

## THE SYMPLECTICITY OF THE MAGNUS REPRESENTATION FOR HOMOLOGY COBORDISMS OF SURFACES

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The Magnus matrix is an algebraic invariant assigned to each homology cobordism of a surface. We show that this matrix satisfies an equality which can be regarded as a non-commutative symplectic relation.

### 1. INTRODUCTION

Let  $\Sigma_{g,1}$  be a compact connected oriented surface of genus  $g \geq 1$  with one boundary component. We consider a triplet  $(M, i_+, i_-)$  consisting of a homology cobordism  $M$  of  $\Sigma_{g,1}$  and two markings  $i_+, i_-$  of  $\partial M$ . More precisely, the compact oriented 3-manifold  $M$  and the two embeddings  $i_+, i_- : \Sigma_{g,1} \rightarrow \partial M$  satisfy

- (1)  $i_+$  is orientation-preserving and  $i_-$  is orientation-reversing,
- (2)  $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$  and  $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$ ,
- (3)  $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$ ,
- (4)  $i_+, i_- : H_*(\Sigma_{g,1}) \rightarrow H_*(M)$  are isomorphisms.

We call such an object a *homology cylinder (over  $\Sigma_{g,1}$ )*, which has its origin in Habiro [4], Garoufalidis–Levine [3] and Levine [7]. Two homology cylinders are said to be isomorphic if there exists an orientation-preserving diffeomorphism between the underlying 3-manifolds which is compatible with the markings. The set  $\mathcal{C}_{g,1}$  of all isomorphism classes of homology cylinders has a monoid structure defined by

$$M \cdot N = (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-)$$

for two homology cylinders  $M = (M, i_+, i_-)$ ,  $N = (N, j_+, j_-)$ . The unit of  $\mathcal{C}_{g,1}$  is  $(\Sigma_{g,1} \times I, \text{id} \times 1, \text{id} \times 0)$ , where collars of  $i_+(\Sigma_{g,1})$  and  $i_-(\Sigma_{g,1})$  are stretched halfway along  $(\partial\Sigma_{g,1}) \times I$ . Typical examples of homology cylinders come from the mapping class group  $\mathcal{M}_{g,1}$  of  $\Sigma_{g,1}$ ; that is the group of isotopy classes of diffeomorphisms of  $\Sigma_{g,1}$  which restrict to the identity on  $\partial\Sigma_{g,1}$ . For each mapping class  $\varphi \in \mathcal{M}_{g,1}$ , we have a homology

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cylinder  $(\Sigma_{g,1} \times I, \text{id} \times 1, \varphi \times 0)$ . In fact, this construction gives an injective monoid homomorphism  $\mathcal{M}_{g,1} \rightarrow \mathcal{C}_{g,1}$ . We can also construct homology cylinders by using homology 3-spheres and pure string links (see [3, 7]). From these facts, we can expect that  $\mathcal{C}_{g,1}$  will play an important role in the classification of 3-manifolds especially by using its monoid structure.

To study the structure of  $\mathcal{C}_{g,1}$ , we defined its Magnus representation

$$r_k : \mathcal{C}_{g,1} \rightarrow GL(2g, \mathcal{K}_{N_k}) \quad (k = 2, 3, \dots)$$

in [13, 14], where  $\mathcal{K}_{N_k}$  is a certain skew field. This representation extends the Magnus representation for  $\mathcal{M}_{g,1}$  defined by Morita [8], as the Gassner representation for pure string links due to Le Dimet [6] and Kirk, Livingston and Wang [5] does that for the pure braid group. See Birman’s book [1] for generalities of the ordinary Magnus representation.

In this paper, we shall show that the Magnus representation of  $\mathcal{C}_{g,1}$  satisfies a certain equality which can be regarded as a non-commutative symplectic relation (Theorem 2.4). This generalises not only the result of Morita [8] and Suzuki [16] that the same equality holds for the Magnus representation of  $\mathcal{M}_{g,1}$ , but also Kirk, Livingston and Wang’s result [5] that the (reduced) Gassner representation for pure string links satisfies some unitary relation. In fact, our proof derives an explicit equality describing the unitarity of the Gassner representation, while their result is given in an implicit form.

Note that the statement of Theorem 2.4 has been already announced in [14], where we observe that the condition obtained from our theorem for a matrix to be written as the Magnus matrix of a homology cylinder is a strong one by considering a relationship to the theory of higher-order Alexander invariants.

## 2. THE MAGNUS REPRESENTATION FOR HOMOLOGY CYLINDERS

We take a base point  $p$  on  $\partial\Sigma_{g,1}$ , and take  $2g$  loops  $\gamma_1, \dots, \gamma_{2g}$  of  $\Sigma_{g,1}$  as shown in Figure 1. We consider them to be an embedded bouquet  $R_{2g}$  of  $2g$  circles tied at  $p \in \partial\Sigma_{g,1}$ . Then  $R_{2g}$  and the boundary loop  $\zeta$  of  $\Sigma_{g,1}$  together with one 2-cell make up a standard cell decomposition of  $\Sigma_{g,1}$ . The fundamental group  $\pi_1\Sigma_{g,1}$  of  $\Sigma_{g,1}$  is the free group  $F_{2g}$  of rank  $2g$  generated by  $\gamma_1, \dots, \gamma_{2g}$ , where  $\zeta = \prod_{i=1}^g [\gamma_i, \gamma_{g+i}]$ .

For a group  $G$ , we denote by  $N_k(G) := G/(\Gamma^k G)$  the  $k$ -th nilpotent quotient of  $G$ , where we define  $\Gamma^1 G := G$  and  $\Gamma^l G := [\Gamma^{l-1} G, G]$  for  $l \geq 2$ . For simplicity, we write  $N_k(X)$  for  $N_k(\pi_1 X)$  where  $X$  is a connected topological space, and write  $N_k$  for  $N_k(F_{2g}) = N_k(\Sigma_{g,1})$ .

Let  $(M, i_+, i_-)$  be a homology cylinder. By definition,  $i_+, i_- : \pi_1\Sigma_{g,1} \rightarrow \pi_1 M$  are both 2-connected, namely they induce isomorphisms on  $H_1$  and epimorphisms on  $H_2$ . Then, by Stallings’ theorem [15],  $i_+, i_- : N_k \xrightarrow{\cong} N_k(M)$  are isomorphisms for each  $k \geq 2$ .

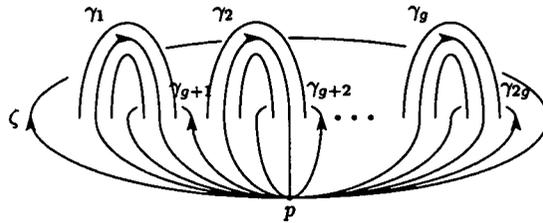


Figure 1: A standard cell decomposition of  $\Sigma_{g,1}$

Using them, we obtain a monoid homomorphism

$$\sigma_k : C_{g,1} \rightarrow \text{Aut } N_k \quad ((M, i_+, i_-) \mapsto (i_+)^{-1} \circ i_-).$$

2.1. DEFINITION OF THE MAGNUS REPRESENTATION FOR HOMOLOGY CYLINDERS.

We first summarise our notation. For a matrix  $A$  with entries in a ring  $R$ , and a ring homomorphism  $\varphi : R \rightarrow R'$ , we denote by  ${}^\varphi A$  the matrix obtained from  $A$  by applying  $\varphi$  to each entry.  $A^T$  denotes the transpose of  $A$ . When  $R = \mathbb{Z}G$  for a group  $G$  or its right field of fractions (if it exists), we denote by  $\bar{A}$  the matrix obtained from  $A$  by applying the involution induced from  $(x \mapsto x^{-1}, x \in G)$  to each entry. For a module  $M$ , we write  $M^n$  for the module of column vectors with  $n$  entries.

For a finite cell complex  $X$ , we denote by  $\tilde{X}$  its universal covering. We take a base point  $p$  of  $X$  and a lift  $\tilde{p}$  of  $p$  as a base point of  $\tilde{X}$ .  $\pi := \pi_1(X, p)$  acts on  $\tilde{X}$  from the right through its deck transformation group. Namely, the lift of  $\gamma \in \pi$  starting from  $\tilde{p}$  reaches  $\tilde{p}\gamma^{-1}$ . We regard the  $\mathbb{Z}\pi$ -cellular chain complex  $C_*(\tilde{X})$  of  $\tilde{X}$  as a collection of free right  $\mathbb{Z}\pi$ -modules consisting of column vectors together with differentials given by left multiplications of matrices. For each  $\mathbb{Z}\pi$ -bimodule  $A$ , the twisted chain complex  $C_*(X; A)$  is given by the tensor product of the right  $\mathbb{Z}\pi$ -module  $C_*(\tilde{X})$  and the left  $\mathbb{Z}\pi$ -module  $A$ , so that  $C_*(X; A)$  and  $H_*(X; A)$  are right  $\mathbb{Z}\pi$ -modules.

Hereafter, we fix an integer  $k \geq 2$ . The following construction is based on Kirk-Livingston-Wang's work of the Gassner representation for string links in [5].

Let  $(M, i_+, i_-)$  be a homology cylinder.  $p \in \partial\Sigma_{g,1}$  is a common base point of  $\Sigma_{g,1}$  and  $M$ . By Stallings' theorem,  $N_k$  and  $N_k(M)$  are isomorphic. Since  $N_k$  is a finitely generated torsion-free nilpotent group for each  $k \geq 2$ , we can embed  $\mathbb{Z}N_k$  into the right field of fractions

$$\mathcal{K}_{N_k} := \mathbb{Z}N_k(\mathbb{Z}N_k - \{0\})^{-1}$$

(see [2, 11] for details). Similarly, we have

$$\mathbb{Z}N_k(M) \hookrightarrow \mathcal{K}_{N_k(M)} := \mathbb{Z}N_k(M)(\mathbb{Z}N_k(M) - \{0\})^{-1}.$$

By a standard argument (see for instance [5, Proposition 2.1], [13, Lemma 5.11]),

we can show that

$$i_+, i_- : H_*(\Sigma_{g,1}, p; i_{\pm}^* \mathcal{K}_{N_k(M)}) \rightarrow H_*(M, p; \mathcal{K}_{N_k(M)})$$

are isomorphisms as right  $\mathcal{K}_{N_k(M)}$ -vector spaces. Since  $R_{2g} \subset \Sigma_{g,1}$  is a deformation retract, we have

$$H_1(\Sigma_{g,1}, p; i_{\pm}^* \mathcal{K}_{N_k(M)}) \cong H_1(R_{2g}, p; i_{\pm}^* \mathcal{K}_{N_k(M)}) = C_1(\widetilde{R}_{2g}) \bigotimes_{\mathbb{Z}\pi_1 R_{2g}} i_{\pm}^* \mathcal{K}_{N_k(M)} \cong \mathcal{K}_{N_k(M)}^{2g}$$

with a basis

$$\{\widetilde{\gamma}_1 \otimes 1, \dots, \widetilde{\gamma}_{2g} \otimes 1\} \subset C_1(\widetilde{R}_{2g}) \bigotimes_{\mathbb{Z}\pi_1 R_{2g}} i_{\pm}^* \mathcal{K}_{N_k(M)}$$

as a right  $\mathcal{K}_{N_k(M)}$ -vector space. Here we fix a lift  $\widetilde{p}$  of  $p$  as a base point of  $\widetilde{R}_{2g}$ , and denote by  $\widetilde{\gamma}_i$  the lift of the oriented edge  $\gamma_i$  starting from  $\widetilde{p}$  and ending at  $\widetilde{p}\gamma_i^{-1}$ .

**DEFINITION 2.1:** For each  $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ , we write  $r'_k(M) \in GL(2g, \mathcal{K}_{N_k(M)})$  for the representation matrix of the right  $\mathcal{K}_{N_k(M)}$ -isomorphism

$$\begin{aligned} \mathcal{K}_{N_k(M)}^{2g} \cong H_1(\Sigma_{g,1}, p; i_-^* \mathcal{K}_{N_k(M)}) &\xrightarrow[i_-]{\cong} H_1(M, p; \mathcal{K}_{N_k(M)}) \\ &\xrightarrow[i_+^{-1}]{\cong} H_1(\Sigma_{g,1}, p; i_+^* \mathcal{K}_{N_k(M)}) \cong \mathcal{K}_{N_k(M)}^{2g} \end{aligned}$$

(2) The Magnus representation for  $\mathcal{C}_{g,1}$  is the map  $r_k : \mathcal{C}_{g,1} \rightarrow GL(2g, \mathcal{K}_{N_k})$  which assigns to  $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$  the matrix  $i_+^{-1} r'_k(M)$ .

**EXAMPLE 2.2.** For  $\varphi \in \mathcal{M}_{g,1} \hookrightarrow \text{Aut } F_{2g}$ , we can easily check that

$$r_k((\Sigma_{g,1} \times I, \text{id} \times 1, \varphi \times 0)) = \left( \frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)_{i,j}^{\rho_k},$$

where  $\rho_k : \mathbb{Z}F_{2g} \rightarrow \mathbb{Z}N_k \subset \mathcal{K}_{N_k}$  is the natural homomorphism and  $\partial/\partial\gamma_i$  are free differentials. From this, we see that  $r_k$  extends the Magnus representation for  $\mathcal{M}_{g,1}$  (see [8, 16]).

While we call  $r_k$  the Magnus “representation”, it is actually a crossed homomorphism. Namely, we have that following.

**THEOREM 2.3.** ([14]) For  $M_1, M_2 \in \mathcal{C}_{g,1}$ , we have

$$r_k(M_1 \cdot M_2) = r_k(M_1) \cdot \sigma_k(M_1) r_k(M_2).$$

**2.2. MAIN THEOREM.** The following is the main theorem of this paper, which implies the symplecticity (in a twisted sense) of the Magnus representation for  $\mathcal{C}_{g,1}$ .



3. PROOF OF THEOREM 2.4

Let  $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ . The proof of Theorem 2.4 is divided into the following three steps.

STEP 1. Triangulate  $\Sigma_{g,1}$  by taking a refinement of the cell decomposition in Figure 1. We decompose  $\partial\Sigma_{g,1}$  into two segments  $A$  and  $B$  such that  $A \cup B = \partial\Sigma_{g,1}$ ,  $\partial A = \partial B = A \cap B = \{p, q\}$  as in Figure 2.

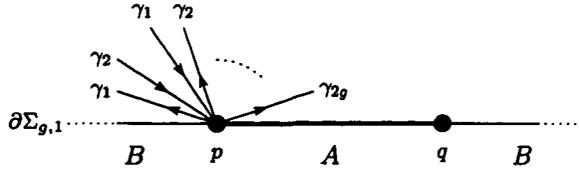


Figure 2: Decomposing  $\partial\Sigma_{g,1}$  into  $A$  and  $B$

Then we have isomorphisms

$$\begin{aligned}
 H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) &\cong H_1(\Sigma_{g,1}, A; \mathcal{K}_{N_k}), \\
 H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) &\cong H_1(\Sigma_{g,1}, A; \mathcal{K}_{N_k}) \cong H_1(\Sigma_{g,1}, q; \mathcal{K}_{N_k}) \cong H_1(\Sigma_{g,1}, B; \mathcal{K}_{N_k}).
 \end{aligned}$$

Using them, we obtain a pairing

$$\begin{aligned}
 \langle \cdot, \cdot \rangle_{\Sigma_{g,1}} : H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) \otimes H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) &\xrightarrow{\cong} H_1(\Sigma_{g,1}, A; \mathcal{K}_{N_k}) \otimes H_1(\Sigma_{g,1}, B; \mathcal{K}_{N_k}) \\
 &\rightarrow \mathcal{K}_{N_k}
 \end{aligned}$$

where the second map is the intersection pairing given, for instance, in [12, Definition 4.66]. More precisely, we need to generalise the pairing in the book to the case of pairs of manifolds and enlarge its coefficients from  $\mathbb{Z}N_k$  to  $\mathcal{K}_{N_k}$ . In our notation, the formula for the corresponding chain-level intersection

$$\langle \cdot, \cdot \rangle_{\Sigma_{g,1}} : (C_1(\widetilde{\Sigma_{g,1}}, \overline{A}) \otimes_{\mathbb{Z}\pi_1\Sigma_{g,1}} \mathcal{K}_{N_k}) \otimes (D_1(\widetilde{\Sigma_{g,1}}, \overline{B}) \otimes_{\mathbb{Z}\pi_1\Sigma_{g,1}} \mathcal{K}_{N_k}) \rightarrow \mathcal{K}_{N_k}$$

of the simplicial 1-chains

$$C_1(\Sigma_{g,1}, A; \mathcal{K}_{N_k}) = C_1(\widetilde{\Sigma_{g,1}}, \overline{A}) \otimes_{\mathbb{Z}\pi_1\Sigma_{g,1}} \mathcal{K}_{N_k}$$

and the dual cellular 1-chains

$$D_1(\Sigma_{g,1}, B; \mathcal{K}_{N_k}) = D_1(\widetilde{\Sigma_{g,1}}, \overline{B}) \otimes_{\mathbb{Z}\pi_1\Sigma_{g,1}} \mathcal{K}_{N_k}$$

(see [9, Chapter 8]), where  $\bar{A}$  and  $\bar{B}$  denote the inverse images of  $A$  and  $B$  in  $\widetilde{\Sigma}_{g,1}$ , is given as follows: For  $u \in C_1(\widetilde{\Sigma}_{g,1}, \bar{A})$ ,  $v \in D_1(\widetilde{\Sigma}_{g,1}, \bar{B})$  and  $r, s \in \mathcal{K}_{N_k}$ ,

$$\langle u \otimes r, v \otimes s \rangle_{\Sigma_{g,1}} = \sum_{\gamma \in \pi_1 \Sigma_{g,1}} (u\gamma, v)\bar{r}\gamma s,$$

where  $(u\gamma, v) \in \mathbb{Z}$  is the usual intersection number of  $u\gamma$  and  $v$  on  $\widetilde{\Sigma}_{g,1}$ .

By definition,  $\langle uf, v \rangle_{\Sigma_{g,1}} = \bar{f}\langle u, v \rangle_{\Sigma_{g,1}}$  and  $\langle u, vf \rangle_{\Sigma_{g,1}} = \langle u, v \rangle_{\Sigma_{g,1}} f$  hold for any  $f \in \mathcal{K}_{N_k}$  and  $u, v \in H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k})$ . Note that this pairing generalises Suzuki’s higher intersection number in [16].

STEP 2. We glue two copies of  $\Sigma_{g,1}$  along  $A$  to construct a surface  $S$  with the natural inclusions  $i'_+, i'_- : \Sigma_{g,1} \hookrightarrow S$ . We orient  $S$  so that  $i'_+$  is orientation-preserving.  $S$  is a surface of genus  $2g$  with the boundary  $i'_+(B) \cup i'_-(B)$ . Capping  $S$  by a disk  $D$ , we obtain a closed surface of genus  $2g$ , which is naturally identified with  $\partial M$  so that  $i_+$  and  $i_-$  are homotopic to  $i'_+$  and  $i'_-$  respectively. We can extend the triangulation of  $\Sigma_{g,1}$  to those of  $\partial M$  and  $M$ .

Similarly to the above, we define a pairing

$$\langle \cdot, \cdot \rangle_S : H_1(S, p; \mathcal{K}_{N_k(M)}) \otimes H_1(S, p; \mathcal{K}_{N_k(M)}) \rightarrow \mathcal{K}_{N_k(M)}$$

by using

$$\begin{aligned} H_1(S, p; \mathcal{K}_{N_k(M)}) &\cong H_1(S, i'_+(B); \mathcal{K}_{N_k(M)}), \\ H_1(S, p; \mathcal{K}_{N_k(M)}) &\cong H_1(S, A; \mathcal{K}_{N_k(M)}) \cong H_1(S, q; \mathcal{K}_{N_k(M)}) \cong H_1(S, i'_-(B); \mathcal{K}_{N_k(M)}). \end{aligned}$$

Then we can check that

$$i^+ \langle u, v \rangle_{\Sigma_{g,1}} = \langle i'_+(u), i'_+(v) \rangle_S, \quad i^- \langle u, v \rangle_{\Sigma_{g,1}} = -\langle i'_-(u), i'_-(v) \rangle_S \in \mathcal{K}_{N_k(M)}$$

by an argument similar to that in Step 3. We also have a pairing

$$\langle \cdot, \cdot \rangle_M : H_1(M, D; \mathcal{K}_{N_k(M)}) \otimes H_2(M, S; \mathcal{K}_{N_k(M)}) \rightarrow \mathcal{K}_{N_k(M)}.$$

To see the relationship between  $\langle \cdot, \cdot \rangle_S$  and  $\langle \cdot, \cdot \rangle_M$ , we recall that the diagram

$$\begin{array}{ccc} H^1(M, D; \mathcal{K}_{N_k(M)}) & \xrightarrow{i^*} & H^1(S, i'_+(B); \mathcal{K}_{N_k(M)}) \\ \downarrow \cap [M, \partial M] & & \downarrow \cap [S, \partial S] \\ H_2(M, S; \mathcal{K}_{N_k(M)}) & \xrightarrow{\partial} & H_1(S, i'_-(B); \mathcal{K}_{N_k(M)}) \end{array}$$

commutes up to sign, where  $i : S \hookrightarrow M$  is the inclusion, the lower horizontal arrow is obtained from the homology exact sequence of the triplet  $(M, S, i'_-(B))$ , and we write  $[M, \partial M] \in H_3(M, \partial M)$  and  $[S, \partial S] \in H_2(S, \partial S)$  for the fundamental classes of  $M$  and  $S$  respectively. From this, we have

$$\langle x, \partial y \rangle_S = -\langle i(x), y \rangle_M$$

for any  $x \in H_1(S, p; \mathcal{K}_{N_k(M)})$  and  $y \in H_2(M, S; \mathcal{K}_{N_k(M)})$ . Consequently,

$$\begin{aligned}
 i^-\langle u, v \rangle_{\Sigma_{g,1}} &= -\langle i'_-(u), i'_-(v) \rangle_S \\
 &= -\langle i'_-(u), i'_-(v) - i'_+(r_k(M)v) \rangle_S \\
 &= \langle i_-(u), X \rangle_M \\
 &= \langle i_+(r_k(M)u), X \rangle_M \\
 &= -\langle i'_+(r_k(M)u), i'_-(v) - i'_+(r_k(M)v) \rangle_S \\
 &= \langle i'_+(r_k(M)u), i'_+(r_k(M)v) \rangle_S \\
 &= i^+\langle r_k(M), r_k(M) \rangle_{\Sigma_{g,1}},
 \end{aligned}$$

where  $\partial X = i'_-(v) - i'_+(r_k(M)v)$ . By applying  $i_+^{-1}$ , we obtain the equality

$$\langle r_k(M)u, r_k(M)v \rangle_{\Sigma_{g,1}} = \sigma_k^{(M)} \langle u, v \rangle_{\Sigma_{g,1}}.$$

STEP 3. We shall describe the pairing  $\langle \cdot, \cdot \rangle_{\Sigma_{g,1}}$  explicitly by calculating the chain-level intersection of cycles representing the basis elements  $\{\tilde{\gamma}_1 \otimes 1, \dots, \tilde{\gamma}_{2g} \otimes 1\}$  of  $H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k})$ . For that, we use Papakyriakopoulos' argument in [10]. In our notation, cycles intersect each other as in Figures 3 and 4, where  $\tilde{\gamma}_i^*$  is a path which represents the class corresponding to  $\tilde{\gamma}_i \otimes 1$  by the isomorphism  $H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) \cong H_1(\Sigma_{g,1}, B; \mathcal{K}_{N_k})$  mentioned in Step 1.

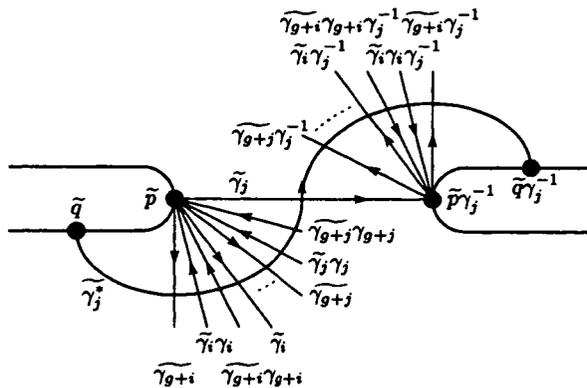


Figure 3: Intersections of lifts of edges (I)

From the figures, for example, we see that

$$\langle \tilde{\gamma}_i \otimes 1, \tilde{\gamma}_j \otimes 1 \rangle_{\Sigma_{g,1}} = 1 - \gamma_i - \gamma_j^{-1} + \gamma_i \gamma_j^{-1} = (1 - \gamma_i)(1 - \gamma_j^{-1})$$

for  $1 \leq j < i \leq g$ . By computing all cases, we obtain the matrix  $\tilde{J}$ . □

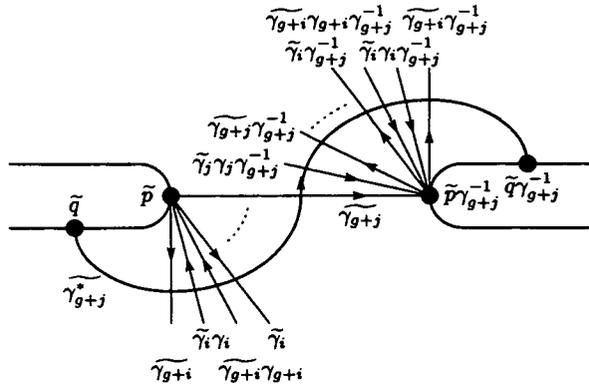


Figure 4: Intersections of lifts of edges (II)

REMARK 3.1. Our equality in Theorem 2.4 does depend on the choices involved in the proof. For example, by changing the generating system of  $\pi_1 \Sigma_{g,1}$  and the rules for defining intersection pairings, we obtain other matrices than  $\tilde{J}$  representing the symplecticity of the Magnus representation.

#### 4. THE GASSNER REPRESENTATION FOR STRING LINKS

Let  $D^2$  be a 2-dimensional disk. Given  $n \geq 1$ , we fix  $n$  points  $p_1, \dots, p_n$  in the interior of  $D^2$ . A *string link* of  $n$  strands is a smooth embedding of the disjoint union of ordered  $n$  unit intervals into  $D^2 \times I$  sending 0 and 1 of  $i$ -th interval to  $p_i \times \{0\}$  and  $p_{\tau(i)} \times \{1\}$ , where  $\tau$  is an element of the symmetric group of degree  $n$ . The set  $\mathcal{SL}_n$  of isotopy classes of string links of  $n$  strands has a natural monoid structure. A string link is said to be *pure* if  $\tau = \text{id}$ , and we denote by  $\mathcal{PSL}_n$  the submonoid of  $\mathcal{SL}_n$  consisting of all pure string links.

Let  $D_n := D^2 - \{p_1, \dots, p_n\}$ .  $\pi_1 D_n$  is the free group generated by  $n$  loops  $\delta_1, \dots, \delta_n$  as in Figure 5. In this section, we put  $N_k := N_k(\pi_1 D_n)$ . We write  $i_0$  and  $i_1$  for the natural embeddings of  $D_n$  into  $D^2 \times \{0\}$  and  $D^2 \times \{1\}$  respectively. It is easily checked that  $i_0, i_1 : \pi_1 D_n \rightarrow \pi_1(D^2 \times I - N(L))$  are 2-connected for each string link  $L$ , where  $N(L)$  denotes a tubular neighbourhood of (the image of)  $L$ . Hence a construction similar to that in Section 2 gives a monoid homomorphism

$$\sigma_{G,k} : \mathcal{SL}_n \rightarrow \text{Aut } N_k \quad (L \mapsto (i_1)^{-1} \circ i_0)$$

for each  $k \geq 2$ . Note that  $\text{Ker } \sigma_{G,2} = \mathcal{PSL}_n$ . By using  $H_1(D_n, p; \mathcal{K}_{N_k})$ , we also have a crossed homomorphism

$$r_{G,k} : \mathcal{SL}_n \rightarrow GL(n, \mathcal{K}_{N_k})$$



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