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Nilpotency of Some Lie Algebras Associated with p-Groups

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Abstract. Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra over a commutative ring with unity in which 2 is invertible. Suppose that L_0 is abelian and L is generated by finitely many homogeneous elements $a_1, \dots, a_k$ such that every commutator in $a_1, \dots, a_k$ is ad-nilpotent. We prove that L is nilpotent. This implies that any periodic residually finite $2'$-group G admitting an involutory automorphism ϕ with $C_G(\phi)$ abelian is locally finite.

0 Introduction

Let p be a prime and G a finitely generated residually finite p-group. Examples of groups constructed in [1], [3], [4], [5], [17] show that in general G can be infinite.

Using the Lazard central series of G one associates to G a Lie algebra $L = L_p(G)$ in such a way that G is finite if and only if L is nilpotent. Some detail of this construction is given in the last section of the paper. Lazard proved that L is generated by a finite set M such that each commutator of elements in M is ad-nilpotent [9]. This enables us to reduce problems on finiteness of certain p-groups to problems on nilpotency of some Lie algebras. For example, the celebrated Zelmanov's solution of the Restricted Burnside Problem is based on his deep results concerning Lie algebras.

In the present paper we prove the following theorem.

Theorem *Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra over a commutative ring with unity in which 2 is invertible. Suppose that L_0 is abelian and L is generated by finitely many homogeneous elements $a_1, \dots, a_k$ such that every commutator in $a_1, \dots, a_k$ is ad-nilpotent. Then L is nilpotent.*

This theorem is proved in Section 4 of the paper. It enabled us to derive that any periodic residually finite $2'$-group G admitting an involutory automorphism ϕ with $C_G(\phi)$ abelian is locally finite (Theorem 5.8).

It should be said that Theorem 5.8 is not a first result stating local finiteness of a periodic group in terms of the centralizer of an automorphism ϕ of G. B. H. Neumann showed that any periodic group admitting a fixed point free involutory automorphism is abelian [10]. Šunkov proved that if G is a periodic group having an involutory automorphism ϕ such that $C_G(\phi)$ is finite then G is locally finite (and contains a solvable subgroup of finite index) [16].

According to Deryabina and Ol'shanski for any positive integer n which has at least one odd divisor there exists an infinite group G having an element of order n with finite centralizer such that all proper subgroups of G are finite [2]. Therefore the result of Šunkov

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cannot be extended to periodic groups having finite centralizer of an element whose order is not a 2-power. N. Rocco and the author proved recently that if G is a periodic residually finite group having a 2-automorphism with finite centralizer then G is locally finite [11]. In [12], [13], [15] we prove local finiteness of some periodic groups admitting a fixed point free four-group of automorphisms.

It seems however that Theorem 5.8 of this paper is the first result of the kind in which the condition imposed on $C_G(\phi)$ does not refer to the order of $C_G(\phi)$.

Most of our notation and terminology is standard. We use $[x_1, \ldots, x_k]$ to denote $\big[\ldots[x_1, x_2], x_3\big], \ldots, x_k]$. An element g of a Lie algebra L is called ad-nilpotent if there exists an integer m such that

$$[x, \underbrace{g, \ldots, g}_{m \text{ times}}] = 0$$

for any $x \in L$. Let $M \subseteq L$ be a subset of a Lie algebra L. By a commutator in elements of M we mean any element that can be obtained by taking products of elements of M with arbitrary system of brackets. A subalgebra of L generated by a set M is denoted by $\langle M \rangle$.

A Lie algebra L is $\mathbb{Z}_2$-graded if it contains subspaces L_0 and L_1 such that $L = L_0 + L_1$ and $[L_i, L_j] \leq L_{i+j}$; $i, j \in \mathbb{Z}_2$. Clearly, L_0 here is a subalgebra. If $L = L_0 + L_1$ is a $\mathbb{Z}_2$-graded Lie algebra then any element $x \in L_0 \cup L_1$ is called homogeneous. Homogeneous elements $x \in L_0$ are called even while those $x \in L_1$ odd.

Suppose that X is either a quotient algebra or a subalgebra generated by a set of homogeneous elements of a $\mathbb{Z}_2$-graded Lie algebra L. Then X is naturally equiped with the $\mathbb{Z}_2$-grading induced by that of L. So X can be viewed a $\mathbb{Z}_2$-graded Lie algebra.

Through out the paper we use freely the following fact. Let $x, g, h_1, \ldots, h_k \in L$ be any elements of a Lie algebra L. Assume that g commutes with all $h_1, \ldots, h_k$. Then

$$[x, h_1, \ldots, h_k, g] = [x, g, h_1, \ldots, h_k].$$

1 Existence of a Nilpotent Ideal

Our goal in this section is to prove the following:

Proposition 1.1 *Let L be a Lie algebra, L_1, L_2, L_3 abelian subalgebras of L such that*

$$L = L_1 + L_2 + L_3$$

and

$$[L_1, L_2] \leq L_3, [L_1, L_3] \leq L_2, [L_2, L_3] \leq L_1.$$

Assume that a is an ad-nilpotent element in L_1 and let N be the ideal of L generated by a. Then N is nilpotent.

This proposition can be deduced from some results scattered in [14], [15] and proved in a somewhat different setting. For the reader's convenience we give a complete and detailed proof here.

Lemma 1.2 *Assume hypothesis of 1.1. Suppose that $b \in L_2$ and $[a, b] = 0$. Let T be the ideal of L generated by $[b, L_1]$. Then $[T, a] = 0$.*

Proof It is sufficient to show that for any set of elements $c_1, c_2, \ldots, c_t$ such that $c_1 \in L_1$ and $c_i \in L_{k_i}$, where each $k_i \in \{1, 2, 3\}$, we have

$$[b, c_1, \ldots, c_t, a] = 0. \tag{1.3}$$

To prove this we shall use induction on t.

If $t = 1$ (1.3) can be proved easily:

Since $a, c_1 \in L_1$ and L_1 is abelian, $c_1 \in C_L(a)$. By hypothesis b also lies in $C_L(a)$, and so (1.3) follows.

Let now $t \geq 2$ and suppose that for smaller values of t (1.3) holds. If $c_2 \in L_1$ we denote $[b, c_1]$ by d and observe that $d \in L_3$ and $[d, a] = 0$. Then

$$[b, c_1, \ldots, c_t, a] = [d, c_2, \ldots, c_t, a].$$

The form of the latter commutator differs from that of the commutator in (1.3) only by the fact that $b \in L_2 \cap C_L(a)$ while $d \in L_3 \cap C_L(a)$. This difference is not very serious of course and we are in a position to apply the induction hypothesis. It yields

$$[d, c_2, c_4, \ldots, c_t, a] = 0$$

and so (1.3) holds.

If $c_2 \in L_3$ then $[b, c_1, c_2] = 0$ because $[b, c_1]$ and c_2 both lie in L_3.

Thus we can assume that $c_2 \in L_2$. If $t = 2$ we see that $[b, c_1, c_2, a] = 0$ because both a and $[b, c_1, c_2]$ lie in L_1. So $t \geq 3$. Consider the three possible cases:

1) $c_3 \in L_1$;
2) $c_3 \in L_2$;
3) $c_3 \in L_3$.

1) Since both c_3 and $[b, c_1, c_2]$ lie in L_1 and L_1 is abelian, $[b, c_1, c_2, c_3] = 0$ and obviously (1.3) holds.

In the calculations below all the underlined commutators $\underline{[x_1, x_2, \ldots, x_s]}$ equal zero as there always exist some positive integer k and some $i \in \{1, 2, 3\}$ such that $[x_1, x_2, \ldots, x_k] \in L_i$ and $x_{k+1} \in L_i$ which, since L_i is abelian, guarantees that $[x_1, \ldots, x_k, x_{k+1}] = 0$. So, we will just discard such commutators.

2) We have

$$[b, c_1, c_2, c_3] = \big[b, [c_1, c_2], c_3\big] + \underline{[b, c_2, c_1, c_3]} = \big[b, [c_1, c_2, c_3]\big] + \underline{\big[b, c_3, [c_1, c_2]\big]}.$$

Put $a_1 = [c_1, c_2, c_3] \in L_1$. Then the commutator in (1.3) can be rewritten in the form

$$[b, a_1, c_4, \ldots, c_t, a]$$

which by the induction hypothesis equals zero.

3) In this case we have

$$[b, c_1, c_2, c_3] = \underline{\big[[b, c_1], c_3, c_2\big]} + \big[b, c_1, [c_2, c_3]\big].$$

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If we put $a_2 = [c_2, c_3] \in L_1$ and $b_1 = [b, c_1] \in L_3 \cap C_L(a)$ then the commutator in (1.3) can be rewritten as

$$[b_1, a_2, c_4, \ldots, c_t, a].$$

The form of the latter commutator differs from that of the commutator in (1.3) only by the fact that $b \in L_2 \cap C_L(a)$ while $b_1 \in L_3 \cap C_L(a)$. We have already remarked that this difference is not very serious and we can apply the induction hypothesis. Therefore

$$[b_1, a_2, c_4, \ldots, c_t, a] = 0.$$

The proof is now complete. ■

Lemma 1.4 *Assume hypothesis of 1.1. Then* $[L, L] \leq \langle L_1, L_2 \rangle$.

Proof Denote $\langle L_1, L_2 \rangle$ by N. The inclusions $[L_1, L_3] \leq L_2$ and $[L_2, L_3] \leq L_1$ show that L_3 normalizes N. Since $L = L_1 + L_2 + L_3$, it follows that N is an ideal in L. The quotient L/N must be abelian because so is L_3. This completes the proof. ■

Proof of 1.1 Let n be the minimal number such that

$$[L, \underbrace{a, \ldots, a}_{n \text{ times}}] = 0.$$

We certainly can assume that $n \geq 1$ and prove nilpotency of N using induction on n. Set

$$M = [L_2, \underbrace{a, \ldots, a}_{n-1 \text{ times}}].$$

Then $M \leq C_L(a) \cap L_2$ if n is odd and $M \leq C_L(a) \cap L_3$ if n is even. Assume that $M \leq L_2$ (the case $M \leq L_3$ is absolutely symmetric).

Let $\overline{L} = L/C_L(N)$, and let $\overline{X}$ denote the image of any subset $X \subseteq L$ in $\overline{L}$. The minimal ideal T of L containing $[M, L_1]$ is spanned by commutators of the form $[b, c_1, \ldots, c_t]$, where $c_1 \in L_1$ and $c_i \in L_1 \cup L_2 \cup L_3$; $i \geq 2$. Therefore, by Lemma 1.2, a centralizes T. This implies $T \leq C_L(N)$ and we obtain $[\overline{M}, \overline{L_1}] = 0$. Since $\overline{M} \leq \overline{L_2}$ and $\overline{L_2}$ is abelian, we conclude that

$$\overline{M} \leq C_{\overline{L}}\big(\langle \overline{L_1}, \overline{L_2} \rangle\big).$$

By Lemma 1.4 $[\overline{L}, \overline{L}] \leq \langle \overline{L_1}, \overline{L_2} \rangle$. Hence $\overline{M}$ centralizes the subalgebra $\overline{R}$ of $\overline{L}$ generated by $[\overline{L}, \overline{L}]$ and $\overline{a}$. Obviously $\overline{N} \leq \overline{R}$ and therefore $\overline{M} \leq Z(\overline{N})$. Bearing in mind that $\overline{L} = L/C_L(N)$ we conclude that $M \leq Z_2(N)$, the second term of the upper central series of N. The same argument with L_2 replaced by L_i shows that

$$[L_i, \underbrace{a, \ldots, a}_{n-1 \text{ times}}] \leq Z_2(N)$$

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for any $i = 1, 2, 3$. Since $L = \sum_{1\le i\le 3} L_i$, we obtain

$$[L, \underbrace{a, \ldots, a}_{n-1 \text{ times}}] \le Z_2(N).$$

Let us consider now the quotient algebra $\bar{\bar{L}} = L/Z_2(N)$. Let $\bar{\bar{X}}$ denote the image of any subset $X \subseteq L$ in $\bar{\bar{L}}$. Then

$$[\bar{\bar{L}}, \underbrace{\bar{\bar{a}}, \ldots, \bar{\bar{a}}}_{n-1 \text{ times}}] = 0$$

and so by induction $\bar{\bar{N}}$ is nilpotent. Since $\bar{\bar{N}} = N/Z_2(N)$, we derive that N is nilpotent as well. Remark that the proof actually shows that N is of nilpotency class at most $2n-1$. The proof is complete. ■

2 A Sufficient Condition for $[L, L]$ to be Nilpotent

From now on the term "Lie algebra" means Lie algebra over some commutative ring $\mathbb{K}$ with unity in which 2 is invertible.

We will prove here

Proposition 2.1 *Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra and assume that there exists an odd element $a \in L_1$ such that $L = \langle L_0, a\rangle$. Suppose that L_0 is abelian and a is ad-nilpotent. Then the subalgebra $\langle L_1\rangle$ of L generated by all odd elements is nilpotent.*

We remark that under hypotheses of the above proposition $\langle L_1\rangle$ is actually an ideal of L containing the derived algebra $[L, L]$. It follows that the derived algebra $[L, L]$ is nilpotent.

Lemma 2.2 *Let i, j be non-negative integers either both odd or both even. Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra with L_0 abelian. Let $h_1, \ldots, h_i$ and $g_1, \ldots, g_j$ be arbitrary elements in L_0 and $a \in L_1$ an arbitrary odd element. Then $[a, h_1, \ldots, h_i]$ and $[a, g_1, \ldots, g_j]$ commute.*

Proof We use induction on $\min\{i, j\}$. Suppose first that $j = 0$ and set $b = [a, h_1, \ldots, h_i]$. We have to prove that $[b, a] = 0$. By the Jacobi identity we have.

$$[a, h_1, \ldots, h_i, a] = [a, h_1, \ldots, h_{i-1}, a, h_i] - \big[a, h_1, \ldots, h_{i-1}, [a, h_i]\big].$$

Note that $[a, h_1, \ldots, h_{i-1}, a, h_i] = 0$ because $[a, h_1, \ldots, h_{i-1}, a,] \in L_0$ and L_0 is abelian. Therefore

$$[a, h_1, \ldots, h_i, a] = -\big[a, h_1, \ldots, h_{i-1}, [a, h_i]\big].$$

Write

$$\big[a, h_1, \ldots, h_{i-1}, [a, h_i]\big] = \big[a, h_1, \ldots, h_{i-2}, [a, h_i], h_{i-1}\big] - \big[a, h_1, \ldots, h_{i-2}, [a, h_i, h_{i-1}]\big].$$

Again we note that $\big[a, h_1, \ldots, h_{i-1}, [a, h_i], h_{i-1}\big] = 0$ because $\big[a, h_1, \ldots, h_{i-1}, [a, h_i]\big] \in L_0$ and L_0 is abelian. Therefore

$$[a, h_1, \ldots, h_i, a] = \big[a, h_1, \ldots, h_{i-2}, [a, h_i, h_{i-1}]\big].$$

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It becomes clear that for any non-negative $k \le i$ we have

$$[a, h_1, \dots, h_i, a] = (-1)^k \big[a, h_1, \dots, h_{i-k}, [a, h_i, h_{i-1}, \dots, h_{i-k+1}]\big].$$

Since i is even, when $k = i$ we obtain

$$[a, h_1, \dots, h_i, a] = \big[a, [a, h_i, \dots, h_1]\big]. \tag{2.3}$$

By hypothesis all $h_1, \dots, h_i$ commute and so

$$[a, h_i, h_{i-1}, \dots, h_1] = [a, h_1, \dots, h_{i-1}, h_i] = b.$$

Combining this with (2.3) we obtain $[b, a] = [a, b]$. Since 2 is invertible in $\mathbb{K}$, it follows that $[b, a] = 0$.

Assume now that $i \ge j \ge 1$. Set $c = [a, g_1, \dots, g_{j-1}]$ and let us show that

$$[c, g_j, b] = 0.$$

We have $[c, g_j, b] = [c, b, g_j] - \big[c, [b, g_j]\big]$. As $[c, b] \in L_0$ and L_0 is abelian, we conclude that $[c, b, g_j] = 0$ while $\big[c, [b, g_j]\big] = 0$ by the induction hypothesis. The proof is now complete. ■

Lemma 2.4 *Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra and assume that there exists an odd element $a \in L_1$ such that $L = \langle L_0, a\rangle$. Then L_1 is spanned by all commutators of the form $[a, h_1, \dots, h_k]$, where $h_1, \dots, h_k \in L_0$.*

Proof Since $L = \langle L_0, a\rangle$, it follows that L is spanned by all commutators of the form

$$[x_1, \dots, x_s], \tag{2.5}$$

where each x_i is either even or equals a. It is immediate that L_1 is spanned by all commutators (2.5) in which the number of occurrences of a is odd. Assume that $[x_1, \dots, x_s]$ is such a typical generator of L_1 and let j be the maximal number such that $x_j = a$. Then the number of occurrences of a in $[x_1, \dots, x_{j-1}]$ is even and therefore $[x_1, \dots, x_{j-1}] \in L_0$. Set $h_1 = [x_1, \dots, x_{j-1}]$. Since j is the maximal number with $x_j = a$, we conclude that $x_{j+1}, \dots, x_s \in L_0$. We obtain

$$[x_1, \dots, x_s] = -[a, h_1, x_{j+1}, \dots, x_s]$$

and the lemma follows. ■

Proof of 2.1 Set

$$M_1 = \langle [a, h_1, \dots, h_k]; h_1, \dots, h_k \in L_0; k \text{ is even}\rangle,$$
$$M_2 = \langle [a, h_1, \dots, h_k]; h_1, \dots, h_k \in L_0; k \text{ is odd}\rangle$$

and $M_3 = L_0$. Then by 2.4 $L_1 = M_1 + M_2$. Therefore $L = M_1 + M_2 + M_3$. Lemma 2.2 tells us that M_1 and M_2 are abelian while M_3 is so by hypotheses. Also we have $[M_1, M_2] \le M_3$, $[M_1, M_3] \le M_2$ and $[M_2, M_3] \le M_1$. Note that $a \in M_1$. Thus, we are in a position to use Proposition 1.1 according to which the ideal N of L generated by a is nilpotent. Lemma 2.4 shows that $L_1 \le N$. This completes the proof. ■

3 Existence of Commuting Ideals

We start this section with a generalization of Lemma 2.4.

Lemma 3.1 *Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra and assume that $L = \langle X, Y\rangle$, where $X \subseteq L_0$ and $Y \subseteq L_1$. Then L_1 is spanned by commutators of the form $[y, h_1, \dots, h_k]$ for suitable $y \in Y$ and $h_1, \dots, h_k \in L_0$.*

Proof Since $L = \langle X, Y\rangle$, it follows that L is spanned by all commutators of the form

$$[z_1, \dots, z_s],$$

where each $z_i \in X \cup Y$. It is immediate that L_1 is spanned by all such commutators in which the number of occurrences of elements of Y is odd. Assume that $[z_1, \dots, z_s]$ is such a typical generator of L_1 and let j be the maximal number such that $z_j \in Y$. Then the number of occurrences of elements of Y in $[z_1, \dots, z_{j-1}]$ is even and therefore $[z_1, \dots, z_{j-1}] \in L_0$. Set $h_1 = [z_1, \dots, z_{j-1}]$. Since j is the maximal number with $z_j = a$, we conclude that $z_{j+1}, \dots, z_s \in X$. We obtain

$$[z_1, \dots, z_s] = -[z_j, h_1, z_{j+1}, \dots, z_s]$$

and the lemma follows. ■

Lemma 3.2 *Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra with L_0 abelian. Let $B \leq L_1$ be any set of odd elements and $g \in L_0$ such an even element that $[B, g, B] = 0$. Let D be the set of all commutators of the form $[b, h_1, \dots, h_k]$, $b \in B$ and $h_1, \dots, h_k \in L_0$. Assume that $a \in L_1 \cap C_L(g)$. Then for any non-negative integer m we have*

$$[a, \underbrace{D, \dots, D}_{m \text{ times}}, g] = 0.$$

Proof Induction on m. If $c \in L_1$ is any odd element we observe that $[a, c, g] = 0$ because $[a, c] \in L_0$ and L_0 is abelian. Since $D \subseteq L_1$, for $m = 1$ the assertion follows. Suppose now that $m = 2$. We will prove first that

$$\text{if } c_1, c_2 \in B \quad \text{then} \quad [a, c_1, c_2, g] = 0. \tag{3.3}$$

Write

$$[a, c_1, c_2, g] = [a, c_1, g, c_2] + \big[a, c_1, [c_2, g]\big].$$

Note that $[a, c_1, g, c_2] = 0$ because $[a, c_1, g] = 0$. Therefore

$$[a, c_1, c_2, g] = \big[a, c_1, [c_2, g]\big].$$

Since $c_1, c_2 \in B$ and $[B, g, B] = 0$, we obtain

$$\big[a, c_1, [c_2, g]\big] = \big[a, [c_2, g], c_1\big] = \Big[[a, c_2, g] + \big[c_2, [a, g]\big], c_1\Big].$$

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Both commutators $[a, c_2, g]$ and $\big[c_2, [a, g]\big]$ equal zero. The first one is zero because $[a, c_2] \in L_0$ and L_0 is abelian while the second one is zero because $[a, g] = 0$ by hypothesis. Thus, (3.3) holds.

Now write

$$c_1 = [b_1, f_1, \ldots, f_{k_1}],\ c_2 = [b_2, h_1, \ldots, h_{k_2}],$$

where $b_1, b_2 \in B$ and $f_1, \ldots, f_{k_1}, h_1, \ldots, h_{k_2} \in L_0$. Suppose that $k_1 > 0$ and write

$$[a, c_1] = \big[a, [b_1, f_1, \ldots, f_{k_1}]\big].$$

Using the same argument as in the proof of Lemma 2.2 we obtain

$$[a, c_1] = (-1)^{k_1}[a, f_1, \ldots, f_{k_1}, b_1].$$

Set $a_1 = [a, f_1, \ldots, f_{k_1}]$ and observe that $a_1 \in L_1 \cap C_L(g)$. It now suffices to prove that $[a_1, b_1, c_2, g] = 0$. This argument shows that without any loss of generality we can assume that $k_1 = 0$.

Taking this into account and using the fact that $[a, b_1] \in L_0$ and hence $[a, b_1]$ commutes with all $h_1, \ldots, h_{k_2}$ we compute

$$\begin{aligned}[a, c_1, c_2, g] &= \big[a, b_1, [b_2, h_1, \ldots, h_{k_2}], g\big] = -\big[b_2, h_1, \ldots, h_{k_2}, [a, b_1], g\big] \\ &= -\big[b_2, [a, b_1], h_1, \ldots, h_{k_2}, g\big] = [a, b_1, b_2, h_1, \ldots, h_{k_2}, g].\end{aligned}$$

We already know that $[a, b_1, b_2, g] = 0$. Since g commutes with $h_1, \ldots, h_{k_2}$, it follows that $[a, c_1, c_2, g] = 0$.

Thus, it remains to handle the case $m \geq 3$. Let $c_1, c_2, \ldots, c_m$ be arbitrary elements in D. We have to show that

$$[a, c_1, \ldots, c_m, g] = 0.$$

Set $a_2 = [a, c_1, c_2]$ and observe that $a_2 \in L_1$ and by the preceding paragraph $a_2 \in C_L(g)$. The induction hypothesis yields

$$[a_2, c_3, \ldots, c_m, g] = 0.$$

This completes the proof. ■

Lemma 3.4 *Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra with L_0 abelian. Let $A, B \subseteq L_1$ be sets of odd elements such that $L = \langle L_0, A, B\rangle$ and $[A, L_0] \subseteq A$. Suppose that $g \in L_0$ and $a \in L_1$ are such even and odd elements respectively that $[A, a] = 0$ and $[B, g, B] = 0$. Let N and G be the ideals of L generated by a and g respectively. If $[a, g] = 0$ then $[N, G] = 0$.*

Proof Assume that $[a, b] = 0$. Let D be the set of all commutators of the form $[b, h_1, \ldots, h_i]$ where $b \in B$ and $h_1, \ldots, h_i \in L_0$. Then by 3.1 L_1 is spanned by A and D. Then N is spanned by commutators of the form $[a, d_1, \ldots, d_s]$, where $d_1, \ldots, d_s \in L_0 \cup A \cup D$. Therefore it suffices to prove that we always have

$$[a, d_1, \ldots, d_s, g] = 0. \tag{3.5}$$

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This will be proved by induction on s and some other parameters introduced when necessary. We observe that the set $L_0 \cup A \cup D$ is closed with respect to the multiplication in L.

Obviously (3.5) holds if $d_1 \in A$ because $[A, a] = 0$, so we may assume that $d_1 \notin A$. Suppose that at least one of $d_2, \dots, d_s$ belongs to A. Let j be the smallest number such that $d_j \in A$ and let us use induction on j. Write

$$[a, d_1, \dots, d_{j-1}, d_j, d_{j+1}, \dots, d_s, g] = \big[a, d_1, \dots, [d_{j-1}, d_j], d_{j+1}, \dots, d_s, g\big] \\ + [a, d_1, \dots, d_j, d_{j-1}, d_{j+1}, \dots, d_s, g].$$

The first of the above commutators is zero by the induction on s because $[d_{j-1}, d_j] \in L_0 \cup A \cup D$ while the second one is zero by the induction on j. The conclusion is that there is no loss of generality in assuming that none of $d_1, \dots, d_s$ lies in A. Suppose now that some of them lie in L_0 and let k denote the smallest number such that $d_k \in L_0$. We will use now induction on k. If $k = 1$ we write $a_1 = [a, d_1]$. Since L_0 normalizes A, it follows that $[A, a_1] = 0$. Also we note that $[a_1, g] = 0$ because L_0 is abelian. Substituting now in (3.5) $[a, d_1]$ by a_1 and using induction on s we deduce (3.5). This enables us to assume that $k \geq 2$. Since $d_{k-1} \notin A$, it follows that $d_{k-1} \in D$ and so $[d_{k-1}, d_k] \in D$. We have

$$[a, d_1, \dots, d_{k-1}, d_k, d_{k+1}, \dots, d_s, g] = \big[a, d_1, \dots, [d_{k-1}, d_k], d_{k+1}, \dots, d_s, g\big] \\ + [a, d_1, \dots, d_k, d_{k-1}, d_{k+1}, \dots, d_s, g].$$

The first of the above commutators is zero by the induction on s because $[d_{k-1}, d_k] \in D$ while the second one is zero by the induction on k.

The conclusion is that without any loss of generality we may assume that all $d_1, \dots, d_s$ lie in D. In this case (3.5) is immediate from Lemma 3.2. The proof is complete. ■

4 Main Theorem

Given any Lie algebra L, let $R(L)$ denote the sum of all locally nilpotent ideals of L. Thus, $R(L)$ is the maximal locally nilpotent ideal of L.

Lemma 4.1 *Let L be a Lie algebra generated by finitely many elements $a_1, \dots, a_k$ such that each commutator in $a_1, \dots, a_k$ is ad-nilpotent. If L is not nilpotent then there exists a proper ideal N in L such that $R(L/N) = 0$.*

Proof Let us consider an ascending chain of ideals R_α of L defined as follows. Set $R_0 = 0$. If α is not a limit ordinal and $R_{\alpha-1}$ is already defined then $R_\alpha/R_{\alpha-1} = R(L/R_{\alpha-1})$. If α is a limit ordinal then $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$. Let N be the union of all ideals R_α. Then obviously $R(L/N) = 0$. We wish to show that $L \neq N$.

Suppose that $L = N$. Since L is finitely generated, it cannot be a union of proper ideals. Therefore $L = R_\alpha$, where the ordinal α is not limit. Then $L/R_{\alpha-2}$ is not nilpotent and so without any loss of generality we can assume that $\alpha = 2$. Since L/R_1 is locally nilpotent and finitely generated, we conclude that L/R_1 is nilpotent. Let c be a commutator in $a_1, \dots, a_k$ of maximal possible weight such that $c \notin R_1$. Then $c + R_1$ is central in L/R_1 and therefore $\langle c, R_1\rangle$ is an ideal in L. By [7, 1.3.1] $\langle c, R_1\rangle$ is locally nilpotent. The definition of R_1 now implies that $\langle c, R_1\rangle = R_1$ and $c \in R_1$. This contradicts our choice of c and completes the proof. ∎

Lemma 4.2 *Let L be a Lie algebra with $R(L) = 0$. Let N be an ideal of L and $C = C_L(N)$. Let $\bar{L} = L/C$ and $\bar{N}$ be the image of N in $\bar{L}$. Then $R(\bar{L}) = 0$ and $C_{\bar{L}}(\bar{N}) = 0$.*

Proof Since $N \cap C$ is an abelian ideal of L, it follows that $N \cap C = 0$. Suppose that $\bar{R} = R/C$ is a locally nilpotent ideal of $\bar{L}$. Then $\bar{N} \cap \bar{R}$ is a locally nilpotent ideal of $\bar{L}$ contained in $\bar{N}$. Since $N \cap C = 0$, we conclude that $N \cap R$ is locally nilpotent. Therefore $N \cap R = 0$ and $R \leq C$. This means that $\bar{R} = 0$.

Next, let $\bar{D} = D/C = C_{\bar{L}}(\bar{N})$. By the preceding paragraph $\bar{N} \cap \bar{D} = 0$. Therefore $N \cap D \leq C$. Now $N \cap C = 0$ implies $N \cap D = 0$. It follows that $D \leq C$ and $\bar{D} = 0$, as required. ∎

Lemma 4.3 *Let $L = L_0 + L_1$ be a non-abelian nilpotent $\mathbb{Z}_2$-graded Lie algebra generated by a set $Y \subseteq L_1$. Then there exists a non-zero commutator h in elements of Y such that $h \in L_0 \cap Z_2(L)$, where $Z_2(L)$ denotes the second term of the upper central series of L.*

Proof Let L be of nilpotency class $c \geq 2$ and let d be the even of the numbers $c - 1$, c. Then there exist elements $y_1, \ldots, y_d \in Y$ such that $h = [y_1, \ldots, y_d] \neq 0$ (otherwise, the nilpotency class of L would be less than c). We observe that $h \in L_0 \cap Z_2(L)$, as required. ∎

Theorem 4.4 *Let $L = L_0 + L_1$ be a $\mathbb{Z}_2$-graded Lie algebra over a commutative ring with unity in which 2 is invertible. Suppose that L_0 is abelian and L is generated by finitely many homogeneous elements $a_1, \ldots, a_k$ such that every commutator in $a_1, \ldots, a_k$ is ad-nilpotent. Then L is nilpotent.*

Proof Let L be a counterexample to the theorem. By 4.1 we can assume that $R(L) = 0$.

Our reasoning in the case $k = 2$ is slightly different from that in the case $k \geq 3$ so we will consider these cases separately. Suppose first that $k = 2$. It is convenient to denote a_1 by a and a_2 by b, so that $L = \langle a, b\rangle$. If any of a, b belongs to L_0 then L is generated by L_0 and a single ad-nilpotent odd element. Then by Proposition 2.1 the derived algebra $[L, L]$ is nilpotent. This contradicts the assumption $R(L) = 0$. Therefore $a, b \in L_1$.

Let A and B be the subspaces of L_1 spanned by all commutators of the form $[a, h_1, \ldots, h_i]$ and $[b, h_1, \ldots, h_i]$ respectively, where $h_1, \ldots, h_i \in L_0$. Then by Lemma 3.1 $L_1 = A + B$. Proposition 2.1 tells us that the subalgebras $\langle A\rangle$ and $\langle B\rangle$ generated by A and B respectively are nilpotent. Let $\langle B\rangle$ be nilpotent of class c and suppose first that $c \geq 2$. Then by 4.3 there exists a non-zero element $g \in Z_2(\langle B\rangle) \cap L_0$. Let G be the ideal of L generated by g. By Lemma 4.2 we can assume that $C_L(G) = 0$. Choose

$$0 \neq d \in L_1 \cap C_L(A) \cap C_L(g).$$

If N is the ideal of L generated by d then Lemma 3.4 says that $[N, G] = 0$. Hence $N = 0$, a contradiction.

We conclude that

$$L_1 \cap C_L(A) \cap C_L(g) = 0.$$

Recall that $g \in L_0$ and so g normalizes $L_1 \cap C_L(A)$. If d is any element in $L_1 \cap C_L(A)$ then for each positive integer m the commutator

$$[d, \underbrace{g, \ldots, g}_{m \text{ times}}]$$

also lies in $L_1 \cap C_L(A)$. Since g is ad-nilpotent, we can choose m such that

$$[d, \underbrace{g, \ldots, g}_{m \text{ times}}] \in C_L(g).$$

The conclusion is that

$$L_1 \cap C_L(A) = 0.$$

Recall that $\langle A \rangle$ is a non-zero nilpotent subalgebra generated by certain homogeneous elements. Therefore $Z(\langle A \rangle)$ contains some non-zero homogeneous element h_0. Our assumptions imply that $h_0 \in L_0$. Now observe that

$$[b, h_0] \in L_1 \cap C_L(A).$$

Indeed, the inclusion $[b, h_0] \in L_1$ is obvious. Let $a_1 = [a, h_1, \ldots, h_i]$ be a generator of $\langle A \rangle$ with $h_1, \ldots, h_i \in L_0$. We have

$$\big[a_1, [b, h_0]\big] = [a_1, b, h_0] + \big[b, [a_1, h_0]\big].$$

The first of the commutators on the right hand side above is zero because $[a_1, b] \in L_0$ and L_0 is abelian while the second one is zero because $h_0 \in Z(\langle A \rangle)$ and therefore $[a_1, h_0] = 0$. The assumption that $L_1 \cap C_L(A) = 0$ implies now that $[b, h_0] = 0$. We have just noted that $h_0 \in Z(\langle A \rangle)$ and therefore $[a, h_0] = 0$. So, we conclude that $h_0 \in Z(L)$. This contradicts the assumption that $R(L) = 0$.

Thus, we obtain a contradiction in all cases when $\langle B \rangle$ is not abelian. Consider now what happens if $\langle B \rangle$ is abelian. By a symmetric argument we can assume also that $\langle A \rangle$ is abelian. So, $\langle A \rangle = A$ and $\langle B \rangle = B$. Set now $g_0 = [a, b] \in L_0$. Let G_0 be the ideal of L generated by g. By Lemma 4.2 we can assume that $C_L(G_0) = 0$. Since g_0 is ad-nilpotent and normalizes $A \leq L_1$, it follows that $A \cap C_L(g_0)$ contains a non-zero element a_2. Since A and B are abelian, we observe that

$$[A, a_2] = [B, g_0, B] = 0.$$

We are again in a position to use Lemma 3.4. Let N be the ideal of L generated by a_2. By Lemma 3.4 $[N, G] = 0$, whence $N = 0$. This contradicts the choice of a_2 and completes the proof in the case $k = 2$.

Suppose now that $k \geq 3$. Without any loss of generality we assume that a_{k-1} and a_k do not commute. Set $a = a_1$, $b_1 = a_2, \ldots, b_{k-1} = a_k$. Arguing by induction on k we assume that the subalgebra $\langle b_1, \ldots, b_{k-1} \rangle$ is nilpotent. Let $B = \langle b_1, \ldots, b_{k-1} \rangle \cap L_1$. If $\langle b_1, \ldots, b_{k-1} \rangle \leq L_0$ then $L = \langle L_0, a \rangle$. In this case Proposition 2.1 tells us that $[L, L]$ is nilpotent which yields a contradiction. It follows that $B \neq 0$. Let us show that there exists a non-zero $g_1 \in L_0$ such that

$$[B, g_1, B] = 0.$$

If the subalgebra $\langle B \rangle$ is not abelian, then existence of g_1 with required properties follows from Lemma 4.3. If $\langle B \rangle$ is abelian then, since $\langle b_1, \dots, b_{k-1} \rangle$ is not abelian (recall that b_{k-2} and b_{k-1} do not commute), it follows that either b_{k-2} or b_{k-1} lies in L_0. For clarity's sake assume that $b_{k-1} \in L_0$. Note that $[B, b_{k-1}, B] = 0$ because b_{k-1} normalizes B and B generates an abelian subalgebra. Also it is clear that b_{k-1} satisfies the other conditions so we can take $g_1 = b_{k-1}$.

Thus, we proved that there exists an element $g_1 \in L_0$ such that $[B, g_1, B] = 0$. The remaining part of the proof practically does not differ from the settled above case $k = 2$.

Denote by A the subspace of L_1 spanned by all commutators of the form $[a, h_1, \dots, h_i]$ for suitable $h_1, \dots, h_i \in L_0$. By Proposition 2.1 $\langle A \rangle$ is nilpotent. Again we can show that existence of a non-zero element $d \in L_1 \cap C_L(A)$ leads to a contradiction. So, we assume

$$L_1 \cap C_L(A) = 0.$$

Arguing exactly as in the case $k = 2$ above, we choose $0 \neq h_0 \in Z(\langle A \rangle)$ and observe that

$$[L, h_0] \subseteq L_1 \cap C_L(A) = 0.$$

Since $Z(L) = 0$, it follows that $h_0 = 0$. This contradicts the choice of h_0 and completes the proof. ■

5 On residually finite groups

Let G be any group and p a prime. For $i \geq 1$ set

$$D_i(G) = \prod_{jp^k \geq i} \gamma_j(G)^{p^k},$$

where $\gamma_j(G)$ stands for the j-th term of the lower central series of G and for any subgroup $H \leq G$ the symbol H^{p^k} denotes the subgroup of H generated by the set $\{h^{p^k}; h \in H\}$. It follows that $\gamma_i(G) \leq D_i(G)$ for all $i \geq 1$ and that $\big(D_i(G)\big)_{i \geq 1}$ is a descending series of characteristic subgroups of G. This series is called the Lazard p-series of G.

Lemma 5.1 **([6, p. 250])** *For any group G and all i, $j \geq 1$ we have*

$$\big[D_i(G), D_j(G)\big] \leq D_{i+j}(G).$$

The above lemma shows that the Lazard p-series $\big(D_i(G)\big)_{i \geq 1}$ is a strongly central series of G in the sense of [6]. This fact enables us to associate a Lie algebra $L_p(G)$ to G. This is done as follows.

To simplify notation we write D_i in place of $D_i(G)$. For all $i \geq 1$ the quotient group D_i/D_{i+1} can be viewed as a vector space over the field with p elements $\mathbb{F}_p$. Let $L(G)$ denote their direct sum,

$$L(G) = \bigoplus_{i=1}^{\infty} D_i/D_{i+1}.$$

For arbitrary cosets $aD_{i+1} \in D_i/D_{i+1}$ and $bD_{j+1} \in D_j/D_{j+1}$ we define a bracket product

$$[aD_{i+1}; bD_{j+1}] = [a,b]D_{i+j+1}, \tag{5.2}$$

where $[a,b]$ denotes the group commutator $a^{-1}b^{-1}ab$. Lemma 5.1 implies that the product above is well defined in the sense that the right hand side of (5.2) does not depend on the coset representatives a, b. Extending now the product (5.2) linearly to the whole $L(G)$ we give $L(G)$ a structure of Lie algebra over the field $\mathbb{F}_p$. The subalgebra of $L(G)$ generated by D_1/D_2 will be denoted by $L_p(G)$.

The following lemma is due to M. Lazard [9, p. 128].

Lemma 5.3 *Let $a \in D_i - D_{i+1}$ and suppose that a is of order p^k. Let $\tilde{a} = aD_{i+1}$ be the corresponding element in $L_p(G)$. Then*

$$\bigl[L_p(G), \underbrace{\tilde{a}, \ldots, \tilde{a}}_{p^k \text{ times}}\bigr] = 0.$$

Next, we remark that if ϕ is an automorphism of G then ϕ naturally acts on each quotient D_i/D_{i+1}. This action induces an automorphism of $L_p(G)$. Slightly abusing terminology we will say that ϕ acts on $L_p(G)$.

Let G be a periodic group. For any element $x \in G$ of odd order let us use $x^{\frac{1}{2}}$ to denote the element $y \in \langle x \rangle$ such that $y^2 = x$. If G is a periodic $2'$-group and ϕ is an involutory automorphism of G then $x(x^{-\phi}x)^{-\frac{1}{2}} \in C_G(\phi)$. Indeed, let $h = x(x^{-\phi}x)^{-\frac{1}{2}}$. Using that $(x^{-\phi}x)^{\frac{1}{2}} = x^{-\phi}x(x^{-\phi}x)^{-\frac{1}{2}}$ we compute

$$h^{\phi} = x^{\phi}(x^{-1}x^{\phi})^{-\frac{1}{2}} = x^{\phi}(x^{-\phi}x)^{\frac{1}{2}} = x^{\phi}x^{-\phi}x(x^{-\phi}x)^{-\frac{1}{2}} = x(x^{-\phi}x)^{-\frac{1}{2}} = h.$$

Lemma 5.4 *Let G be a periodic $2'$-group admitting an involutory automorphism ϕ. Let I be the set of all elements $x \in G$ such that $x^{\phi} = x^{-1}$. Then $G = C_G(\phi)I$.*

Proof Let x be an arbitrary element of G, and let $h = x(x^{-\phi}x)^{-\frac{1}{2}}$. We know that $h \in C_G(\phi)$. Clearly, $g = (x^{-\phi}x)^{\frac{1}{2}} \in I$. Since $x = hg$, the lemma follows. ■

Lemma 5.5 *Let G be a periodic $2'$-group having an involutory automorphism ϕ. Suppose that N is normal ϕ-invariant subgroup of G. Then*

$$C_G(\phi)N/N = C_{G/N}(\phi).$$

Proof Let $x \in G$ and $xN \in C_{G/N}(A)$. Then $x^{-\phi}x \in N$. Since

$$x = x(x^{-\phi}x)^{-\frac{1}{2}}(x^{-\phi}x)^{\frac{1}{2}} \quad \text{and} \quad x(x^{-\phi}x)^{-\frac{1}{2}} \in C_G(\phi),$$

we obtain that $xN \in C_G(A)N/N$.

Thus, we showed that $C_{G/N}(\phi) \leq C_G(\phi)N/N$. The reverse inclusion is obvious. ■

Lemma 5.6 *Let G be a periodic $2'$-group having an involutory automorphism ϕ such that $C_G(\phi)$ is abelian. Assume that p is a prime. Then also the centralizer of ϕ in $L_p(G)$ is abelian.*

Proof This is immediate from 5.5 and the definition of $L_p(G)$. ■

Proposition 5.7 *Let G be a finitely generated residually finite p-group admitting an involutory automorphism ϕ such that $C_G(\phi)$ is abelian. Then $L_p(G)$ is nilpotent.*

Proof Let $L = L_p(G)$. The previous lemma allows us to view ϕ as an automorphism of L such that $C_L(\phi)$ is abelian. Set $L_0 = C_L(\phi)$ and $L_1 = \{x \in L; x^\phi = -x\}$. Then $L = L_0 + L_1$ and this provides a $\mathbb{Z}_2$-grading of L with L_0 abelian. By 5.4 G is generated by finitely many elements $a_1, \dots, a_s$ such that either $a_i^\phi = a_i$ or $a_i^\phi = a_i{}^{-1}$. Let $\tilde{a}_1, \dots, \tilde{a}_s$ be the generators of L corresponding to $a_1, \dots, a_s$. Since the subgroups $\langle a_i \rangle$ are ϕ-invariant, we conclude that $\tilde{a}_1, \dots, \tilde{a}_s$ are homogeneous in L. By the Lazard Lemma all commutators in the generators $\tilde{a}_1, \dots, \tilde{a}_s$ are ad-nilpotent. So by Theorem 4.4 L is nilpotent. The proof is complete. ■

Theorem 5.8 *Let G be a periodic residually finite $2'$-group having an involutory automorphism ϕ such that $C_G(\phi)$ is abelian. Then G is locally finite.*

Proof Suppose first that G is residually nilpotent. In this case G decomposes into direct product of maximal p-subgroups. It suffices to prove that any p-subgroup of G is locally finite. Therefore without any loss of generality we may assume that G is a p-group for a prime p. Next, let X be any finite subset of G. Obviously X is contained in some finitely generated ϕ-invariant subgroup S of G. Proposition 5.7 tells us that $L_p(S)$ is nilpotent. Therefore, by the argument of Zelmanov [18, p. 573], S is finite. We derive that any finite subset X of G is contained in a finite subgroup. Therefore G is locally finite.

Now we consider the general case. By 5.5 if Q is an arbitrary ϕ-invariant finite quotient of G then Q admits an involutory automorphism whose centralizer in Q is abelian. A result of Kovács and Wall [8] now yields that the derived group Q' is nilpotent. Together with residual finiteness of G this leads us to the conclusion that the derived group of G is residually nilpotent. By the preceding paragraph we obtain that G' is locally finite. This implies local finiteness of G. The proof is complete. ■

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