit tori. That is,*

$$\lim_{R \rightarrow \infty} \frac{|\mathcal{G}^R(\Lambda_\omega) \cap I|}{N(\omega, R)} = \int_I f(x) dx$$

for almost every  $\omega \in \mathcal{E}^d$  with respect to an  $SL_2(\mathbb{R})$ -invariant probability measure on  $\mathcal{E}^d$  that is in the same measure class as the Lebesgue measure. Here,  $f(t)$  is the density function for doubled slit tori from the proof of Theorem 1.1.

*Proof.* Consider a typical element  $\omega_1 \in \mathcal{E}^d$ . We obtain a twice-marked torus  $(g, \nu) \in \mathcal{H}(0, 0)$  by projecting to a single copy and forgetting the slit. Use this twice-marked torus to construct a doubled slit torus  $\omega_2$ . Each has the same set of saddle connections ( $\Lambda_{\omega_1} = \Lambda_{\omega_2}$ ) so that their respective gap distributions are the same as well.  $\square$

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#### A. Appendix. Measure computations and estimates

In the appendix we compute an explicit limiting gap distribution for slopes of saddle connections on a doubled slit torus. We also prove estimates on the decay of gaps for affine lattices.

A.1. *Haar-measure computations and estimates for  $\mathcal{W}$ .* In this section, we explicitly calculate the gap distribution function given by

$$m_{\mathcal{W}}(\omega \in \mathcal{W} | \mathcal{R}(\omega) > t),$$

where  $t > 0$ . As a corollary, we compute the decay rate of the tail and show there is support at zero.

We compute the distribution by considering the partition on  $\mathcal{W}$  given by surfaces with projection of a short affine lattice  $\mathcal{W}_{sa}$  and surfaces with projection of a short lattice  $\mathcal{W}_{sl}$ . On either part of the partition, we will have the return time map  $\mathcal{R}$  explicitly. The basic idea of proof will be to use the condition of there being a gap larger than  $t$  (i.e. that the return time  $\mathcal{R}$  is larger than  $t$ ) and see what conditions are imposed on the coordinates where the return time is defined. This will yield explicit integrals whence we can compute the integrals and make arguments with a Taylor series to show their decay.

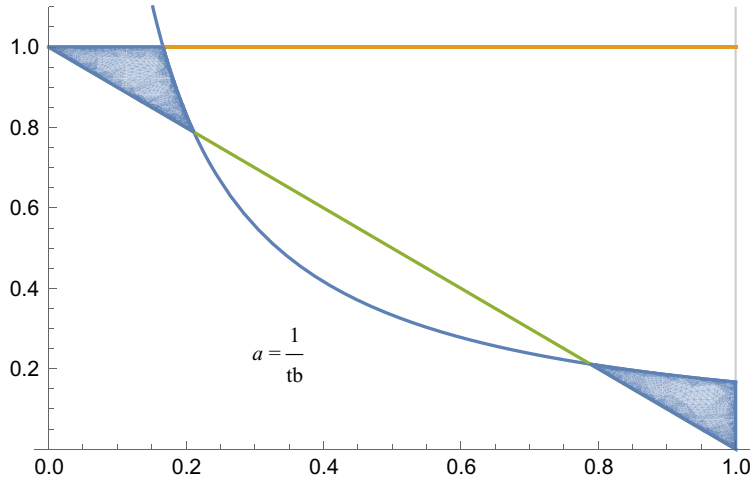


FIGURE A1. The contribution on  $\Omega_1$  when  $t > 4$ .

A.1.1. *Haar-measure computations and estimates for  $\mathcal{W}_{sa}$ .* We compute the contribution on the doubled slit tori coming from a short affine piece. That is,

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sa} | \mathcal{R}(\omega) > t\}).$$

The return time map  $\mathcal{R}$  is a piecewise map on three pieces depending on  $\Pi(\omega)$ . In each piece, we know explicitly the return map and so we can compute the contribution.

*Contribution when  $\Pi(\omega) \in \Omega_1$ .* Suppose  $\Pi(\omega) \in \Omega_1$ . Notice that  $\Omega_1$  can be written as

$$\Omega_1 = \left\{ (a, b, s, \alpha) \in \Omega : b \in (0, 1], \alpha \in (1 - b, 1], a \in (1 - b, \alpha], s \in \left[ 0, \frac{\alpha - a}{ab\alpha} \right) \right\}.$$

We have,

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sa} | \mathcal{R}(\omega) > t, \Pi(\omega) \in \Omega_1\}) = 4m_{\Omega}(\{(a, b, s, \alpha) \in \Omega_1 : R(a, b, s, \alpha) > t\}).$$

Thus, we now compute  $m_{\Omega}(\Omega_1 \cap \{R > t\})$ .

The condition on the return time says

$$t < R(a, b, s, \alpha) = \frac{sa}{\alpha - a} \iff \frac{t(\alpha - a)}{a} < s.$$

Notice that this is a valid lower bound for  $s$  since  $s$  is non-negative. On  $\Omega_1$ ,  $s$  has an upper bound of  $(\alpha - a)/ab\alpha$  so in order for the return map  $R$  to be larger than  $t$ , we must also satisfy the cross-term condition  $t(\alpha - a)/a < s < (\alpha - a)/ab\alpha \iff t < 1/b\alpha$ . We rewrite this as  $\alpha < 1/bt$ . Hence, all we need to understand is how the hyperbola  $\alpha = 1/bt$  intersects the  $(b, \alpha)$ -triangle  $\{(b, \alpha) : b \in (0, 1], \alpha \in (1 - b, 1]\}$ . See Figure A1 for an illustration for the  $(b, a)$  slice of the contribution of  $\Omega_1$  when  $t > 4$ .

If  $t \in (0, 1)$ , then everything lies below the hyperbola  $\alpha = 1/bt$  and our contribution is given by

$$m_{\Omega}(\{R > t\} \cap \Omega_1) = \int_{b=0}^1 \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db.$$

If  $t \in [1, 4)$ , then

$$m_{\Omega}(\{R > t\} \cap \Omega_1) = \int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db + \int_{b=1/t}^1 \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db.$$

If  $t \in [4, \infty)$ , then

$$m_{\Omega}(\{R > t\} \cap \Omega_1) = \int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db + \int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db + \int_{b=(1+\sqrt{1-(4/t)})/2}^1 \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db.$$

By evaluating each integral and then approximating the result by its Taylor series, we can understand how it decays. As an example, we do this for the first integral.

$$\begin{aligned} & \int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db \\ &= \int_{b=0}^{1/t} \frac{b(b(2+b)t - 8) + 2(b(2+t) - 4) \log(1-b)}{4b} \, db \\ &\approx \int_{b=0}^{1/t} \frac{b(b(2+b)t - 8) + 2(b(2+t) - 4)(-b - (b^2/2) - (b^3/3))}{4b} \, db \\ &= -\frac{1}{6} \int_{b=0}^{1/t} b^2(b(2+t) - 1) \, db = -\frac{1}{12t^4} + \frac{1}{72t^3}. \end{aligned}$$

Hence, on this piece, we have a cubic decay. Doing this for the rest of the integrals, we have

$$\begin{aligned} & \int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db \sim t^{-3}, \\ & \int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db \sim t^{-5}, \\ & \int_{b=(1+\sqrt{1-(4/t)})/2}^1 \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=t(\alpha-a)/a}^{(\alpha-a)/ab\alpha} 1 \, ds \, da \, d\alpha \, db \sim t^{-2}. \end{aligned}$$

Hence, the decay rate on the tail of  $\Omega_1$  is quadratic. By multiplying each integral above by 4, we have that contribution of

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sa} | \mathcal{R}(\omega) > t, \Pi(\omega) \in \Omega_1\})$$

and also the decay rate

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{\text{sa}} | \mathcal{R}(\omega) > t, \Pi(\omega) \in \Omega_1\}) \sim t^{-2}.$$

Contribution when  $\Pi(\omega) \in \Omega_3$ . Suppose  $\Pi(\omega) \in \Omega_3$ . Notice that  $\Omega_3$  can be written as

$$\Omega_3 = \left\{ (a, b, s, \alpha) \in \Omega : b \in (0, 1], \alpha \in (0, 1 - b], a \in (1 - b, 1], s \in \left[0, \frac{1}{ab}\right) \right\}.$$

We have

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{\text{sa}} | \mathcal{R}(\omega) > t, \Pi(\omega) \in \Omega_3\}) = 4m_{\Omega}(\{(a, b, s, \alpha) \in \Omega_3 : R(a, b, s, \alpha) > t\}).$$

Thus, we now compute  $m_{\Omega}(\Omega_3 \cap \{R > t\})$ .

On  $\Omega_3$ , we have

$$t < R(a, b, s, \alpha) = \frac{a^{-1} - sb}{b + \alpha} \iff \frac{a^{-1} - t(b + \alpha)}{b} > s.$$

Notice that the left-hand side is always bounded above by  $(ab)^{-1}$ . In order for there to be any valid values, we need the left-hand side to be positive, which is equivalent to

$$a < \frac{1}{t(b + \alpha)}.$$

Hence, we seek the minimum between  $1/t(b + \alpha)$  and  $a$ .

Case I:  $a < \min\{1/t(b + \alpha), 1\} = 1$ . This happens when

$$\frac{1}{t(b + \alpha)} > 1 \iff \alpha < \frac{1}{t} - b.$$

Notice now that  $\alpha < 1 - b$  on  $\Omega_3$  and that

$$\alpha < \min\{1/t - b, 1 - b\} = \begin{cases} 1 - b & \text{if } t < 1, \\ 1/t - b & \text{if } t > 1. \end{cases}$$

Hence, if  $t < 1$ , the integral is

$$\int_{b=0}^1 \int_{\alpha=0}^{1-b} \int_{a=1-b}^1 \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db = \frac{\pi^2}{6} - 1 - \frac{t}{3}.$$

On the other hand, if  $t > 1$ , then  $\alpha < 1/t - b$  and since  $\alpha > 0$ , we must have  $1/t > b$ . Thus, for large  $t$  (larger than 1), the integral is

$$\int_{b=0}^{1/t} \int_{\alpha=0}^{1/t-b} \int_{a=1-b}^1 \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db.$$

A Taylor series argument shows that this decays quadratically.

Case II:  $a < \min\{1/t(b + \alpha), 1\} = 1/t(b + \alpha)$ . This happens when

$$\frac{1}{t(b + \alpha)} < 1 \iff \alpha > \frac{1}{t} - b.$$

Since  $1 - b < \alpha$ , then combining the inequalities, we obtain

$$\alpha < \frac{b^2 - b + 1/t}{1 - b}.$$

By understanding how these functions intersect the  $(b, \alpha)$ -plane

$$\{(b, \alpha) \mid 0 < b \leq 1 \text{ and } 0 < \alpha < 1 - b\},$$

then we can understand how this function grows in  $t$ .

We obtain the following. If  $t \in (0, 1)$ , there is no contribution.

If  $t \in [1, 2)$ , the contribution is

$$\begin{aligned} & m_{\Omega}(\{R > t\} \cap \Omega_3) \\ &= \int_{b=0}^{1-1/t} \int_{\alpha=1/t-b}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \\ &+ \int_{b=1-1/t}^{1/t} \int_{\alpha=1/t-b}^{1-b} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \\ &+ \int_{b=1/t}^1 \int_{\alpha=0}^{1-b} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db. \end{aligned}$$

If  $t \in [2, 4)$ , the contribution is

$$\begin{aligned} & m_{\Omega}(\{R > t\} \cap \Omega_3) \\ &= \int_{b=0}^{1/t} \int_{\alpha=1/t-b}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \\ &+ \int_{b=1/t}^{1-1/t} \int_{\alpha=0}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \\ &+ \int_{b=1-1/t}^1 \int_{\alpha=0}^{1-b} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db. \end{aligned}$$

If  $t > 4$ , the tail on this piece of  $\Omega_3$  is given by

$$\begin{aligned} & m_{\Omega}(\{R > t\} \cap \Omega_3) \\ &= \int_{b=0}^{1/t} \int_{\alpha=t^{-1}-b}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \\ &+ \int_{b=1/t}^{(1-\sqrt{1-4/t})/2} \int_{\alpha=0}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \\ &+ \int_{b=(1+\sqrt{1-4/t})/2}^{1-1/t} \int_{\alpha=0}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \\ &+ \int_{b=1-1/t}^1 \int_{\alpha=0}^{1-b} \int_{a=1-b}^{1/(t(b+\alpha))} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db. \end{aligned}$$

By evaluating each integral and then approximating by its Taylor series, we see that

$$\int_{b=0}^{1/t} \int_{\alpha=t^{-1}-b}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/t(b+\alpha)} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \sim t^{-4},$$

$$\int_{b=1/t}^{(1-\sqrt{1-4/t})/2} \int_{\alpha=0}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/t(b+\alpha)} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \sim t^{-5},$$

$$\int_{b=(1+\sqrt{1-4/t})/2}^{1-1/t} \int_{\alpha=0}^{(b^2-b+1/t)/(1-b)} \int_{a=1-b}^{1/t(b+\alpha)} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \sim t^{-5},$$

$$\int_{b=1-1/t}^1 \int_{\alpha=0}^{1-b} \int_{a=1-b}^{1/t(b+\alpha)} \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \sim t^{-2}.$$

By adding the contributions from both cases, we conclude that

$$m_{\Omega}(\{R > t\} \cap \Omega_3) \sim t^{-2}.$$

By multiplying each integral above by 4, we have that contribution of

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sa} | \mathcal{R}(\omega) > t, \Pi(\omega) \in \Omega_3\})$$

and also the decay rate

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sa} | \mathcal{R}(\omega) > t, \Pi(\omega) \in \Omega_3\}) \sim t^{-2}.$$

*Contribution when  $\Pi(\omega) \in \Omega_2 \cup \Omega_4$ .* Suppose  $\Pi(\omega) \in \Omega_2$ . Notice that  $\Omega_2$  can be written as

$$\Omega_2 = \left\{ b \in (0, 1], \alpha \in (1 - b, 1], a \in (1 - b, \alpha), s \in \left[ (ab)^{-1} \left( \frac{\alpha - a}{\alpha} \right), (ab)^{-1} \right] \right\}.$$

We have

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sa} | \mathcal{R}(\omega) > t, \Pi(\omega) \in \Omega_2\}) = 4m_{\Omega}(\{(a, b, s, \alpha) \in \Omega_2 : (ab)^{-1} - s > t\}).$$

Unpacking a little, we see

$$(ab)^{-1} - s > t \iff (ab)^{-1} - t > s$$

and this is always the minimum upper bound for  $s$  in  $\Omega_2$  since  $t > 0$ . On the other hand, to satisfy the cross-term condition, we need

$$(ab)^{-1} \left( \frac{\alpha - a}{\alpha} \right) < (ab)^{-1} - t \iff \alpha < 1/bt.$$

Hence, we want to understand how the hyperbola  $\alpha = 1/bt$  intersects the triangle

$$\{(b, \alpha) \mid b \in (0, 1], \alpha \in (1 - b, 1]\}.$$

If  $t \in [0, 1)$ , then everything lies below the hyperbola  $\alpha = 1/bt$  and the contribution is

$$m_{\Omega}(\{\mathcal{R} > t\} \cap \Omega_2) = \int_{b=0}^1 \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db.$$

If  $t \in [1, 4)$ , then the hyperbola  $\alpha = 1/bt$  intersects once at  $b = 1/t$  and the contribution is

$$m_{\Omega}(\{\mathcal{R} > t\} \cap \Omega_2) = \int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db \\ + \int_{b=1/t}^1 \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db.$$

If  $t \in [4, \infty)$ , then the hyperbola  $\alpha = 1/bt$  intersects twice and the contribution is

$$m_{\Omega}(\{\mathcal{R} > t\} \cap \Omega_2) \\ = \int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db \\ + \int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db \\ + \int_{(1+\sqrt{1-(4/t)})/2}^1 \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db.$$

By evaluating each integral and approximating the result with a Taylor series, we see that

$$\int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db \sim t^{-2}, \\ \int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db \sim t^{-4}, \\ \int_{(1+\sqrt{1-(4/t)})/2}^1 \int_{\alpha=1-b}^{1/bt} \int_{a=1-b}^{\alpha} \int_{s=(ab)^{-1}((\alpha-a)/\alpha)}^{(ab)^{-1}-t} 1 \, ds \, da \, d\alpha \, db \sim t^{-2}.$$

In total, we have a quadratic tail when  $\Pi(\omega \in \Omega_2)$

$$m_{\Omega}(\{\mathcal{R} > t\} \cap \Omega_2) \sim t^{-2}.$$

Now suppose  $\Pi(\omega) \in \Omega_4$ . Notice that  $\Omega_4$  can be written as

$$\Omega_4 = \{b \in (0, 1], a \in (1 - b, 1], \alpha \in (1 - b, a), s \in [0, (ab)^{-1}]\}.$$

We have

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sa} | \mathcal{R}(\omega) > t, \Pi(\omega) \in \Omega_4\}) = 4m_{\Omega}(\{(a, b, s, \alpha) \in \Omega_4 : (ab)^{-1} - s > t\}).$$

The condition on the return time says

$$(ab)^{-1} - s > t \iff (ab)^{-1} - t > s$$

and this is always the minimum upper bound for  $s$  in  $\Omega_4$  since  $t > 0$ . On the other hand, to satisfy the cross-term condition, we need

$$0 < (ab)^{-1} - t \iff a < 1/bt.$$

Hence, we are interested in the hyperbola  $a = 1/bt$  and how it intersects the triangle  $\{b \in (0, 1], a \in (1 - b, 1]\}$ .



If  $t \in [0, 1)$ , then everything lies below the hyperbola  $\alpha = 1/bt$  and the contribution is

$$m_{\Omega}(\{\mathcal{R} > t\} \cap \Omega_4) = \int_{b=0}^1 \int_{a=1-b}^1 \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db.$$

If  $t \in [1, 4)$ , then the hyperbola  $\alpha = 1/bt$  intersects once at  $b = 1/t$  and the integral is

$$\begin{aligned} m_{\Omega}(\{\mathcal{R} > t\} \cap \Omega_4) &= \int_{b=0}^{1/t} \int_{a=1-b}^1 \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db \\ &\quad + \int_{b=1/t}^1 \int_{a=1-b}^{1/bt} \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db. \end{aligned}$$

If  $t \in [4, \infty)$ , then the hyperbola  $\alpha = 1/bt$  intersects twice and the integral is

$$\begin{aligned} m_{\Omega}(\{\mathcal{R} > t\} \cap \Omega_4) &= \int_{b=0}^{1/t} \int_{a=1-b}^1 \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db \\ &\quad + \int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{a=1-b}^{1/bt} \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db \\ &\quad + \int_{(1+\sqrt{1-(4/t)})/2}^1 \int_{a=1-b}^{1/bt} \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db. \end{aligned}$$

By evaluating each integral and approximating the result with a Taylor series, we see that

$$\begin{aligned} \int_{b=0}^{1/t} \int_{a=1-b}^1 \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db &\sim t^{-2}, \\ \int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{a=1-b}^{1/bt} \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db &\sim t^{-4}, \\ \int_{(1+\sqrt{1-(4/t)})/2}^1 \int_{a=1-b}^{1/bt} \int_{\alpha=1-b}^a \int_{s=0}^{(ab)^{-1-t}} 1 \, ds \, da \, d\alpha \, db &\sim t^{-2}, \\ m_{\Omega}(\{\mathcal{R} > t\} \cap \Omega_4) &\sim t^{-2}. \end{aligned}$$

*Tail contribution of  $\mathcal{W}_{sa}$ .* Hence, combining all the above contributions, we have that the tail on  $\mathcal{W}_{sa}$  has quadratic decay,

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sa} : \mathcal{R}(\omega) > t\}) \sim t^{-2}.$$

*A.1.2. Haar-measure computations and estimates for  $\mathcal{W}_{sl}$ .* We now compute the contribution on the doubled slit tori coming from a short lattice piece. That is,

$$m_{\mathcal{W}}(\{\omega \in \mathcal{W}_{sl} | \mathcal{R}(\omega) > t\}).$$

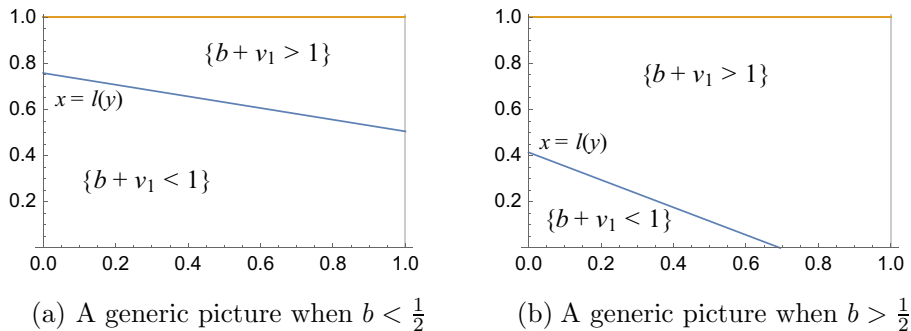


FIGURE A2. A generic picture of the two cases that arise for the return map.

For  $\omega = \omega_{(a,b,v_1,v_2)} \in \mathcal{W}_{sl}$ , we have a piecewise map on two pieces depending on the length of  $b + v_1$ . In each piece, we know explicitly the return map and thus can compute

$$m_{\mathcal{W}}(\omega \in \mathcal{W}_{sl} | \mathcal{R}(\omega) > t) = 4 \iint_{(a,b) \in \Delta} \int_{v \in \mathbb{R}^2 / p_{a,b} \mathbb{Z}^2} \chi_{\{\mathcal{R} > t\}}(a, b, v) \, dv \, db \, da.$$

It will be useful to parameterize  $\mathbb{R}^2 / p_{a,b} \mathbb{Z}^2$  as the set

$$\{(v_1, v_2) = (ax + by, a^{-1}y) \mid (a, b) \in \Delta, (x, y) \in [0, 1)^2\}.$$

The condition that determines what the return map is depends on whether  $b + v_1$  is larger than or smaller than 1. In the coordinates  $(v_1, v_2) = (ax + by, a^{-1}y)$ , we have

$$b + v_1 > 1 \iff x \geq \frac{1-b}{a} - \frac{b}{a}y =: l(y).$$

Hence, there is natural interest in the behavior of  $l$ . Notice

- (1)  $l(0) = (1 - b)/a$  is always less than one for any  $(a, b) \in \Delta$ , and
- (2)  $l(1) < 0 \iff b > 1/2$ .

Hence, we can represent the two cases of our return map through the  $(y, x)$  slice as in Figure A2.

*Case I:*  $b + v_1 > 1$ . Notice that  $l(1) < 0 \iff b > 1/2$ . Moreover,  $l$  hits the  $y$ -axis at  $y = b^{-1} - 1$ . As indicated at the start of this subsection, we break our computation up into the case where  $b > 1/2$  and  $b < 1/2$ . Notice that  $l$  hits the  $y$ -axis at  $y = b^{-1} - 1$ .

*Case IA:*  $b + v_1 > 1$  and  $b < 1/2$ . If  $b < 1/2$ , then our region  $b + v_1 > 1$  can be described as

$$\{(b, a, x, y) \mid b \in (0, 1/2], a \in (1 - b, 1), y \in [0, 1), x \in [l(y), 1)\}.$$

See Figure A2(a).

On this region,

$$\mathcal{R}(\omega) = (ab)^{-1} > t \iff a < \frac{1}{bt}.$$

Hence, all we have to do is understand how this hyperbola intersects the triangle  $\{(b, a) \in \Delta : b \in (1, 1/2), a \in (1 - b, 1)\}$ .

If  $t \in [0, 2)$ , then everything lies below the hyperbola and the contribution is given by

$$\int_{b=0}^{1/2} \int_{a=1-b}^1 \int_{y=0}^1 \int_{x=l(y)}^1 1 \, dx \, dy \, da \, db.$$

If  $t \in [2, 4)$ , then the contribution is given by

$$\int_{b=0}^{1/t} \int_{a=1-b}^1 \int_{y=0}^1 \int_{x=l(y)}^1 1 \, dx \, dy \, da \, db + \int_{b=1/t}^{1/2} \int_{a=1-b}^{1/bt} \int_{y=0}^1 \int_{x=l(y)}^1 1 \, dx \, dy \, da \, db.$$

If  $t \in [4, \infty)$ , then the contribution is given by

$$\begin{aligned} &\int_{b=0}^{1/t} \int_{a=1-b}^1 \int_{y=0}^1 \int_{x=l(y)}^1 1 \, dx \, dy \, da \, db \\ &+ \int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{a=1-b}^{1/bt} \int_{y=0}^1 \int_{x=l(y)}^1 1 \, dx \, dy \, da \, db. \end{aligned}$$

By evaluating each integral and approximating the result with a Taylor series, we see that

$$\begin{aligned} &\int_{b=0}^{1/t} \int_{a=1-b}^1 \int_{y=0}^1 \int_{x=l(y)}^1 1 \, dx \, dy \, da \, db \sim t^{-3}, \\ &\int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{a=1-b}^{1/bt} \int_{y=0}^1 \int_{x=l(y)}^1 1 \, dx \, dy \, da \, db \sim t^{-4}. \end{aligned}$$

Hence, the tail on this region is less than quadratic. More precisely, we have

$$m_{\mathcal{W}}(\{\mathcal{R} > t\} \cap \{b + v_1 > 1\} \cap \{b < 1/2\}) \sim t^{-3}.$$

*Case 1B:*  $b + v_1 > 1$  and  $b > 1/2$ . If  $b > 1/2$ , then our region is the union of

$$\{(b, a, x, y) \mid b \in (1/2, 1), a \in (1 - b, 1), y \in [0, b^{-1} - 1), x \in [l(y), 1)\}$$

with

$$\{(b, a, x, y) \mid b \in (1/2, 1), a \in (1 - b, 1), y \in [b^{-1} - 1, 1), x \in [0, 1)\}.$$

See Figure A2(b).

Now we can implement the return time condition. On this region,

$$R(\omega) = (ab)^{-1} > t \iff (bt)^{-1} > a.$$

Hence, all we have to do is understand how this hyperbola intersects the triangle  $\{(b, a) \in \Delta \mid b \in (1/2, 1), a \in (1 - b, 1)\}$ .

If  $t \in (0, 1)$ , then everything lies below the hyperbola and the integral is

$$\int_{b=1/2}^1 \int_{a=1-b}^1 \left[ \int_{y=0}^{b^{-1}-1} \int_{x=l(y)}^1 1 \, dx \, dy + \int_{y=b^{-1}-1}^1 \int_{x=0}^1 1 \, dx \, dy \right] da \, db.$$

If  $t \in [1, 2)$ , then the integral is

$$\int_{b=1/2}^{1/t} \int_{a=1-b}^1 \left[ \int_{y=0}^{b^{-1}-1} \int_{x=l(y)}^1 1 \, dx \, dy + \int_{y=b^{-1}-1}^1 \int_{x=0}^1 1 \, dx \, dy \right] da \, db$$

$$+ \int_{b=1/t}^1 \int_{a=1-b}^{1/bt} \left[ \int_{y=0}^{b^{-1}-1} \int_{x=l(y)}^1 1 \, dx \, dy + \int_{y=b^{-1}-1}^1 \int_{x=0}^1 1 \, dx \, dy \right] da \, db.$$

If  $t \in [2, 4)$ , then the integral is

$$\int_{b=1/2}^1 \int_{a=1-b}^{1/bt} \left[ \int_{y=0}^{b^{-1}-1} \int_{x=l(y)}^1 1 \, dx \, dy + \int_{y=b^{-1}-1}^1 \int_{x=0}^1 1 \, dx \, dy \right] da \, db.$$

If  $t \in [4, \infty)$ , then the integral is

$$\int_{b=(1+\sqrt{1-(4/t)})/2}^1 \int_{a=1-b}^{1/bt} \left[ \int_{y=0}^{b^{-1}-1} \int_{x=l(y)}^1 1 \, dx \, dy + \int_{y=b^{-1}-1}^1 \int_{x=0}^1 1 \, dx \, dy \right] da \, db.$$

By evaluating this integral and approximating the result with a Taylor series, we see that

$$\int_{b=(1+\sqrt{1-(4/t)})/2}^1 \int_{a=1-b}^{1/bt} \left[ \int_{y=0}^{b^{-1}-1} \int_{x=l(y)}^1 1 \, dx \, dy + \int_{y=b^{-1}-1}^1 \int_{x=0}^1 1 \, dx \, dy \right] da \, db \sim t^{-2}.$$

Hence,

$$m_{\mathcal{W}}(\{\mathcal{R} > t\} \cap \{b + v_1 > 1\} \cap \{b > 1/2\})$$

decays quadratically.

Combining this with the part where  $b < 1/2$ , we see that

$$m_{\mathcal{W}}(\{\mathcal{R} > t\} \cap \{b + v_1 > 1\}) \sim t^{-2}.$$

*Case 2:  $b + v_1 < 1$ .* We are interested in the situation underneath  $l(y)$  in Figure A2. Now we impose the return time condition. On this region (that is, when  $b + v_1 \leq 1$ ), we have

$$R(\omega) = \frac{v_2}{v_1} > t \iff x \leq \left( \frac{1}{a^2t} - \frac{b}{a} \right) y =: L(y).$$

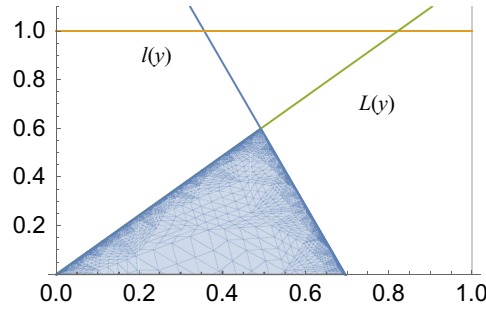
Notice that  $L$  is a line through the origin and since it is an upper bound for  $x$ , then we are only concerned when it has positive slope. That is, when

$$\frac{1}{a^2t} - \frac{b}{a} > 0 \iff \frac{1}{bt} > a.$$

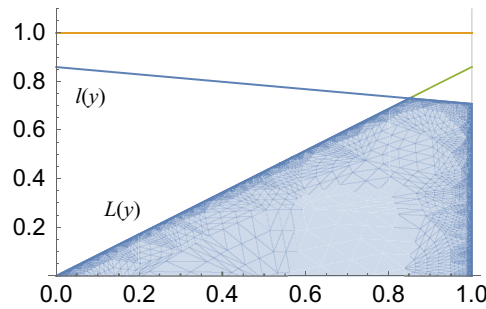
Lastly, we notice to parameterize the  $(y, x)$ -plane, we need to understand how the two lines  $l$  and  $L$  intersect each other. Since they are each lines, it suffices to understand how they interact when  $y = 1$ . We obtain three cases:

- (1)  $b > 1/2$  and  $L(1) \geq l(1)$ ;
- (2)  $b < 1/2$  and  $L(1) \geq l(1)$ ;
- (3)  $b < 1/2$  and  $L(1) < l(1)$ .

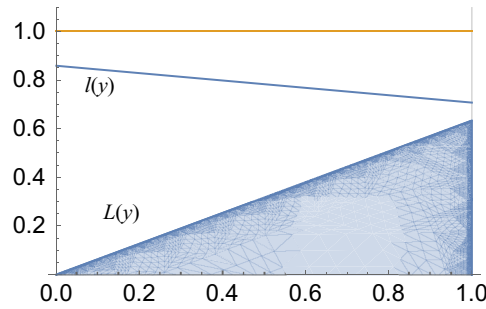
We do not have the case  $b > 1/2$  and  $L(1) < l(1)$  because when  $b > 1/2$ , then we have  $l(1)$  as negative, which forces  $L$  to be negative, but this is not possible since it is an upper



(a)  $b > 1/2$  and  $L(1) \geq l(1)$



(b)  $b < 1/2$  and  $L(1) \geq l(1)$



(c)  $b < 1/2$  and  $L(1) < l(1)$

FIGURE A3. A generic picture of the three cases that arise for the return map.

bound for  $x$ . Lastly, we note that  $L$  and  $l$  intersect at the point  $y = (1 - b)at$ . We illustrate the three cases in Figure A3.

We now compute the contribution on each of the above cases.

Case 2A:  $b + v_1 < 1$ ,  $b > 1/2$ , and  $L(1) \geq l(1)$ . If  $b > 1/2$  and  $L(1) \geq l(1)$ , then the region is the union of

$$\{(b, a, y, x) : b \in (1/2, 1], a \in (1 - b, 1], y \in (0, (1 - b)at], x \in (0, L(y))\}$$

and

$$\{(b, a, y, x) : b \in (1/2, 1], a \in (1 - b, 1], y \in ((1 - b)at, b^{-1} - 1], x \in (0, l(y))\}.$$

The condition  $L(1) \geq l(1)$  is equivalent to  $a \leq 1/t(1 - b)$ . Combining this with the condition from the return map means that we are interested in

$$a \leq \min \left\{ \frac{1}{bt}, \frac{1}{t(1 - b)} \right\}.$$

However,

$$\frac{1}{tb} < \frac{1}{t(1 - b)} \iff 1/2 < b$$

so that actually we only care about how the hyperbola  $a = 1/tb$  intersects the region  $\{(b, a) : b \in (1/2, 1], a \in (1 - b, 1]\}$ .

If  $t \in [0, 1)$ , then everything is under the hyperbola and the integral is

$$\int_{b=1/2}^1 \int_{a=1-b}^1 \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^{b^{-1}-1} \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db.$$

If  $t \in [1, 2)$ , then the integral is

$$\begin{aligned} &\int_{b=1/2}^{1/t} \int_{a=1-b}^1 \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^{b^{-1}-1} \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db \\ &+ \int_{b=1/t}^1 \int_{a=1-b}^{1/bt} \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^{b^{-1}-1} \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db. \end{aligned}$$

If  $t \in [2, 4)$ , then the integral is

$$\int_{b=1/2}^1 \int_{a=1-b}^{1/bt} \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^{b^{-1}-1} \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db.$$

If  $t \in [4, \infty)$ , then the integral is

$$\int_{b=(1+\sqrt{1-(4/t)})/2}^1 \int_{a=1-b}^{1/bt} \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^{b^{-1}-1} \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db.$$

By evaluating the above integral and approximating it with its Taylor series we see that the decay is cubic. That is,

$$m_{\mathcal{W}}(\{\mathcal{R} > t\} \cap \{b + v_1 < 1\} \cap \{b > 1/2\} \cap \{L(1) \geq l(1)\}) \sim t^{-3}.$$

Case 2B:  $b + v_1 < 1$ ,  $b < 1/2$ , and  $L(1) \geq l(1)$ . If  $b < 1/2$  and  $L(1) \geq l(1)$ , then the region is the union of

$$\{(b, a, y, x) : b \in (0, 1/2], a \in (1 - b, 1], y \in (0, (1 - b)at], x \in (0, L(y))\}$$

and

$$\{(b, a, y, x) : b \in (0, 1/2], a \in (1 - b, 1], y \in ((1 - b)at, 1], x \in (0, l(y))\}.$$

As before, the condition  $L(1) \geq l(1)$  is equivalent to  $a \leq 1/t(1 - b)$ . Combining this with the condition from the return map means that we are interested in

$$a \leq \min \left\{ \frac{1}{bt}, \frac{1}{t(1 - b)} \right\}.$$

We have,

$$\frac{1}{tb} > \frac{1}{t(1-b)} \iff 1/2 > b.$$

Consequently, we are interested in how the hyperbola  $a = 1/t(1 - b)$  intersects the triangle  $\{(b, a) : b \in (0, 1/2], a \in (1 - b, 1]\}$ .

If  $t \in [0, 1)$ , then everything is under the hyperbola and the integral is

$$\int_{b=0}^{1/2} \int_{a=1-b}^1 \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^1 \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db.$$

If  $t \in [1, 2)$ , then the integral is

$$\begin{aligned} &\int_{b=1-\sqrt{t^{-1}}}^{1-t^{-1}} \int_{a=1-b}^{1/t(1-b)} \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^1 \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db \\ &+ \int_{b=1-t^{-1}}^{1/2} \int_{a=1-b}^1 \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^1 \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db. \end{aligned}$$

If  $t \in [2, 4)$ , then the integral is

$$\int_{b=1-\sqrt{t^{-1}}}^{1/2} \int_{a=1-b}^{1/t(1-b)} \left[ \int_{y=0}^{(1-b)at} \int_{x=0}^{L(y)} 1 \, dx \, dy + \int_{y=(1-b)at}^1 \int_{x=0}^{l(y)} 1 \, dx \, dy \right] da \, db.$$

If  $t \in [4, \infty)$ , then the hyperbola passes below the triangle and there is no contribution.

Case 2C:  $b + v_1 < 1$ ,  $b < 1/2$ , and  $L(1) < l(1)$ . If  $b < 1/2$  and  $L(1) < l(1)$ , then the region is

$$\{(b, a, y, x) : b \in (0, 1/2], a \in (1 - b, 1], y \in (0, 1], x \in (0, L(y))\}.$$

Using that  $L(1) < l(1)$  is equivalent to  $a > 1/t(1 - b)$ . we have that we are interested in the condition:

$$\max \left\{ 1 - b, \frac{1}{t(1 - b)} \right\} < a < \min \left\{ 1, \frac{1}{bt} \right\}.$$

By understanding this region in the  $(b, a)$ -slice we deduce the following.

If  $t \in [0, 1)$ , the condition that  $a > 1/t(1 - b)$  says there is no contribution here.

If  $t \in [1, 2)$ , the integral is

$$\begin{aligned} &\int_{b=0}^{1-\sqrt{1/t}} \int_{a=1-b}^1 \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db \\ &+ \int_{b=1-\sqrt{1/t}}^{1-(1/t)} \int_{a=1/t(1-b)}^1 \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db. \end{aligned}$$

If  $t \in [2, (3 + \sqrt{5})/2)$ , the integral is

$$\begin{aligned} & \int_{b=0}^{1-(1/\sqrt{t})} \int_{a=1-b}^1 \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db \\ & + \int_{b=1-\sqrt{1/t}}^{1/t} \int_{a=1/t(1-b)}^{1/bt} \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db \\ & + \int_{b=1-\sqrt{1/t}}^{1/2} \int_{a=1/t(1-b)}^{1/bt} \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db. \end{aligned}$$

If  $t \in [(3 + \sqrt{5})/2, 4)$ , the integral is

$$\begin{aligned} & \int_{b=0}^{1/t} \int_{a=1-b}^1 \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db + \int_{b=1/t}^{1-\sqrt{1/t}} \int_{a=1-b}^{1/bt} \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db \\ & + \int_{b=1-\sqrt{1/t}}^{1/2} \int_{a=1/t(1-b)}^{1/bt} \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db. \end{aligned}$$

Lastly, if  $t \geq 4$ , the integral is

$$\begin{aligned} & \int_{b=0}^{1/t} \int_{a=1-b}^1 \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db \\ & + \int_{b=1/t}^{(1-\sqrt{1-(4/t)})/2} \int_{a=1-b}^{1/bt} \int_{y=0}^1 \int_{x=0}^{L(y)} 1 \, dx \, dy \, da \, db. \end{aligned}$$

By evaluating the integrals and approximating it with its Taylor series we see that the decay is cubic.

*A.1.3. Proof of Theorem 4.4, (3) and (4).* Combining the results of the previous two subsections we see that we have quadratic decay for the slope gap distribution of doubled slit tori and support at zero.

An explicit description of the tail distribution  $G$  is given below. Recall, that this is defined as  $G(t) = \frac{1}{6}(3 + \pi^2) - F(t)$  where  $F(t)$  is the cumulative distribution function.

Piecewise, we have:  $G(t) \cdot \chi_{[0,1]} = \frac{1}{6}(3 + \pi^2) - (7t/8)$ ,

$$G(t) \cdot \chi_{(1,2]}$$

$$\begin{aligned} & = \frac{1}{24} \left( 24 \left( \text{Li}_2 \left( \frac{1}{t} \right) - \text{Li}_2 \left( \frac{t-1}{t} \right) \right) - 12 \log^2(t) + 24 \log(t-1) \log(t) + 12 \log(t) \right) \\ & - \frac{5}{2} + \frac{1}{6t^2} + \frac{1}{24} t (-24 \log(t-1) + 24 \log(t) - 4) + \frac{24 \log(t-1) + 54 \log(t) + 51}{24t}, \end{aligned}$$

$$G(t) \cdot \chi_{(2,(3+\sqrt{5})/2]}$$

$$= \frac{1}{48} \left( -48 \left( \text{Li}_2 \left( \frac{1}{t} \right) - \text{Li}_2 \left( \frac{t-1}{t} \right) \right) - 3 \log(t^3) - 24 \log^2(t) \right) + \frac{1}{2t^{3/2}}$$



$$\begin{aligned}
 & + \frac{12 \log(1 - (1/\sqrt{t})) - 36 \log(t - 1) + 48 \log(t) - 12 \log(t - \sqrt{t})}{48t^2} \\
 & + \frac{72 \coth^{-1}(1 - 2t) - 18}{48t^2} + \frac{24 \log(1 - (1/\sqrt{t})) + 24 \log(t) - 36}{48\sqrt{t}} \\
 & + \frac{1}{48} \left( -12 \log \left( 1 - \frac{1}{\sqrt{t}} \right) - 3(7 + \log(256)) \log \left( \frac{4}{t} \right) \right) \\
 & + \frac{1}{24} (-63 - 45 \log(2) - \log(8) \log(256)) \\
 & + \frac{1}{48} t(-48 \log(t - 1) + 48 \log(t) - 16) \\
 & + \frac{-12 \log(1 - (1/\sqrt{t})) + 48 \log(t - 1) + 96 \log(t) + 192}{48t} \\
 & + \frac{1}{48} (48 \log(t - 1) \log(t) + 24(3 + \log(2)) \log(t)), \\
 G(t) \cdot \chi_{((3+\sqrt{5})/2, 4]} \\
 & = \frac{1}{48} \left( -48 \left( \text{Li}_2 \left( \frac{1}{t} \right) - \text{Li}_2 \left( \frac{t-1}{t} \right) \right) - 3 \log(t^3) - 24 \log^2(t) \right) \\
 & + \frac{-12 \log(1 - t) - 24 \log(t - 1) + 12 \log(-t) + 24 \log(t)}{48t^2} \\
 & + \frac{72 \coth^{-1}(1 - 2t) - 24}{48t^2} \\
 & + \frac{1}{48} \left( -12 \log \left( 1 - \frac{1}{\sqrt{t}} \right) - 144 - 2 \log(8)(15 + \log(256)) \right) \\
 & + \frac{1}{48} \left( 12 \log(\sqrt{t} - 1) + 3(7 + \log(256)) \log \left( \frac{4}{t} \right) \right) \\
 & + \frac{24 \log(1 - (1/\sqrt{t})) - 24 \log(\sqrt{t} - 1) + 12 \log(t)}{48\sqrt{t}} \\
 & + \frac{1}{48} t(-48 \log(t - 1) + 48 \log(t) - 16) \\
 & + \frac{-12 \log(1 - (1/\sqrt{t})) + 12 \log(\sqrt{t} - 1) + 24 \log(t - 1) + 24 \log((t - 1)/\sqrt{t})}{48t} \\
 & + \frac{114 \log(t) + 198}{48t} \\
 & + \frac{1}{48} (48 \log(t - 1) \log(t) + 6(13 + \log(16)) \log(t)).
 \end{aligned}$$

The formula for  $G(t)$  when  $t > 4$  is extremely long so we do not write them down. The main characteristic is that it has quadratic tail decay as shown in Theorem 4.4(3).

A.2. *Haar measure computations and estimates for  $\Omega$ .* We prove the estimate of Theorem 4.4(1). We also show that there is support at zero for the density function.

*Proof of Theorem 4.4(1).* We compute the cumulative distribution function for the gaps of slopes

$$m_{\Omega}(\{R > t\} \cap \Omega_i)$$

separately for each  $i = 1, \dots, 4$  where  $\Omega_i$  are the regions where the return map is the same.

We notice that much of these computations were needed in computing the gap distribution on  $\mathcal{W}$ , specifically the ones for the tail of  $\Omega_1$  and  $\Omega_3$ . As such, we do not reproduce them here.

For  $\Omega_1$  and  $\Omega_3$  we can compute exact asymptotics, while we only compute upper and lower bounds for  $\Omega_2$  and  $\Omega_4$ . This is because the return time function includes a floor function in these cases which make explicit computations difficult.

*Tail on  $\Omega_1$ .* This computation was implicit in the section of measure estimates on  $\mathcal{W}$ . It was shown there that for  $t > 4$ , we have

$$m_{\Omega}(\{R > t\} \cap \Omega_1) \sim t^{-2}.$$

Hence, the decay on  $\Omega_1$  is quadratic.

*Tail in  $\Omega_2$ .* Due to the floor function in the formula for the return map on  $\Omega$ , we only compute lower and upper estimates on the decay. We first notice that  $\Omega_2$  can be written as

$$\Omega_2 = \left\{ b \in (0, 1], a \in (1 - b, 1], \alpha \in (a, 1), s \in \left[ (ab)^{-1} \left( \frac{\alpha - a}{\alpha} \right), (ab)^{-1} \right] \right\}.$$

We compute an upper bound for the contribution on  $\Omega_2$ . Recall that  $j = \lfloor (1 + a - \alpha)/b \rfloor$  is in the formula for the return map on  $\Omega_2$ . An upper bound is found by

$$\begin{aligned} R|_{\Omega_2} &= \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} < \frac{((1 + a - \alpha)/b)(a^{-1} - sb) + sa}{\alpha - a + jb} \\ &= \frac{(1 + a - \alpha)(ab)^{-1} - s(1 - \alpha)}{\alpha - a + jb} < \frac{2(ab)^{-1}}{\alpha - a + jb}. \end{aligned}$$

Using that

$$\alpha - a + jb > \alpha - a + \left( \frac{1 + a - \alpha}{b} - 1 \right) b = 1 - b$$

and continuing our estimate from before, we find

$$R|_{\Omega_2} < \frac{2(ab)^{-1}}{\alpha - a + jb} < \frac{2}{ab(1 - b)} =: U.$$

To compute the contribution of  $m_\Omega(\{t < U\})$ , we notice that

$$t < U \iff a < \frac{2}{tb(1-b)}.$$

Thus, we need to understand how the curve  $1/tb(1-b)$  intersects the triangle  $\{(b, a) : b \in (0, 1], a \in (1-b, 1]\}$ . Due to this, the following points will be of interest

$$b_k := \frac{2}{3} \left( \cos \left( \frac{1}{3} \arccos \left( \frac{54}{2t} - 1 \right) - \frac{2\pi * k}{3} \right) + 1 \right)$$

where  $k = 1, 2$ . For  $t$  large, we note that  $b_1$  is close to 1 and  $b_2$  is close to 0. These points correspond to where the curve  $a = 1/tb(1-b)$  intersects the line  $a = 1-b$ .

Hence,

$$\begin{aligned} m_\Omega(\{t < U\}) &= \int_{b=0}^{(1-\sqrt{1-(8/t)})/2} \int_{a=1-b}^1 \int_{\alpha=a}^1 \int_{s=(\alpha-a)/ab\alpha}^{1/ab} 1 \, ds \, d\alpha \, da \, db \\ &+ \int_{b=(1-\sqrt{1-(8/t)})/2}^{b_2} \int_{a=1-b}^{1/(bt(1-b))} \int_{\alpha=a}^1 \int_{s=(\alpha-a)/ab\alpha}^{1/ab} 1 \, ds \, d\alpha \, da \, db \\ &+ \int_{b=b_1}^{(1+\sqrt{1-(8/t)})/2} \int_{a=1-b}^{1/(bt(1-b))} \int_{\alpha=a}^1 \int_{s=(\alpha-a)/ab\alpha}^{1/ab} 1 \, ds \, d\alpha \, da \, db \\ &+ \int_{b=(1+\sqrt{1-(8/t)})/2}^1 \int_{a=1-b}^1 \int_{\alpha=a}^1 \int_{s=(\alpha-a)/ab\alpha}^{1/ab} 1 \, ds \, d\alpha \, da \, db. \end{aligned}$$

By evaluating each integral and approximating the result with a Taylor series, we see that

$$\begin{aligned} \int_{b=0}^{(1-\sqrt{1-(8/t)})/2} \int_{a=1-b}^1 \int_{\alpha=a}^1 \int_{s=(\alpha-a)/ab\alpha}^{1/ab} 1 \, ds \, d\alpha \, da \, db &\sim t^{-2}, \\ \int_{b=(1-\sqrt{1-(8/t)})/2}^{b_2} \int_{a=1-b}^{1/(bt(1-b))} \int_{\alpha=a}^1 \int_{s=(\alpha-a)/ab\alpha}^{1/ab} 1 \, ds \, d\alpha \, da \, db &\sim t^{-1}, \\ \int_{b=b_1}^{(1+\sqrt{1-(8/t)})/2} \int_{a=1-b}^{1/(bt(1-b))} \int_{\alpha=a}^1 \int_{s=(\alpha-a)/ab\alpha}^{1/ab} 1 \, ds \, d\alpha \, da \, db &\sim t^{-1}, \\ \int_{b=(1+\sqrt{1-(8/t)})/2}^1 \int_{a=1-b}^1 \int_{\alpha=a}^1 \int_{s=(\alpha-a)/ab\alpha}^{1/ab} 1 \, ds \, d\alpha \, da \, db &\sim t^{-1}. \end{aligned}$$

Hence, an upper bound for the decay is

$$m_\Omega(\{t < R\} \cap \Omega_2) < m_\Omega(\{t < U\}) \sim t^{-1}.$$

We now compute a lower bound for the contribution on  $\Omega_2$ . Using that the denominator of  $R$  is bounded above by 1 and that all the terms in the numerator are positive, we have

that

$$R|_{\Omega_2} = \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} > \frac{sa}{1}.$$

Finally, using the lower bound of  $s$  in  $\Omega_2$  yields that

$$R|_{\Omega_2} > \frac{\alpha - a}{b\alpha} =: L.$$

To compute the contribution of  $m_{\Omega}(\{t < L\})$ , we notice that

$$t < L \iff a < \alpha(1 - bt),$$

which indicates interest in how the curve  $a = \alpha(1 - bt)$  intersects the  $(\alpha, a)$  triangle. A computation confirms that we only have points in this triangle whenever  $b < 1/t$ . Of those points, we only get a contribution whenever  $1 - tb > 1$ , which gives the integral

$$\int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=(\alpha-a)/ab\alpha}^{(ab)^{-1}} 1 \, ds \, da \, d\alpha \, db.$$

Evaluating and using Taylor series shows that this integral has quadratic decay that is

$$\int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=1-b}^{\alpha} \int_{s=(\alpha-a)/ab\alpha}^{(ab)^{-1}} 1 \, ds \, da \, d\alpha \, db \sim t^{-2}.$$

*Tail on  $\Omega_3$ .* This computation was implicit in the section of measure estimates on  $\mathcal{W}$ . It was shown there that for  $t > 4$ , we have

$$m_{\Omega}(\{R > t\} \cap \Omega_3) \sim t^{-2}.$$

Hence, the decay on  $\Omega_3$  is quadratic.

*Tail on  $\Omega_4$ .* We first notice that  $\Omega_4$  can be written as

$$\Omega_4 = \{b \in (0, 1], a \in (1 - b, 1], \alpha \in (1 - b, a], s \in [0, (ab)^{-1}]\}.$$

Second, we notice that the upper estimate from  $\Omega_2$  works. Namely,

$$R|_{\Omega_4} < \frac{2}{ab(1 - b)} =: U.$$

Moreover,

$$t < U \iff a < \frac{2}{tb(1 - b)}.$$

Thus, we need to understand how the curve  $1/tb(1 - b)$  intersects the triangle

$$\{(b, a) : b \in (0, 1], a \in (1 - b, 1]\}.$$

As before, we will be interested in

$$b_k := \frac{2}{3} \left( \cos \left( \frac{1}{3} \arccos \left( \frac{54}{2t} - 1 \right) - \frac{2\pi * k}{3} \right) + 1 \right),$$

where  $k = 1, 2$ . For  $t$  large, we note that  $b_1$  is close to 1 and  $b_2$  is close to 0. Lastly, these points correspond to where the curve  $a = 1/tb(1 - b)$  intersects the line  $a = 1 - b$ .

An upper bound for the contribution is given by

$$\begin{aligned}
 m_{\Omega}(\{t < U\}) &= \int_{b=0}^{(1-\sqrt{1-(8/t)})/2} \int_{a=1-b}^1 \int_{\alpha=1-b}^a \int_{s=0}^{1/ab} 1 \, ds \, d\alpha \, da \, db \\
 &+ \int_{b=(1-\sqrt{1-(8/t)})/2}^{b_2} \int_{a=1-b}^{1/(bt(1-b))} \int_{\alpha=1-b}^a \int_{s=0}^{1/ab} 1 \, ds \, d\alpha \, da \, db \\
 &+ \int_{b=b_1}^{(1+\sqrt{1-(8/t)})/2} \int_{a=1-b}^{1/(bt(1-b))} \int_{\alpha=1-b}^a \int_{s=0}^{1/ab} 1 \, ds \, d\alpha \, da \, db \\
 &+ \int_{b=(1+\sqrt{1-(8/t)})/2}^1 \int_{a=1-b}^1 \int_{\alpha=1-b}^a \int_{s=0}^{1/ab} 1 \, ds \, d\alpha \, da \, db.
 \end{aligned}$$

We find the decay of each term to be

$$\begin{aligned}
 \int_{b=0}^{(1-\sqrt{1-(8/t)})/2} \int_{a=1-b}^1 \int_{\alpha=1-b}^a \int_{s=0}^{1/ab} 1 \, ds \, d\alpha \, da \, db &\sim t^{-2}, \\
 \int_{b=(1-\sqrt{1-(8/t)})/2}^{b_2} \int_{a=1-b}^{1/(bt(1-b))} \int_{\alpha=1-b}^a \int_{s=0}^{1/ab} 1 \, ds \, d\alpha \, da \, db &\sim t^{-1}, \\
 \int_{b=b_1}^{(1+\sqrt{1-(8/t)})/2} \int_{a=1-b}^{1/(bt(1-b))} \int_{\alpha=1-b}^a \int_{s=0}^{1/ab} 1 \, ds \, d\alpha \, da \, db &\sim t^{-1}, \\
 \int_{b=(1+\sqrt{1-(8/t)})/2}^1 \int_{a=1-b}^1 \int_{\alpha=1-b}^a \int_{s=0}^{1/ab} 1 \, ds \, d\alpha \, da \, db &\sim t^{-1}.
 \end{aligned}$$

We now compute a lower bound for the contribution on  $\Omega_4$ . Using that the denominator of  $R$  is bounded above by 1 and that all the terms in the numerator are positive, we have that

$$R|_{\Omega_4} = \frac{j(a^{-1} - sb) + sa}{\alpha - a + jb} > \frac{sa}{1} =: L.$$

To compute the contribution of  $m_{\Omega}(\{L > t\})$ , we notice that

$$L > t \iff s > t/a.$$

This is a valid lower bound for  $s$  since  $t/a$  is always positive. The upper bounds on  $s$  yield the condition

$$t/a < (ab)^{-1} \iff b < t^{-1},$$

which yields the integral

$$m_{\Omega}(\{R > t\} \cap \Omega_4) > m_{\Omega}(\{L > t\}) = \int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=\alpha}^1 \int_{s=t/a}^{1/ab} 1 \, ds \, da \, d\alpha \, db.$$

Evaluating this integral and approximating the result by its Taylor series, we have that

$$\int_{b=0}^{1/t} \int_{\alpha=1-b}^1 \int_{a=\alpha}^1 \int_{s=t/a}^{1/ab} 1 \, ds \, da \, d\alpha \, db \sim t^{-2},$$

so that a lower bound on the contribution of  $\Omega_4$  is given by a quadratic function. □

As a corollary of the measure estimates from above, we obtain that the distribution function for gaps of an affine lattice are supported near 0.

*Proof of Theorem 4.4(2).* Note that the volume of  $\Omega_3$  is given by  $m_\Omega(\Omega_3) = (\pi^2/6) - 1$ . Notice that in the measure estimate of the tail of  $\Omega_3$  in case I, yields that

$$\begin{aligned} m_\Omega(\{R > t\} \cap \Omega_3) &\geq \int_{b=0}^1 \int_{\alpha=0}^{1-b} \int_{a=1-b}^1 \int_{s=0}^{(a^{-1}-t(b+\alpha))/b} 1 \, ds \, da \, d\alpha \, db \\ &= \frac{\pi^2}{6} - 1 - \frac{t}{3} \end{aligned}$$

whenever  $t < 1$ . We only have an inequality because there may be contribution from the other pieces  $\Omega_i$ . Hence,

$$\int_0^\varepsilon g(t) dt > m_\Omega(\{R < \varepsilon\} \cap \Omega_3) = \frac{\pi^2}{6} - 1 - m_\Omega(\{R > \varepsilon\} \cap \Omega_3) \geq \varepsilon/3.$$

This contribution is positive for any  $\varepsilon > 0$  and so we conclude that there is support at zero.  $\square$

#### REFERENCES

- [1] A. Arbieto, C. Matheus and C. G. Moreira. *The Remarkable Effectiveness of Ergodic Theory in Number Theory (Ensaio Matemáticos [Mathematical Surveys], 17)*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2009.
- [2] J. S. Athreya. Gap distributions and homogeneous dynamics. *Geometry, Topology, and Dynamics in Negative Curvature (London Mathematical Society Lecture Notes Series, 425)*. Cambridge University Press, Cambridge, 2016, pp. 1–31.
- [3] J. S. Athreya and J. Chaika. The distribution of gaps for saddle connection directions. *Geom. Funct. Anal.* **22**(6) (2012), 1491–1516.
- [4] J. S. Athreya, J. Chaika and S. Lelièvre. The gap distribution of slopes on the golden L. *Recent Trends in Ergodic Theory and Dynamical Systems (Contemporary Mathematics, 631)*. American Mathematical Society, Providence, RI, 2015, pp. 47–62.
- [5] J. S. Athreya and Y. Cheung. A Poincaré section for the horocycle flow on the space of lattices. *Int. Math. Res. Not. IMRN* **2014**(10) (2014), 2643–2690.
- [6] Y. Cheung. Hausdorff dimension of the set of nonergodic directions. *Ann. of Math. (2)* **158**(2) (2003), 661–678. With an appendix by M. Boshernitzan.
- [7] Y. Cheung, P. Hubert and H. Masur. Dichotomy for the Hausdorff dimension of the set of nonergodic directions. *Invent. Math.* **183**(2) (2011), 337–383.
- [8] S. G. Dani. Invariant measures and minimal sets of horospherical flows. *Invent. Math.* **64**(2) (1981), 357–385.
- [9] B. Dozier. Equidistribution of saddle connections on translation surfaces. *J. Mod. Dyn.* **14** (2019), 87–120.
- [10] N. D. Elkies and C. T. McMullen. Gaps in  $\sqrt{n} \bmod 1$  and ergodic theory. *Duke Math. J.* **123**(1) (2004), 95–139.
- [11] A. Eskin. Unipotent flows and applications. *Homogeneous Flows, Moduli Spaces and Arithmetic (Clay Mathematics Proceedings, 10)*. American Mathematical Society, Providence, RI, 2010, pp. 71–129.
- [12] A. Eskin, J. Marklof and D. W. Morris. Unipotent flows on the space of branched covers of Veech surfaces. *Ergod. Th. & Dynam. Sys.* **26**(1) (2006), 129–162.
- [13] A. Eskin and H. Masur. Asymptotic formulas on flat surfaces. *Ergod. Th. & Dynam. Sys.* **21**(2) (2001), 443–478.
- [14] H. Furstenberg. The unique ergodicity of the horocycle flow. *Lect. Notes Math.* **318** (1973), 95–115.
- [15] B. Heersink. Poincaré sections for the horocycle flow in covers of  $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$  and applications to Farey fraction statistics. *Monat. Math.* **179**(3) (2016), 389–420.

- [16] P. Hubert and T. A. Schmidt. An introduction to Veech surfaces. *Handbook of Dynamical Systems*. Vol. 1B. Elsevier B. V., Amsterdam, 2006, pp. 501–526.
- [17] J. Marklof and A. Strömbergsson. The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems. *Ann. of Math. (2)* **172**(3) (2010), 1949–2033.
- [18] H. Masur. Closed trajectories for quadratic differentials with an application to billiards. *Duke Math. J.* **53**(2) (1986), 307–314.
- [19] H. Masur. Ergodic theory of translation surfaces. *Handbook of Dynamical Systems*. Vol. 1B. Elsevier B. V., Amsterdam, 2006, pp. 527–547.
- [20] H. Masur and S. Tabachnikov. Rational billiards and flat structures. *Handbook of Dynamical Systems*. Vol. 1A. North-Holland, Amsterdam, 2002, pp. 1015–1089.
- [21] C. T. McMullen. Dynamics of  $SL_2(\mathbb{R})$  over moduli space in genus two. *Ann. of Math. (2)* **165**(2) (2007), 397–456.
- [22] M. Ratner. On measure rigidity of unipotent subgroups of semisimple groups. *Acta Math.* **165**(3–4) (1990), 229–309.
- [23] M. Ratner. Strict measure rigidity for unipotent subgroups of solvable groups. *Invent. Math.* **101**(2) (1990), 449–482.
- [24] M. Ratner. On Raghunathan’s measure conjecture. *Ann. of Math. (2)* **134**(3) (1991), 545–607.
- [25] M. Schmoll. On the asymptotic quadratic growth rate of saddle connections and periodic orbits on marked flat tori. *Geom. Funct. Anal.* **12**(3) (2002), 622–649.
- [26] M. Schmoll. Spaces of elliptic differentials. *Algebraic and Topological Dynamics (Contemporary Mathematics, 385)*. American Mathematical Society, Providence, RI, 2005, pp. 303–320.
- [27] A. Strömbergsson. An effective Ratner equidistribution result for  $ASL(2, R)$ . *Duke Math. J.* **164**(5) (2015), 843–902.
- [28] D. Taha. On cross sections to the geodesic and horocycle flows on quotients of  $SL(2, R)$  by Hecke triangle groups  $G_q$ . *Preprint*, 2020.
- [29] C. Uyanik and G. Work. The distribution of gaps for saddle connections on the octagon. *Int. Math. Res. Not. IMRN* **2016**(18) (2016), 5569–5602.
- [30] Y. B. Vorobets. Plane structures and billiards in rational polygons: The Veech alternative. *Uspekhi Mat. Nauk* **51**(311) (1996), 3–42.
- [31] G. Work. A transversal for horocycle flow on  $H(\alpha)$ . *Geom. Dedicata* **205** (2020), 21–49.
- [32] A. Zorich. Flat surfaces. *Frontiers in Number Theory, Physics, and Geometry*. Vol. I. Springer, Berlin, 2006, pp. 437–583.