13

Challenges and extensions

In this final chapter we look to the future of ACSV, discussing the most important challenges and extensions of current results. Work attacking several of these problems is ongoing. The breadth of behavior exhibited by multivariate generating functions is vast, and new applications arise constantly that require additional techniques.

13.1 Contributing singularities and diagonals

Let $F(\mathbf{z})$ be the generating function of a sequence $(a_{\mathbf{r}})$. Theorems 7.20 and 7.35 represent $a_{\mathbf{r}}$ as an integer sum of saddle point integrals near critical points of F, which can be analyzed to determine asymptotics of $a_{\mathbf{r}}$. Unfortunately, identifying the integer coefficients in this sum seems to be extremely difficult, if not undecidable. Even identifying the contributing singularities of F, which are the critical points of highest height with non-zero coefficients, is currently only possible in general for minimal critical points, in two dimensions, or when F is the product of linear factors.

Being able to identify the contributing singularities of a general rational function would be an important theoretical breakthrough for ACSV. One (topological) approach is to generalize the two-dimensional algorithm discussed in Section 9.3 to higher dimensions. Another (computational) approach is to use software for D-finite functions. Recall from Section 8.4.2 that for any fixed $\mathbf{r} \in \mathbb{Z}^d$ the $\mathbf{r}$-diagonal of a multivariate rational function is D-finite, and the methods of creative telescoping produce a D-finite equation satisfied by the diagonal. Thus, it is possible to study rational diagonals using both ACSV and techniques for D-finite functions. In fact, these methods are complementary: ACSV often determines asymptotics up to *unknown integers that are difficult to determine*, while D-finite techniques often determine asymptotics up

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to *unknown complex numbers that can be rigorously approximated*. The combination of these two representations was used in Example 11.50 to determine asymptotics for a multivariate generating function with cone singularities, and it is natural to ask how far this combined method can be pushed – both to characterize the behavior of multivariate rational diagonals, and to attack the connection problem for the asymptotics of sequences with D-finite generating functions.

Problem 13.1. Classify the types of rational functions for which this hybrid ACSV and D-finite numeric method applies.

Remark 13.1. The diagonal of any bivariate rational function is algebraic, so asymptotics in any fixed direction can always be decided by computing the minimal polynomial for the diagonal and applying univariate techniques. However, the complexity of the computation to reduce to the one-dimensional case can increase with the size of the integers representing a direction of interest, and does not work for general direction. Thus, it is interesting to study even bivariate rational diagonals using multivariate methods.

Example 13.2. The (a, b)-diagonal of the bivariate generating function

$$F(x, y) = \frac{1}{1 - x - y - xy}$$

for the Delannoy numbers has the representation

$$\begin{aligned} G(x) &= \left[t^0\right] F\left(x^{1/a}/t^b, t^a\right) \\ &= \frac{1}{2\pi i} \int_{\gamma_x} \frac{F\left(x^{1/a}/t^b, t^a\right)}{t} dt \\ &= \frac{1}{2\pi i} \int_{\gamma_x} \frac{t^{b-1}}{t^b - x^{1/a} - t^{a+b} - x^{1/a} t^a} dt \,, \end{aligned}$$

for x sufficiently close to the origin and γ_x a circle around the origin that approaches the origin as $x \to 0$. This integrand has a single pole $t = s(x)$ satisfying $\lim_{x \to 0} s(x) = 0$, which is a pole of order b, so

$$\begin{aligned} G(x) &= \operatorname*{Res}_{t=s(x)} F\left(x^{1/a}/t^b, t^a\right)/t \\ &= \lim_{t \to s(x)} \frac{1}{(b-1)!} \partial_t^{b-1} \left((t - s(x))^b F\left(x^{1/a}/t^b, t^a\right)/t\right). \end{aligned}$$

The product rule gives this limit as an algebraic expression in $s(x)$, which can be combined with the defining algebraic equation for $s(x)$ to give an algebraic equation satisfied by G, however this expression is extremely unwieldy for large a and b, and the complexity of the operations grows with a and b. In contrast, asymptotics of the Delannoy numbers in all diagonal directions was given in Example 9.11 using ACSV. ◁

13.2 Phase transitions

Our asymptotic approximations typically hold uniformly as $\boldsymbol{r} \to \infty$ with $\hat{\boldsymbol{r}}$ staying in certain cones of directions, corresponding to contributing points at which the local geometry of $\mathcal{V}$ does not change. When the local geometry of $\mathcal{V}$ does change, asymptotic behavior is no longer uniform. For instance, consider the situation of Example 9.39 in Chapter 9: asymptotic behavior grows like a constant times $r^{-1/2}$ and is uniform in any direction bounded away from the main diagonal, while asymptotic behavior on the main diagonal grows like a constant times $r^{-1/3}$. Without some kind of result to bridge the gap we cannot, for instance, conclude that

$$\limsup \frac{\log a_{\boldsymbol{r}}}{\log |\boldsymbol{r}|} = -1/3\,. \tag{13.1}$$

A similar issue for trivariate functions arises in the analysis of spacetime generating functions for two-dimensional quantum random walks, where the logarithmic Gauss map maps a 2-torus to a simply connected subset Ξ of the plane. Such a map must have entire curves on which it folds over itself, and some points of greater degeneracy where such curves meet or fold on themselves.

There is some work in this area. A combinatorial generating function with the behavior of Example 9.39 was discussed in [Ban+01] under the name *Airy phenomena* (in the rescaled window $s = \lambda r + O(r^{1/3})$, the leading term converges to an Airy function). A start on a general formulation of such asymptotics in dimension $d = 2$ was made by Lladser [Lla03], and (13.1) follows from [Lla03, Corollary 6.12]. Lladser [Lla06] also shows that if there is a change of degree of the amplitude and the phase does not change degree, then we can derive a uniform formula for the coefficients in the expansion.

Problem 13.2. Characterize asymptotic transitions in more than two variables.

13.3 Degenerate phase

Most of our results in previous chapters have relied on reduction to a stationary phase integral for which the phase is quadratically nondegenerate at an isolated critical point, and hence amenable to a Complex Morse Lemma argument, but more complicated situations can arise.

Example 13.3. Recall from Example 10.69 the generating function

$$F(x, y) = \sum_{r,s} a_{rs} x^r y^s = \frac{1}{(1 - xy)(1 - x/2 - y/2)}$$

with main diagonal terms $a_{rr} = \sum_{j=0}^{r} 4^{-j} \binom{2j}{j}$.

The critical point equations for the factor $H_1 = 1 - x/2 - y/2$ in the direction $(1, 1)$ have a unique solution $(1, 1)$, while every point on the torus $|x| = |y| = 1$ satisfies the critical point equations for the factor $H_2 = 1 - xy$. The point $(1, 1)$ is therefore a minimal, but not strictly minimal, critical point which is a double point of the singular variety. In addition to a torus of singularities with the same coordinate-wise modulus, the varieties $\mathcal{V}(H_1)$ and $\mathcal{V}(H_2)$ intersect tangentially at $(1, 1)$.

A systematic application of the surgery approach reduces the problem of finding asymptotics of a_{rr} as $r \to \infty$ to finding asymptotics of

$$\frac{1}{2\pi} \int_D A(\theta, t) \exp\left(-\lambda\phi(\theta, t)\right)\, d\mu$$

as $\lambda \to \infty$, where

$$A(\theta, t) = \frac{2}{2 - e^{i\theta}}$$
$$\phi(\theta, t) = -\log\left(1 - t\left(1 - \frac{e^{-i\theta}}{2 - e^{i\theta}}\right)\right),$$

the domain of integration $D = [-\pi, \pi] \times [0, 1] \subset \mathbb{R}^2$, and μ is the Lebesgue measure. Note that $\operatorname{Re}\phi$ is nonnegative on D, with minimum value 0. The stationary points of ϕ on D are $(0, t)$ for $0 \le t \le 1$, and $(\theta, 0)$ for $-\pi \le \theta \le \pi$. Not only is the phase not equivalent via a smooth change of variables to $t^2 + \theta^2$ (it looks more like $t\theta^2$), the stationary phase set consists of more than a single point, being a 1-dimensional T-shaped subset of the rectangle. ◁

Example 10.69 derived asymptotics for Example 13.3 using an *ad hoc* approach.

Example 13.4. Develop a systematic theory for such degenerate integrals. ◁

Example 13.5. If S denotes the set of $n \times n$ Hermitian matrices with Frobenius norm 1, and E denotes the subset of S containing matrices with repeated eigenvalues, then E has positive codimension in S but has a volume with respect to the Riemannian metric inherited from S. It turns out[1] that this volume is, up

[1] K. Kozhasov (personal communication)

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to an easily computed constant, the coefficient of $(z_1z_2z_3z_4)^n$ in the generating function

$$M(z_1, z_2, z_3, z_4) = \frac{\prod_{1\le i<j\le 4} (z_i - z_j)}{1 - e_1(\boldsymbol{z})^2 + 4e_2(\boldsymbol{z})},$$

where e_j is the jth elementary symmetric function in the variables z_1, z_2, z_3, z_4. While symmetry initially makes the problem tractable, the stationary phase set is a union of curves rather than points and, to make things worse, the numerator vanishes to different orders on these curves and their intersections. By computing a D-finite equation satisfied by the main diagonal of M, and using the numeric methods discussed in Section 8.4.2 and Section 13.1 above, it can be shown that the coefficient of interest has asymptotic behavior $Cn^{-5/2}64^n$ as $n \to \infty$ for a constant $C \approx 0.4527$. It would be interesting to derive this result using multivariate techniques. ◁

In the case of real phase, it is possible to compute a degenerate integral as a Laplace integral by determining volumes of level sets. Suppose we wish to compute an integral of the form

$$\int_D \exp(-\lambda\phi(\boldsymbol{x}))A(\boldsymbol{x})\, d\boldsymbol{x},$$

where $D = [0,1]^d \subset \mathbb{R}^d$ in some dimension $d \geq 1$, the parameter λ is large, and ϕ and A are analytic functions on D. Fubini's Theorem tells us that for a nonnegative measurable function f defined on a measure space (X, μ) we have

$$\int_X f(\boldsymbol{x})\, d\mu(\boldsymbol{x}) = \int_0^\infty \mu\left(\{\boldsymbol{x} : f(\boldsymbol{x}) \geq z\}\right) dz,$$

and the change of variable $z = \exp(-u)$ converts this integral to

$$\int_{-\infty}^\infty e^{-u}\mu\left(\{\boldsymbol{x} : -\log f(\boldsymbol{x}) \leq u\}\right) du.$$

Let V_u denote the measure of the *sub-level set* $\mu\left(\{\boldsymbol{x} : -\log f(\boldsymbol{x}) \leq u\}\right)$. Letting $f(\boldsymbol{x}) = e^{-\lambda\phi(\boldsymbol{x})}$, we obtain

$$V_u = \int_D \mathbf{1}_{\phi(\boldsymbol{x})\leq u/\lambda} d\mu(\boldsymbol{x})$$

and, for simple enough ϕ, it may be possible to compute V_u explicitly. Proposition 4.7 can then be applied to determine asymptotics.

Exercise 13.1. Compute asymptotics as $\lambda \to \infty$ for

$$\int_0^1 \int_0^1 e^{-\lambda xy^2}\, xy\, dx\, dy.$$

Degeneracies of phase can also, in principle, be handled by resolving singularities to obtain a *normal crossing*, using a change of coordinates mapping the phase function into a monomial, expanding the resulting amplitude function into a power series, and then applying known exact asymptotics to each term. Resolution of singularities, together with the methods of [Var77], implies that the possible asymptotic behaviors for any rational asymptotics fall in a limited set of leading terms.

13.4 Critical points at infinity

Let $T = T(\boldsymbol{x})$, where $\boldsymbol{x}$ is in some component B of the complement of the amoeba of the denominator Q of some rational function. As seen in Chapter 7, non-existence of CPAI in the direction $\hat{\boldsymbol{r}}$ is a sufficient condition for $[T]$ to be representable as the sum of cycles corresponding to attachments at affine critical points in direction $\hat{\boldsymbol{r}}$. However, this non-existence is by no means necessary. For instance,

- The trajectories flowing to a CPAI may not be trajectories of any stratified gradient-like flow.
- A trajectory flowing to a CPAI may be a gradient-like flow, but the particular torus $[T]$ may flow down to an affine critical point and not be pulled down a path leading to this CPAI.
- $[T]$ may not flow down without being pulled to infinity, but there may be a cobordism between T and a cycle lower than any CVAI.
- Even if $[T]$ is pulled to infinity by some gradient-like vector field, the CPAI there may not alter the topology and it might be possible to deform $[T]$ to "come back from infinity."

Problem 13.3. a) Can there be unreachable CPAI, meaning that there is a downward gradient-like field that has no flows reaching a CPAI?

b) If so, how can we compute which CPAI are like this?

c) Do all CPAI alter the topology of $\mathcal{V}_*$? Is there an attachment theory for CPAI, giving a way to compute the topological effect of each CPAI?

d) If there is a local attachment, can we find a cycle α representing it and determine $\int_\alpha \boldsymbol{z}^{-\boldsymbol{r}-\boldsymbol{1}} F(\boldsymbol{z})\, d\boldsymbol{z}$?

Another useful tool would be the capability to rule out the existence of CPAI in some direction without an overly long computer algebra computation. An early conjecture, which turned out not to be true, was that all CPAI must be parallel to some face of the Newton polytope of Q. Some ongoing work proves that this conjecture is true when the polytope is *schön*, a property defined in terms of compactifications via toric varieties [Huh13, Definition 3.6]. The set of directions parallel to a face of the Newton polytope is a small set (it has positive codimension) so it is useful to know when CPAI are restricted to these directions. Generally, we would like to find computable restrictions on the possible directions of CPAI.

13.5 Algebraic GFs

Section 12.6 showed how to study algebraic generating functions by embedding them as diagonals of higher-dimensional rational functions. The simplest embedding method, due to Furstenberg, is easy to apply when it works, but does not apply to any algebraic generating function intersected by one of its algebraic conjugates at the origin. There are several known methods for resolving singularities in such cases, for instance the algorithm of Safonov [Saf00] mentioned in Theorem 2.41 and the less constructive procedure of Denef and Lipshitz [DL87].

Example 13.6. Let $f(x) = x/\sqrt{1-x}$ be an algebraic function with minimal polynomial $P(x, y) = (1-x)y^2 - x^2$. Because $f(x) = x + O(x^2)$ and its algebraic conjugate $-f(x) = -x + O(x^2)$ intersect at the origin, the embedding method of Furstenberg does not apply. Safonov's procedure subtracts some initial terms via the substitution $y = xz+x$, yielding a minimal polynomial $(1-x)(z+1)^2-1$ to which Proposition 2.34 now applies. Converting back to the original variables then gives f as the main diagonal of

$$F(x, y) = \frac{\left(2y^3x + 3y^2x - 2y^2 + 2yx - 3y + x - 2\right)yx}{y^2x + 2yx - y + x - 2}.$$

We remark that F has no affine critical points in the main diagonal direction, so it admits critical points at infinity which determine asymptotic behavior, making the analysis difficult. An alternative embedding, following the method of [DL87], is obtained through the much less obvious substitution $y = x/(1-z)$, expressing $f(x)$ as the main diagonal of

$$G(x, y) = \frac{2xy}{1-x-y}.$$

In contrast to the difficult behavior of F, the function G is combinatorial and has a smooth contributing critical point at $(1/2, 1/2)$, allowing for an easy asymptotic analysis. ◁

Problem 13.4. Give a complete analysis of algebraic generating functions $f(x)$ with quadratic minimal polynomials.

Problem 13.5. Can an algebraic series $f(x)$ with nonnegative coefficients always be embedded as the diagonal of a combinatorial rational function? Find an efficient algorithm that converts coefficient extraction for algebraic functions to coefficient extraction for rational functions in a way that preserves the combinatorial nature of the problem.

Recent work [Gre+22] develops software to analyze a variety of algebraic generating functions, ultimately cataloguing 20 combinatorial examples. An alternative approach being developed [BJP23] integrates algebraic generating functions directly. If f is an algebraic function defined by $P(\boldsymbol{z}, f) = 0$ and $f(\mathbf{0}) = c$, with $y = c$ a simple zero of $P(\mathbf{0}, y)$, coefficients of the power series expansion of f at the origin are given by

$$a_{\boldsymbol{r}} = \frac{1}{(2\pi i)^d} \int_T f(\boldsymbol{z})\boldsymbol{z}^{-\boldsymbol{r}-\mathbf{1}}\, d\boldsymbol{z}\,, \tag{13.2}$$

where T is a torus about the origin, sufficiently small so the polydisk with the same radii contains no singularities of f. Because $(\mathbf{0}, c)$ is a simple zero of P, there is a neighborhood $\mathcal{N}$ of $(\mathbf{0}, c)$ in $\mathbb{C}^{d+1}$ such that projection π onto the first d coordinates of the hypersurface $\mathcal{V}_P$ in $\mathbb{C}^{d+1}$ restricted to this neighborhood is a bi-analytic map to a neighborhood of the origin in $\mathbb{C}^d$. Choosing T smaller if necessary so as to be contained in this neighborhood of the origin, the set $C = \pi^{-1}(T)$ is a small torus in $\mathcal{V}_P$ and (13.2) becomes

$$a_{\boldsymbol{r}} = \frac{1}{(2\pi i)^d} \int_C y\boldsymbol{z}^{-\boldsymbol{r}-\mathbf{1}}\, d\boldsymbol{z}. \tag{13.3}$$

Aside from the high negative powers of $z_1, \ldots, z_d$, the integrand y has no denominator, however the coefficients $a_{\boldsymbol{r}}$ may be recovered the same way as one recovers coefficients of rational functions. In the absence of critical points at infinity, the d-dimensional complex variety $\mathcal{V}_* = \mathcal{V}_P \cap \mathbb{C}^d_*$ has a Morse theoretic decomposition into cycles attaining their maximum height near critical points of the height function $h_{(\hat{\boldsymbol{r}},0)}(\boldsymbol{z}) = \sum_{j=1}^d r_j \log|z_j|$. One then resolves the chain C in this basis. For instance, if the surface $\mathcal{V}_P$ is smooth and there are no CPAI for P in the direction $(\hat{\boldsymbol{r}}, 0)$ then the asymptotics of $a_{\boldsymbol{r}}$ are given by some linear combination of $\Phi_{\boldsymbol{w}}$ from (9.4) over critical points $\boldsymbol{w}$ of P in the direction $(\hat{\boldsymbol{r}}, 0)$.

Example 13.7. Let $f(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the Catalan generating function, with minimal polynomial $P(x, y) = xy^2 - y + 1 = 0$. The point $(0, f(0)) = (0, 1)$ is a smooth point of $\mathcal{V}_P$, and a small circle about $(0, 1)$ in $\mathcal{V}_*$ projects by π to a

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small circle about the origin in $\mathbb{C}^1$. The smooth surface $\mathcal{V}_P$ has precisely one critical point $\boldsymbol{p} = (1/4, 2)$ in the direction $(1, 0)$, defined by the simultaneous vanishing of P and P_y. There is a critical point at infinity because the value of P_y goes to zero as $x \to 0$, however the height function goes to infinity as $x \to 0$ so Morse theory tells us that the initial curve C can be deformed in $\mathcal{V}_*$ to a smooth contour γ in P passing through $\boldsymbol{p}$ so that the minimum of $\log|\boldsymbol{x}|$ on γ occurs at $\boldsymbol{p}$. A standard stationary phase integral for $\int_\gamma x^{-n-1} y dx$ leads to an asymptotic series for the Catalan numbers. Because the amplitude y is stationary at $\boldsymbol{p}$, there will be one more negative power of n than the usual $n^{-1/2}$ obtained from a univariate saddle point integral. This recovers without computation the fact that the nth Catalan number is $\Theta\left(n^{-3/2} 4^n\right)$. ◁

Exercise 13.6 below explores what happens when $(0, c)$ is not a simple pole of P. We conclude with an exercise illustrating a multivariate algebraic function.

Exercise 13.2. Let

$$f(x, y) = \frac{1 + x(y-1) - \sqrt{1 - 2x(y+1) + x^2(y-1)^2}}{2}$$

be the Narayana bivariate generating function from Example 2.37, defined by the minimal polynomial

$$P(x, y, w) = w^2 - w\left[1 + x(y-1)\right] + xy$$

and the fact that $f(0, 0) = 0$.

(a) Show that P is smooth.
(b) Find $c = f(0, 0)$ and determine whether $(0, 0, c)$ is a simple zero of P.
(c) Find the critical points for P in the direction $(2, 1, 0)$.
(d) Among the critical points, which have finite height?
(e) Are there critical points at infinity in the direction $(2, 1, 0)$?
(f) What asymptotics for $a_{2n,n}$ do you get from integrating $\int_\gamma w x^{-2n-1} y^{-n-1}\, dx dy$ over an appropriate contour γ?

13.6 Asymptotic formulae

In Section 9.4.2 we presented a geometric interpretation, in terms of Gaussian curvature, of the first term in our basic smooth point formula.

Problem 13.6. Give a coordinate-free formula for the next term in the basic smooth point asymptotic expansion. Give similar formulae for arrangement points.

Theorem 10.38 and Corollary 10.41 in Chapter 10 imply that the asymptotic contribution of a minimal arrangement point is unchanged when the factors in the denominator are replaced by their first-order terms.

Problem 13.7. Let $Q(\mathbf{z})$ be any polynomial in d variables and let $\mathbf{p}$ be a zero of Q such that the homogeneous part $H(\mathbf{z})$ of Q at $\mathbf{p}$ (in the sense of Definition 6.46) factors into $k < d$ linearly independent factors. Under what conditions is the dominant asymptotic contribution of $\mathbf{p}$ to the series coefficients of $1/Q(\mathbf{z})$ the same as the dominant asymptotic contribution of $\mathbf{p}$ to the series coefficients of $1/H(\mathbf{z})$?

An approach to Problem 13.7 is provided by a series of results in [BP11]. Lemma 2.24 of that paper shows that in the interior of the normal cone at $\mathbf{p}$, the function $1/Q(\mathbf{z})$ can be expanded in negative powers of $H(\mathbf{z})$, while Lemma 6.3 there shows that the Cauchy integral for the leading negative power is the inverse Fourier transform. These results are stated for points with specific types of local factorizations, but in fact appear to be much more general. As summarized in Chapter 11 of this text, the construction of a conical contour, and the error estimates that follow, rely only on the direction $\hat{\mathbf{r}}$ being nonobstructed. In fact the types of local divisors allowed are Lorentzian quadratics and smooth divisors, which as a degenerate case (having no quadratics) include arrangement points. Thus, solving Problem 13.7 for a large class of functions should be possible by an application of the results in [BP11].

Some caution is indicated due to the asymptotics for two smooth tangential divisors, worked out in a special case in Proposition 10.68, and the fact that asymptotics for the square of a single smooth divisor are a special case but do not capture the results of Proposition 10.68 in general. The difficulty here may be traced back to the fact that the expansion in [BP11, Lemma 2.24] only holds in the interior of a cone where the homogeneous part does not vanish; because two tangential curves cannot be separated by a cone, the expansion does not hold near where Q vanishes.

13.7 Symmetric functions

Multivariate generating functions often possess some degree of symmetry. For example, the denominators in the Delannoy generating function, the cube grove generating function, the Friedrichs–Lewy–Szegő generating function, and the Gillis–Reznick–Zeilberger generating functions are all symmetric polynomials. The denominator in the Aztec Diamond generating function is symmetric in two of its variables.

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A symmetric function Q always has critical points in the main diagonal direction, since $\nabla_{\log} Q(\boldsymbol{z}) \parallel \mathbf{1}$ whenever $\boldsymbol{z} = (w, \ldots, w)$ and w is a root of the univariate polynomial $Q(z, \ldots, z)$. When Q is symmetric and *multi-affine*, meaning Q has degree 1 in each variable, then there must be a minimal critical point.

Theorem 13.8 ([Bar+18, Lemma 15])**.** *Let Q be a multi-affine elementary symmetric function and let δ^Q denote the univariate diagonalization $\delta^Q(z) := Q(z, \ldots, z)$. If w is a root of δ^Q of minimal modulus then $(w, \ldots, w)$ is a minimal point for Q in the main diagonal direction.*

Proof Denote the roots of δ^Q by $\alpha_1, \ldots, \alpha_k$, where $|\alpha_1|$ is a root of least modulus, and let

$$M(\boldsymbol{z}) = \prod_{j=1}^{k} (z_j - \alpha_j) .$$

For any $\varepsilon > 0$, the polynomial M has no zeros in the polydisk $\mathcal{D}$ centered at the origin whose radii are all $|\alpha_1| - \varepsilon$. For any d-variable polynomial P, denote its *symmetrization* by

$$P_*(\boldsymbol{z}) := \frac{1}{d!} \sum_{\pi \in \mathcal{S}_d} P\left(z_{\pi(1)}, \ldots, z_{\pi(d)}\right) .$$

Then $M_* = Q$ and the Borcea–Brändén symmetrization lemma [BB09, Theorem 2.1] implies that Q has no zeros in the polydisk $\mathcal{D}$. Because $\varepsilon > 0$ was arbitrary, we conclude that $(\alpha_1, \ldots, \alpha_1)$ is a minimal point of Q. □

Exercise 13.3. In d variables, the jth *elementary symmetric function* is the polynomial

$$e_j(\boldsymbol{z}) = \sum_{S \in \mathcal{E}_j} \prod_{i \in S} z_i ,$$

where $\mathcal{E}_j$ consists of all subsets of $\{1, \ldots, d\}$ with j elements. Use Theorem 13.8 to find minimal points for the following denominators without resorting to geometric arguments.

(a) In 2 variables, $Q(x, y) = 1 - e_1(x, y) - e_2(x, y)$ (the Delannoy generating function).
(b) In 3 variables, $Q(x, y, z) = 3 - e_1(x, y, z) - e_2(x, y, z) + 3e_3(x, y, z)$ (the cube grove generating function).
(c) In 4 variables, $Q(\boldsymbol{z}) = 1 - e_1(\boldsymbol{z}) + 27e_4(\boldsymbol{z})$ (the GRZ generating function with critical parameter).

Problem 13.8. Find a way analogous to Theorem 13.8 to conclude minimality in some off-diagonal direction for some class of multi-affine polynomials. Replace the multi-affine hypothesis in Theorem 13.8 by something weaker so that the conclusion still holds.

Naively applying Gröbner basis methods typically breaks symmetry, but recent research in computer algebra has given effective methods for solving polynomial systems with symmetric polynomials, including critical point systems [HL16; Fau+23].

Problem 13.9. Incorporate software for symmetric polynomial solving into packages for ACSV computations.

Exercise 13.4. In four variables, let $G = 1 - e_1 + 27e_4$ be the Gillis–Reznick–Zeilberger denominator, let $K = 1 - e_1 + 2e_3 + 4e_4$ be the Kauers–Zeilberger denominator, let $S = e_3(1-x, 1-y, 1-z, 1-w)$ be the Szegő denominator, and let $L = e_2(1-x, 1-y, 1-z, 1-w)$ be the Lewy–Askey denominator.

(a) Express G, K, S, and L as polynomials $P_1, \dots, P_4$ in the elementary symmetric functions $e_1, \dots, e_4$.
(b) Compute a Gröbner basis for $\langle P_1, \dots, P_4 \rangle$ as polynomials in the variables $e_1, \dots, e_d$ and describe the variety $\mathcal{V}_e$ defined by the points $(e_1, \dots, e_d)$ where the P_j simultaneously vanish.
(c) Use this computation to find the elements of $\mathcal{V}(G, K, S, L)$.

13.8 Conclusion

This book aims to illustrate effective methods for computing asymptotic approximations to the coefficients of multivariate generating functions. Such methods have many applications, and have already been used to study problems arising in, among other areas, dynamical systems, bioinformatics, number theory, statistical physics, algebraic statistics, string theory, information theory, and queueing theory. We expect the number of applications to grow steadily. While many (most?) applied problems can be tackled by a smooth point analysis, there are many interesting problems that involve much more complicated local geometry, such as the tiling models discussed in Section 11.4.

From the standpoint of mathematical analysis, many of the tools required to extend the basic ACSV results already exist. Problems for which minimal points control asymptotics usually sidestep complicated topology, and the Morse-theoretic intuition behind our results can often be ignored in such cases

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by casual users seeking to solve a specific problem. However, substantial topological difficulties can arise when dealing with contributing points that are not necessarily minimal. We believe that to make further progress in this area, substantial work in the Morse-theoretic framework will be required.

This book is certainly not the last word on the subject, but rather an invitation to the reader to join in further development of this research area, which combines beauty, utility, and tractability, and which has given the current authors considerable challenge and enjoyment over many years.

Notes

ACSV was the subject of an AMS-sponsored Mathematical Research Community in 2020–2022, and a 2022 workshop at the American Institute of Mathematics. Among the topics discussed at these events, which still have active research collaborations, are characterizations of CPAI [Gil22], software for ACSV [LMS22], rational embeddings of algebraic functions [Gre+22], and work in progress by Drmota and Pak on multivariate characterizations of so-called $\mathbb{N}$-algebraic functions. The methodology for algebraic functions given at the end of Section 13.5 is contemporaneous to this edition and appears in the preprint [BJP23], along with a new formula for coefficient asymptotics in terms of the defining algebraic function.

Proving minimality by conventional means in Exercise 13.3 (c) above is quite challenging; it is the basis of Problem B5 on the 2020 Putnam examination. The approach of Exercise 13.4 was suggested by Brendan Rhoades, and Example 13.2 is adapted from [Sta99, Section 6.3].

Additional exercises

Exercise 13.5. (binomial transition) Consider the binomial coefficient generating function $(1-x-y)^{-1}$, and compute first-order asymptotics for the coefficient a_{rs}, where $s/r \to 0$ as $r, s \to \infty$. How many different cases are there in the analysis?

Exercise 13.6. Let $P(x,y) = (1-x)y^2 - x^2$ as in Example 13.6.

(a) Show that for sufficiently small $\varepsilon > 0$ there are two liftings by π^{-1} of the centered circle of radius ε in $\mathbb{C}_*$ to a contour in $\mathcal{V}_*$ and that one of them describes the positive square root $y = x/\sqrt{1-x}$.
(b) Find all affine critical points of $\mathcal{V}_*$ in direction $(1,0)$.
(c) Find all critical points at infinity of $\mathcal{V}_*$ in direction $(1,0)$.

(d) Which of these critical points are at finite height?
(e) Which is more of a problem for computation, the double zero of P or the existence of a critical point at infinity?

Exercise 13.7. Find a general formula for $\det \mathbf{\Gamma}_\Psi$ in terms of the partial derivatives of Q when Q vanishes to degree 3 and is locally the product of three transversely intersecting smooth divisors.

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