J. Austral. Math. Soc. **19** (Series A) (1975), 292–296.

HEREDITARILY UNIVERSAL SETS

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(Received 16 April 1973)

Communicated by J. N. Crossley

Abstract

An immune set is found such that the recursive equivalence type of its infinite subsets are universal in a very strong sense.

1. Introduction

Let ω be the non-negative integers and for $\xi \subseteq \omega$ let $\langle \xi \rangle$ be the recursive equivalence type of ξ. Λ is of course the isols.

THEOREM 1. *There is an immune $\eta \subseteq \omega$ such that for every infinite $\xi \subseteq \eta$ and $R \subseteq \omega \times \omega$ the graph of a function r, if $(\exists z \in \Lambda)(\langle \xi \rangle, z) \in R_\Lambda$ then r is eventually recursive combinatorial.*

THEOREM 2. *There is an immune $\tau \subseteq \omega$ such that for every infinite $\xi \subseteq \tau$ and $R \subseteq \omega \times \omega$ the graph of a function r, if $(\exists z \in \Lambda)(\langle \xi \rangle, z) \in R_\Lambda$ then r is eventually recursive increasing.*

THEOREM 3. *η may be taken to be Δ_2^1 and τ may be taken to be Π_1^0 (and retraceable).*

Theorem 3 is a rather curious result. Theorems 1 and 2 look very much alike, the requirements on η appearing only slightly stronger than those on τ. We have no idea as to what degree the of η might be, our τ on the other hand is of degree $0'$. As an open problem we ask if better upper bounds or perhaps some lower bounds could be found for η and τ?

2. Details

Use lower case Greek letters for subsets of ω and let $\varnothing$ be the empty set. Define $(\alpha, \beta)^\omega = \{\alpha \cup \xi \mid \xi \subseteq \beta \wedge \xi \text{ is infinite}\}$, $(\alpha, \beta)^{<\omega} = \{\alpha \cup \xi \mid \xi \subseteq \beta \wedge \xi$ is

* Supported by The Institute for Advanced Study

finite}, $P = (\varnothing, \omega)^\omega$, $Q = (\varnothing, \omega)^{<\omega}$. A set $S \subseteq 2^\omega = P \cup Q$ is *completely Ramsey* if for every $\alpha \in Q$ and $\beta \in P$ there is a $\xi \in (\varnothing, \beta)^\omega$ such that $(\alpha, \xi)^\omega \subseteq S$ or $(\alpha, \xi)^\omega \subseteq 2^\omega - S$. The Galvin-Prikry theorem asserts that every Borel set is completely Ramsey (cf. Galvin and Prikry (1973)). We refer to this result as GP.

Let $R \subseteq \omega \times \omega$ be the graph of a function which is not eventually recursive combinatorial and let F be a recursive R-frame. We use standard frame notation from Nerode (1961). If $\gamma \in F^*$ and $i < 2$ let $C_F^i(\gamma)$ be the ith coordinate of $C_F(\gamma)$. *dom* and *rng* denote domain and range respectively. If $(\gamma, \varnothing) \in F^*$ put $\phi(\gamma) = C_F^0(\gamma, \varnothing)$ and then define

$$B(F) = \{\xi \in P \mid (\exists \gamma \in (\varnothing, \xi)^{<\omega})(\forall \delta \in (\gamma, \xi)^{<\omega})\phi(\delta) \subseteq \delta\}.$$

Note that we always have $\delta \subseteq \phi(\delta)$ provided $\delta \in \operatorname{dom}(\phi)$. Let $\alpha \in Q$ and $\beta \in P$. Since $B(F)$ is clearly Borel, GP gives us an $\eta \in (\varnothing, \beta)^\omega$ such that $(\alpha, \eta)^\omega \subseteq B(F)$ or $(\alpha, \eta)^\omega \subseteq 2^\omega - B(F)$. That the latter always holds is given by

LEMMA 1. $(\alpha, \eta)^\omega \subseteq 2^\omega - B(F)$.

PROOF. Assume $(\alpha, \eta)^\omega \subseteq B(F)$. We strive for a contradiction. Now $\alpha \cup \eta \in B(F)$ and hence there is a $\gamma \in (\varnothing, \alpha \cup \eta)^{<\omega}$ such that $\phi(\delta) = \delta$ for all $\delta \in (\gamma, \alpha \cup \eta)^{<\omega}$. Without loss of generality we may assume that $\alpha \subseteq \gamma$ so that $\phi(\delta) = \delta$ for all $\delta \in (\gamma, \eta)^{<\omega}$. By shrinking η slightly we may also assume that $\gamma \cap \eta = \varnothing$. For the moment let δ range over $(\gamma, \eta)^{<\omega}$ and define $\psi(\delta) = C_F^1(\delta, \varnothing)$. Then $(\delta, \psi(\delta)) \in F$. Let $|\delta|$ be the cardinality of δ. Since R is single valued $|\delta| = |\delta'|$ implies $|\psi(\delta)| = |\psi(\delta')|$. Also

$$(\delta \cap \delta', \varnothing) \leqq (\delta, \psi(\delta)) \wedge (\delta', \psi(\delta')) = (\delta \cap \delta', \psi(\delta) \cap \psi(\delta')) \in F$$

and thus $\psi(\delta \cap \delta') \subseteq \psi(\delta) \cap \psi(\delta')$. Since $(\delta \cap \delta', \psi(\delta \cap \delta')) \in F$ as well we have $\psi(\delta \cap \delta') = \psi(\delta) \cap \psi(\delta')$. Let p be a one-one function mapping ω onto η. Define θ on Q by $\theta(\lambda) = \psi(\gamma \cup p(\lambda))$. Then $|\lambda| = |\lambda'|$ implies $|\theta(\lambda)| = |\theta(\lambda')|$ and $\theta(\lambda \cap \lambda') = \theta(\lambda) \cap \theta(\lambda')$. These properties are inherited from the corresponding ones for ψ. θ is therefore a combinatorial operator inducing a combinatorial function $r: \omega \to \omega$ such that $(x + |\gamma|, r(x)) \in R$ for $x \in \omega$. Thus R is the graph of an eventually combinatorial function. Let $B = \{(\lambda, \mu) \in Q \times Q \mid \lambda \cap \gamma = \varnothing \wedge (\gamma \cup \lambda, \mu) \in F\}$ and $S = \{(x, y) \in \omega \times \omega \mid (\exists (\lambda, \mu) \in B) x = |\lambda| \wedge y = |\mu|\}$. B and hence S are r.e., the latter being the graph of r. Thus R is the graph of an eventually recursive combinatorial function. Since R was initially specified as not being such a relation, we have the desired contradiction.

Let $R \subseteq \omega \times \omega$ be the graph of a function and let F be a recursive R-frame. ϕ is as above and define

$$D(F) = \{\xi \in P \mid (\forall \gamma \in (\varnothing, \xi)^{<\omega})\, \phi(\gamma) \subseteq \xi\}.$$

Let $\alpha \in Q$ and $\beta \in P$. Since $D(F)$ is clearly Borel, GP gives us an $\eta \in (\varnothing, \beta)^\omega$ such that $(\alpha,\eta)^\omega \subseteq D(F)$ or $(\alpha,\eta)^\omega \subseteq 2^\omega - D(F)$. We relate $D(F)$ to the previous lemma by

LEMMA 2. *If* $(\alpha,\eta)^\omega \subseteq 2^\omega - B(F)$ *then* $(\alpha,\eta)^\omega \nsubseteq D(F)$.

PROOF. Assume $(\alpha,\eta)^\omega \subseteq 2^\omega - B(F)$, $\xi \in (\alpha,\eta)^\omega$ and $(\alpha\eta,)^\omega \subseteq D(F)$. Since $\xi \in 2^\omega - B(F)$ there is a $\delta \in (\alpha,\xi)^{<\omega}$ such that $\delta \notin \mathrm{dom}(\phi)$ or $\phi(\delta) \nsubseteq \delta$. In the former case $\xi \notin D(F)$ and in the latter $\delta \cup (\xi - \phi(\delta)) \in (\alpha,\eta)^\omega - D(F)$, both of which contradict $(\alpha,\eta)^\omega \subseteq D(F)$.

Let $E(F) = \{\xi \in P \mid (\exists\zeta)(\xi,\zeta)$ is attainable from $F\}$.

LEMMA 3. $2^\omega - D(F) \subseteq 2^\omega - E(F)$.

PROOF. An immediate consequence of definitions.

Let $S_n \subseteq 2^\omega$ be a sequence such that for each $n \in \omega$, $\alpha \in Q$ and $\beta \in P$ there is an $\eta \in (\varnothing,\beta)^\omega$ such that $(\alpha,\eta)^\omega \subseteq 2^\omega - S_n$. That we can find a uniform η is given by

LEMMA 4. *For each* $\alpha \in Q$ *and* $\beta \in P$ *there is an* $\eta \in (\varnothing,\beta)^\omega$ *such that* $(\alpha,\eta)^\omega \subseteq 2^\omega - S_n$ *for every* $n \in \omega$.

PROOF. Shrink β slightly so that every element of α is less than every element of β. Let $\alpha_0 = \alpha$ and choose $\eta_0 \in (\varnothing,\beta)^\omega$ so that $(\alpha_0,\eta_0)^\omega \subseteq 2^\omega - S_0$. Now suppose we have defined α_n and η_n such that every element of α_n is less than every element of η_n. Let a_n be the least element of η_n. Set $\alpha_{n+1} = \alpha_n \cup \{a_n\}$ and choose $\eta_{n+1} \in (\varnothing,\eta_n - \{a_n\})^\omega$ so that for each $\alpha_0 \subseteq \gamma \subseteq \alpha_{n+1}$ we have $(\gamma,\eta_{n+1})^\omega \subseteq 2^\omega - S_{n+1}$. Then $\eta = \cup\alpha_n$ has the required property.

PROOF OF THEOREM 1. Let F_n be an enumeration of all recursive R-frames where $R \subseteq \omega\times\omega$ is the graph of a which is not eventually recursive combinatorial. Start with an immune set β and use GP and lemma 1 to get an $\eta \in (\varnothing,\beta)^\omega$ such that $(\varnothing,\eta)^\omega \subseteq 2^\omega - B(F_n)$ and either $(\varnothing,\eta)^\omega \subseteq D(F_n)$ or $(\varnothing,\eta)^\omega \subseteq 2^\omega D(F_n)$. By lemma 2, $(\varnothing,\eta)^\omega \subseteq 2^\omega - D(F_n)$ and by lemma 3, $(\varnothing,\eta)^\omega \subseteq 2^\omega - E(F_n)$. Lemma 4 gives an η which uniformly works for all $n \in \omega$. Thus if R is as above and $\xi \in (\varnothing,\eta)^\omega$ then for no recursive R-frame F and ζ [an (ξ,ζ) be attainable from F. This is the contrapositive of our theorem.

Let j be the usual pairing function with k, l as its first, second inverse. Order the elements of Q according to their canonical indices so that we can effectively speak of a first, second... element of Q. Let $q_n(\alpha)$ be a partial recursive function of $n \in \omega$ and $\alpha \in Q$ which with index n enumerates partial recursive functions mapping subsets of Q into ω. Put $q_n^s(\alpha) = y$ if $q_n(\alpha) = y$ in s or fewer computation stages, otherwise we say that $q_n^s(\alpha)$ is undefined. Denote the largest element in $\alpha \in Q$ by $\max(\alpha)$. A retraceable function, t, is called *hereditarily* 1-*meager* if for every $e \in \omega$ there is an $m \in \omega$ such that for all $n > m$ and $\alpha \subseteq \{t(i) \mid i < n\}$ $q_e(\alpha)$ is undefined or $q_e(\alpha) < t(n)$. The following lemma is closely related to our proof (cf. Ellentuck (1973)) of McLaughlin's theorem on the existence of hereditarily retraceable isols (cf. McLaughlin (1967)).

LEMMA 5. *There exists a hereditarily* 1-*meager function with cosimple range.*

PROOF. Our proof is a stage by stage construction of functions $t^s(n)$ whose limit $t(n) = \lim_s t^s(n)$ is hereditarily 1-meager.

Stage $s = 0$: Let $t^0(0) = 1$ and then go on to stage 1.

Stage $s+1$: As inductive hypothesis assume at the end of stage s that we have defined $t^s(n)$ for $n \leqq s$, that $t^s(0) = 1$, and that $kt^s(n+1) = t^s(n)$ for $n < s$. Search for the least $n \leqq s$, and for it the least $m < n$, and for them the least $\alpha \subseteq \{t^s(i) \mid i < n\}$ such that

$$q^s_m(\alpha) \text{ is defined and } t^s(n) \leqq q^s_m(\alpha).$$

If there is no such (n, m, α) go to case A below, otherwise go to case B.

CASE A. Let $t^{s+1}(x) = t^s(x)$ for $x \leqq s$, $t^{s+1}(s+1) = j(t^s(s), 0)$.

CASE B. Find the least y such that

$$\max\{q^s_m(\alpha), t^s(s)\} < j(t^s(n-1), y)$$

(note that $n > 0$) and let $t^{s+1}(x) = t^s(x)$ for $x < n$, $t^{s+1}(n) = j(t^s(n-1), y)$, and $t^{s+1}(x+1) = j(t^{s+1}(x), 0)$ for $n \leqq x \leqq s$. This completes stage $n+1$ of the construction. Now go on to stage $s+2$. It is easy to see that our inductive hypothesis is maintained as we pass through stages. $t(n) = \lim_s t^s(n)$ exists for every n because $t^s(0) = 1$ for every s, and once $t^s(n-1)$ has reached its final value $t^s(n)$ changes its value at most $n \cdot 2^n$ times. $t(0) = 1$ and $kt(n+1) = t(n)$ by our inductive hypothesis and t is one-one since $t(0) \neq 0$. Thus t is retraceable, and the construction in case B insures that $x \notin \mathrm{rng}(t)$ if and only if $(\exists x > s) x \notin \mathrm{rng}(t^s)$. This makes $\mathrm{rng}(t)$ co-r.e. The immunity of $\mathrm{rng}(t)$ follows from the meagerness of t. We demonstrate the latter. Let $m < n$ and choose a stage r so large that $t^r(i)$ for $i \leqq n$ have reached their final values. There can be no $\alpha \subseteq \{t(i) \mid i < n\}$ such that $t(n) \leqq q_m(\alpha)$, otherwise $t^r(n)$ would subsequently change its value.

PROOF OF THEOREM 2. Let $\zeta = \mathrm{rng}(t)$, $\sigma \in (\emptyset, \tau)^\omega$ and s_n a strictly increasing enumeration of σ. Let $R \subseteq \omega \times \omega$ be the graph of a function r for which $(\exists z \in \Lambda)(\langle\sigma\rangle, z) \in R_\Lambda$. Then there is an isolated ζ and a recursive R-frame F such that (σ, ζ) is attainable from F. If $(\alpha, \emptyset) \in F^*$ put $\phi(\alpha) = \max C^0_F(\alpha, \emptyset)$ and let $A = \{\alpha \in Q \mid C^0_F(\alpha, \emptyset) = \alpha\}$. By applying Lemma 5 to ϕ we see that there is an $m \in \omega$ such that $\{s_i \mid i < n\} \in A$ for any $n > m$. Let $\psi(\alpha) = C^1_F(\alpha, \emptyset)$ for $\alpha \in A$. A is a r.e. family of finite sets, ψ is a partial recursive function taking finite sets into finite sets and $(\alpha, \psi(\alpha)) \in F$ for every $\alpha \in A$. If $\alpha, \alpha' \in A$ and $\alpha \subseteq \alpha'$ then $(\alpha, \emptyset) \leqq (\alpha', \psi(\alpha')$ and hence $\psi(\alpha) \subseteq \psi(\alpha')$. Let $S = \{(a, b) \mid (\exists \alpha \in A) a =$

$|\alpha| \wedge b = |\psi(\alpha)|\}$. S is r.e. subset of R and the graph of a partial function whose domain contains all $n > m$. It is also the graph of an eventually increasing function by the monotonicity of ψ. Thus r is eventually recursive increasing.

PROOF OF THEOREM 3. We have already dealt with τ. For η notice that $(\emptyset, \eta)^\omega \subseteq 2^\omega - B(F)$ is a Π_1^1 predicate. Since '*R is the graph of a function which is not eventually recursive combinatorial*' is an arithmetical predicate, and there is an arithmetical enumeration of all recursive frames, we see that the condition required of η in the proof of Theorem 1 is Π_1^1. By Addison's modification of the Kondo theorem (cf. Rogers (1967)) η may be chosen as Δ_2^1.

We had originally hoped to get η recursive in the ordinal notations. We have not been able to do so; however, such an attempt seems promising.

References

F. Galvin and K. Prikry 1973), 'Borel sets and Ramsey's theorem', J. Symbolic Logic **38**, 193–198.

E. Ellentuck (1973), 'On the degrees of universal regressive isols,' *Math. Scand.* **32**, 145–164.

T. McLaughlin (1967), 'Hereditarily regtaceable isols', *Bull. Amer. Math. Soc.* **73**, 113–115.

A. Nerode (1961), 'Extensions to isols', *Ann. of Math.* **73**, 362–403.

H. Rogers Jr. (1967), *Theory of Recursive Finctions and Effective Computability*, (*McGraw-Hill*, New York, 1967).

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