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ARITHMETIC PROPERTIES OF (k, ℓ)-REGULAR BIPARTITIONS

LIUQUAN WANG

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Abstract

Let $B_{k,\ell}(n)$ denote the number of (k, ℓ)-regular bipartitions of n. Employing both the theory of modular forms and some elementary methods, we systematically study the arithmetic properties of $B_{3,\ell}(n)$ and $B_{5,\ell}(n)$. In particular, we confirm all the conjectures proposed by Dou ['Congruences for (3,11)-regular bipartitions modulo 11', *Ramanujan J.* **40**, 535–540].

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1. Introduction

A partition of n is a nonincreasing sequence of positive integers whose sum equals n. For any positive integer ℓ, a partition is called ℓ-regular if none of its parts are divisible by ℓ. For example, $4 + 4 + 2 + 1$ is a 5-regular partition of 11 but $5 + 4 + 1 + 1$ is not 5-regular. We denote by $b_\ell(n)$ the number of ℓ-regular partitions of n and agree that $b_\ell(0) = 1$. The generating function of $b_\ell(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell; q^\ell)_\infty}{(q; q)_\infty}, \tag{1.1}$$

where

$$(a; q)_\infty := \prod_{k=1}^{\infty}(1 - aq^{k-1}), \quad |q| < 1.$$

Arithmetic properties of $b_\ell(n)$ have been extensively investigated (see [2, 4–6, 8, 10], [11, 14–18, 21–24]).

A bipartition (λ_1, λ_2) of n is an ordered pair of partitions (λ_1, λ_2) such that the sum of all the parts equals n. We say that (λ_1, λ_2) is (k, ℓ)-regular if λ_1 is k-regular and λ_2 is ℓ-regular and we denote by $B_{k,\ell}(n)$ the number of (k, ℓ)-regular bipartitions of n.

If $k = \ell$, $B_{k,k}(n)$ is also sometimes denoted by $B_k(n)$. From (1.1) it is easy to see that the generating function of $B_{k,\ell}(n)$ satisfies

$$\sum_{n=0}^{\infty} B_{k,\ell}(n)q^n = \frac{(q^k;q^k)_\infty(q^\ell;q^\ell)_\infty}{(q;q)_\infty^2}.$$

In 2015, by using Ramanujan's two modular equations of degree seven, Lin [12] found an infinite family of congruences for $B_7(n)$ modulo three: for $\alpha \geq 2$ and $n \geq 0$,

$$B_7\Big(3^\alpha n + \frac{5\cdot 3^{\alpha-1}-1}{2}\Big) \equiv 0 \pmod 3.$$

Lin [13] also proved an analogous result for $B_{13}(n)$: for $\alpha \geq 2$ and $n \geq 0$,

$$B_{13}(3^\alpha n + 2\cdot 3^{\alpha-1} - 1) \equiv 0 \pmod 3.$$

Recently, Wang [22] established numerous congruences for $B_5(n)$ modulo powers of 5. For instance, he proved that, for any integers $\alpha \geq 1$ and $n \geq 0$,

$$B_5\Big(5^{2\alpha-1}n + \frac{2\cdot 5^{2\alpha-1}-1}{3}\Big) \equiv 0 \pmod{5^\alpha}.$$

Fewer results for $B_{k,\ell}(n)$ are known when $k \neq \ell$. Dou [7] showed that

$$B_{3,11}\Big(3^\alpha n + \frac{5\cdot 3^{\alpha-1}-1}{2}\Big) \equiv 0 \pmod{11}$$

for all $\alpha \geq 2$ and $n \geq 0$ and conjectured that

$$\begin{aligned} B_{5,7}(7n+6) &\equiv 0 \pmod 7,\\ B_{3,7}(An+B) &\equiv 0 \pmod 2,\\ B_{3,7}(4n+3) &\equiv 0 \pmod 3,\\ B_{3,7}(Cn+D) &\equiv 0 \pmod 9, \end{aligned}$$

for $(A,B) \in \{(14,4),(14,10),(16,1),(28,6),(32,21)\}$ and $(C,D) \in \{(7,3),(7,4),(14,13),(21,6),(21,20),(25,3),(25,13),(25,18),(25,23)\}$.

The main goal of this paper is to prove Dou's congruences. To do this, we systematically study the arithmetic properties of $B_{k,\ell}(n)$ for $k \in \{3,5\}$. Along with many other interesting Ramanujan-type congruences, we confirm and improve Dou's conjectures using both elementary approaches and the theory of modular forms (see Theorem 1.3).

Firstly, we establish some double series representations for the generating functions of $B_{k,\ell}(n)$ modulo five and nine.

Theorem 1.1. *Let $m \geq 1$ be an integer and let $p \geq 5$ be a prime.*

(1) *We have*

$$\sum_{n=0}^{\infty} B_{3,m}(n)q^{24n+m+1} \equiv \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}(3i+1)q^{(6i+1)^2+m(6j+1)^2} \pmod 9. \quad (1.2)$$

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(2) *If* $(-m/p) = -1$ *and* n_0 *is an integer such that* $0 \le n_0 \le p-1$ *and* $24n_0 + m + 1 \equiv 0 \pmod{p}$*, then*

$$\sum_{n=0}^{\infty} B_{3,m}(pn+n_0)q^{24pn+24n_0+m+1} \equiv \sum_{\substack{6i+1=px,\\ x\in\mathbb{Z}}} \sum_{\substack{6j+1=py,\\ y\in\mathbb{Z}}} (-1)^{i+j}(3i+1)q^{p^2(x^2+my^2)} \pmod{9}. \tag{1.3}$$

(3) *We have*

$$\sum_{n=0}^{\infty} B_{3,5}(n)q^{8n+2} \equiv \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j}(2j+1)q^{(6i+1)^2+(2j+1)^2} \pmod{5}. \tag{1.4}$$

(4) *If* $p \equiv 3 \pmod{4}$ *and* n_0 *is an integer such that* $0 \le n_0 \le p-1$ *and* $4n_0 + 1 \equiv 0 \pmod{p}$*, then*

$$\sum_{n=0}^{\infty} B_{3,5}(pn+n_0)q^{8pn+8n_0+2} \equiv \sum_{\substack{6i+1=px,\\ x\in\mathbb{Z}}} \sum_{\substack{2j+1=py,\\ y\in\mathbb{Z}^+}} (-1)^{i+j}(2j+1)q^{p^2(x^2+y^2)} \pmod{5}. \tag{1.5}$$

(5) *If* $p \equiv 3 \pmod{4}$ *and* n_0 *is an integer such that* $0 \le n_0 \le p-1$ *and* $12n_0 + 5 \equiv 0 \pmod{p}$*, then*

$$\begin{aligned}\sum_{n=0}^{\infty} & B_{3,5}(3(pn+n_0)+1)q^{24pn+24n_0+10} \\ & \equiv -3 \sum_{\substack{6i+1=px,\\ x\in\mathbb{Z}}} \sum_{\substack{2j+1=py,\\ y\in\mathbb{Z}^+}} (-1)^{i+j}(2j+1)q^{p^2(x^2+9y^2)} \pmod{5}. \end{aligned} \tag{1.6}$$

(6) *We have*

$$\sum_{n=0}^{\infty} B_{5,m}(n)q^{24n+m+3} \equiv \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{i+j}(2i+1)q^{3(2i+1)^2+m(6j+1)^2} \pmod{5}. \tag{1.7}$$

(7) *If* $(-3m/p) = -1$ *and* n_0 *is an integer such that* $0 \le n_0 \le p-1$ *and* $24n_0 + m + 3 \equiv 0 \pmod{p}$*, then*

$$\begin{aligned}\sum_{n=0}^{\infty} & B_{5,m}(pn+n_0)q^{24(pn+n_0)+m+3} \\ & \equiv \sum_{\substack{2i+1=px,\\ x\in\mathbb{Z}^+}} \sum_{\substack{6j+1=py,\\ y\in\mathbb{Z}}} (-1)^{i+j}(2i+1)q^{p^2(3x^2+my^2)} \pmod{5}. \end{aligned} \tag{1.8}$$

THEOREM 1.2. *Let* $m \ge 1$ *and* $n \ge 0$ *be any integers.*

(1) *If* $24n+1$ *is a quadratic nonresidue modulo* m*, then* $B_{3,m}(n) \equiv 0 \pmod{9}$.
(2) *Under the conditions of Theorem 1.1(2), if* $24n + (24n_0 + m + 1)/p \not\equiv 0 \pmod{p}$*, then* $B_{3,m}(pn+n_0) \equiv 0 \pmod{9}$.

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(3) *Under the conditions of Theorem 1.1(4), if* $4n + (4n_0 + 1)/p \not\equiv 0 \pmod{p}$*, then* $B_{3,5}(pn + n_0) \equiv 0 \pmod{5}$.

(4) *Under the conditions of Theorem 1.1(5), if* $12n + (12n_0 + 5)/p \not\equiv 0 \pmod{p}$*, then* $B_{3,5}(3pn + 3n_0 + 1) \equiv 0 \pmod{5}$.

(5) *If* $\gcd(m, 3) = 1$ *and* $8n + 1$ *is a quadratic nonresidue modulo m, then* $B_{5,m}(n) \equiv 0 \pmod{5}$.

(6) *Under the conditions of Theorem 1.1(7), if* $24n + (24n_0 + m + 3)/p \not\equiv 0 \pmod{p}$ *and* $\gcd(m, 3) = 1$*, then* $B_{5,m}(pn + n_0) \equiv 0 \pmod{5}$.

THEOREM 1.3. *Let* $n \geq 0$ *be any integer.*

(1)

$$B_{3,5}(3n + 2) \equiv 0 \pmod{5}, \tag{1.9}$$
$$B_{5,11}(5n + 4) \equiv 0 \pmod{5}, \tag{1.10}$$
$$B_{3,5}(5n + 3) \equiv B_{3,5}(5n + 4) \equiv 0 \pmod{9}, \tag{1.11}$$
$$B_{3,5}(121n + 11r + 8) \equiv 0 \pmod{9}, \quad \forall r, 0 \leq r \leq 10 \text{ and } r \neq 2, \tag{1.12}$$
$$B_{3,7}(7n + r) \equiv 0 \pmod{9}, \quad r \in \{3, 4, 6\}, \tag{1.13}$$
$$B_{3,7}(25n + r) \equiv 0 \pmod{9}, \quad r \in \{3, 13, 18, 23\}, \tag{1.14}$$
$$B_{3,5}(147n + 21r + 19) \equiv 0 \pmod{5}, \quad r \in \{0, 1, 3, 4, 5, 6\}, \tag{1.15}$$
$$B_{5,7}(7n + r) \equiv 0 \pmod{5}, \quad r \in \{2, 4, 5\}. \tag{1.16}$$

(2)

$$B_{3,7}(4n + 1) \equiv -B_{3,7}(n) \pmod{3}, \tag{1.17}$$
$$B_{3,7}\left(2^{2\alpha}n + \frac{5 \cdot 2^{2\alpha-1} - 1}{3}\right) \equiv 0 \pmod{3}, \quad \forall \alpha \geq 1. \tag{1.18}$$

(3)

$$B_{3,5}(27n + 11) \equiv 0 \pmod{25}, \tag{1.19}$$
$$B_{3,7}(14n + r) \equiv 0 \pmod{2}, \quad r \in \{4, 6, 10\}, \tag{1.20}$$
$$B_{3,7}(16n + 1) \equiv 0 \pmod{2}, \tag{1.21}$$
$$B_{3,7}(32n + 21) \equiv 0 \pmod{2}, \tag{1.22}$$
$$B_{5,7}(7n + 6) \equiv 0 \pmod{7}, \tag{1.23}$$
$$B_{5,7}(49n + 20) \equiv 0 \pmod{49}. \tag{1.24}$$

The paper is organised as follows. In Section 2, we prove Theorems 1.1 and 1.2 and the first group of congruences in Theorem 1.3. In Section 3, we use q-series techniques to prove the second group of congruences in Theorem 1.3. In Section 4, we apply Radu's modular approach to complete the proof of Theorem 1.3.

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2. Proof of Theorems 1.1, 1.2 and Theorem 1.3(1)

LEMMA 2.1.

$$(q;q)_\infty \sum_{n=1}^{\infty} \left(\frac{n}{3}\right)\frac{q^n}{1-q^n} = \sum_{j=-\infty}^{\infty} (-1)^j j q^{j(3j+1)/2}.$$

PROOF. By Jacobi's triple product identity (see, for example, [3, Theorem 1.3.3]),

$$\sum_{j=-\infty}^{\infty} (-1)^j z^j q^{j(3j+1)/2} = (zq^2;q^3)_\infty (q/z;q^3)_\infty (q^3;q^3)_\infty.$$

Logarithmic differentiation with respect to z yields

$$\frac{\sum_{j=-\infty}^{\infty}(-1)^j j z^{j-1} q^{j(3j+1)/2}}{(zq^2;q^3)_\infty (q/z;q^3)_\infty (q^3;q^3)_\infty} = \sum_{n=0}^{\infty}\left(\frac{z^{-2}q^{3n+1}}{1-z^{-1}q^{3n+1}} - \frac{q^{3n+2}}{1-zq^{3n+2}}\right),$$

and the lemma follows by setting $z=1$. □

PROOF OF THEOREM 1.1. (1) From [1, Entry 18.2.20, page 406],

$$\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} = 1 - 3a(q) + 9a(q^3), \tag{2.1}$$

where

$$a(q) = \sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n}.$$

By (2.1),

$$(q^3;q^3)_\infty \equiv (q;q)_\infty^3 (1+3a(q)) \pmod 9.$$

By Lemma 2.1 and Euler's pentagonal number theorem,

$$\begin{aligned}
&\frac{(q^3;q^3)_\infty (q^m;q^m)_\infty}{(q;q)_\infty^2}\\
&\quad\equiv (q;q)_\infty (1+3a(q))(q^m;q^m)_\infty\\
&\quad\equiv \left(\sum_{i=-\infty}^{\infty}(-1)^i q^{i(3i+1)/2} + 3\sum_{i=-\infty}^{\infty}(-1)^i i q^{i(3i+1)/2}\right)\sum_{j=-\infty}^{\infty}(-1)^j q^{mj(3j+1)/2}\\
&\quad\equiv \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}(3i+1)q^{i(3i+1)/2+mj(3j+1)/2} \pmod 9.
\end{aligned}$$

This proves (1.2).

(2) Since $(-m/p) = -1$,

$$(6i+1)^2 + m(6j+1)^2 \equiv 0 \pmod p \Leftrightarrow 6i+1 \equiv 6j+1 \equiv 0 \pmod p.$$

Extracting those terms on both sides of (1.2) for which the powers of q are divisible by p yields (1.3).

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(3) By the binomial theorem and Jacobi's identity (see [3, Theorem 1.3.9]),

$$\sum_{n=0}^{\infty} B_{3,5}(n)q^n = \frac{(q^3;q^3)_\infty (q^5;q^5)_\infty}{(q;q)_\infty^2} \equiv (q^3;q^3)_\infty (q;q)_\infty^3 \tag{2.2}$$

$$\equiv \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j}(2j+1)q^{3i(3i+1)/2+j(j+1)/2} \pmod{5}, \tag{2.3}$$

from which (1.4) follows.

(4) Since $(-1/p) = -1$,

$$(6i+1)^2 + (2j+1)^2 \equiv 0 \pmod{p} \quad \Leftrightarrow \quad 6i+1 \equiv 2j+1 \equiv 0 \pmod{p}.$$

Extracting those terms on both sides of (1.4) for which the powers of q are divisible by p, yields (1.5).

(5) Let

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} a(n)q^n.$$

By Jacobi's identity,

$$(q;q)_\infty^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2}.$$

Extracting those terms on both sides for which the powers of q are of the form $3n+1$,

$$\sum_{n=0}^{\infty} a(3n+1)q^{3n+1} = -3q \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{9k(k+1)/2} = -3q(q^9;q^9)_\infty^3.$$

From the above equation and (2.2),

$$\sum_{n=0}^{\infty} B_{3,5}(3n+1)q^n \equiv -3(q;q)_\infty (q^3;q^3)_\infty^3$$

$$\equiv -3 \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j}(2j+1)q^{i(3i+1)/2+3j(j+1)/2} \pmod{5},$$

and hence

$$\sum_{n=0}^{\infty} B_{3,5}(3n+1)q^{24n+10} \equiv -3 \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j}(2j+1)q^{(6i+1)^2+9(2j+1)^2} \pmod{5}. \tag{2.4}$$

Since $p \equiv 3 \pmod{4}$, $(-9/p) = -1$ and thus

$$(6i+1)^2 + 9(2j+1)^2 \equiv 0 \pmod{p} \Leftrightarrow 6i+1 \equiv 2j+1 \equiv 0 \pmod{p}.$$

Extracting those terms on both sides of (2.4) for which the powers of q are divisible by p, we obtain (1.6).

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(6) By the binomial theorem,

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,m}(n)q^n &= \frac{(q^5;q^5)_\infty (q^m;q^m)_\infty}{(q;q)_\infty^2} \equiv (q;q)_\infty^3 (q^m;q^m)_\infty \\ &\equiv \sum_{i=0}^{\infty} (-1)^i (2i+1) q^{i(i+1)/2} \sum_{j=-\infty}^{\infty} (-1)^j q^{mj(3j+1)/2} \pmod 5, \end{aligned} \tag{2.5}$$

from which (1.7) follows.

(7) Since $(-3m/p) = -1$,

$$3(2i+1)^2 + m(6j+1)^2 \equiv 0 \pmod p \Leftrightarrow 2i+1 \equiv 6j+1 \pmod p.$$

Extracting those terms on both sides of (1.7) for which the powers of q are divisible by p, we obtain (1.8). □

Proof of Theorem 1.2. If $24n+1$ is a quadratic nonresidue modulo m, then there are no integers i, j such that

$$24n + m + 1 = (6i+1)^2 + m(6j+1)^2.$$

Hence by (1.2) we see that $B_{3,m}(n) \equiv 0 \pmod 9$. This proves Theorem 1.2(1). A similar argument involving (1.7) yields Theorem 1.2(5).

It follows from Theorem 1.1(2) that if $24pn + 24n_0 + m + 1 \not\equiv 0 \pmod{p^2}$, that is, if $24n + (24n_0 + m + 1)/p \not\equiv 0 \pmod p$, then $B_{3,m}(pn + n_0) \equiv 0 \pmod 9$. This proves Theorem 1.2(2). Similar arguments involving parts (4), (5) and (7) of Theorem 1.1 yield parts (3), (4) and (6) of Theorem 1.2. □

Proof of Theorem 1.3(1). Observe that for any integer j, the residue of $j(j+1)/2$ modulo three is zero or one. Thus from (2.3) we deduce theorem 1.3.

To prove (1.10), we set $m = 11$ in (1.7). Since $(\frac{-3}{5}) = -1$,

$$3(2i+1)^2 + 11(6j+1)^2 \equiv 0 \pmod 5 \quad \Leftrightarrow \quad 2i+1 \equiv 6j+1 \equiv 0 \pmod 5.$$

From (1.7) we conclude that $B_{5,11}(5n+4) \equiv 0 \pmod 5$.

Set $m = 5$ in Theorem 1.2(1). If $n \equiv 3$ or $4 \pmod 5$, then $24n+1 \equiv 2$ or $3 \pmod 5$, which are quadratic nonresidues modulo five. This proves (1.11).

Note that $(\frac{-5}{11}) = -1$. Let $p = 11$ and $m = 5$ in Theorem 1.2(2). We choose $n_0 = 8$ so that $24n_0 + m + 1 \equiv 0 \pmod{11}$. Then, for any $0 \le r \le 10$ and $r \ne 2$, $24(11n + r) + 18 \not\equiv 0 \pmod{11}$. From Theorem 1.2(2) we deduce (1.12).

Similarly, setting $m = 7$ in Theorem 1.2(1), $(p, m, n_0) = (5, 7, 3)$ in Theorem 1.2(2), $(p, n_0) = (7, 6)$ in Theorem 1.2(4) and $m = 7$ in Theorem 1.2(5), we get (1.13), (1.14), (1.15) and (1.16), respectively. □

Remark 2.2. By making different choices of p, m and n_0 in Theorem 1.2, we can obtain other analogous sets of congruences.

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3. Proof of Theorem 1.3(2)

LEMMA 3.1. *Define* $c(n)$ *by*

$$(q;q)_\infty(q^7;q^7)_\infty := \sum_{n=0}^{\infty} c(n)q^n.$$

Then, for all $n \geq 0$,

$$c(4n+1) = -c(n) \quad \text{and} \quad c(4n+3) = 0.$$

PROOF. Throughout the proof, we shall use the notation

$$[z;q]_\infty = (z;q)_\infty(q/z;q)_\infty \quad \text{and} \quad [z_1, z_2, \ldots, z_n; q]_\infty = [z_1;q]_\infty[z_2;q]_\infty \cdots [z_n;q]_\infty.$$

We will employ the addition formula (see [9, Exercise 2.16, page 61])

$$[x\lambda, x/\lambda, \mu\nu, \mu/\nu; q]_\infty = [x\nu, x/\nu, \lambda\mu, \mu/\lambda; q]_\infty + \frac{\mu}{\lambda}[x\mu, x/\mu, \lambda\nu, \lambda/\nu; q]_\infty. \tag{3.1}$$

Note that

$$(q;q)_\infty(q^7;q^7)_\infty = (q^2;q^2)_\infty(q^{14};q^{14})_\infty[q, q^3, q^5, q^7; q^{14}]_\infty.$$

Setting $(x, \lambda, \mu, \nu, q) \to (-q^4, q^3, q^4, -q, q^{14})$ in (3.1),

$$[q, q^3, q^5, q^7; q^{14}]_\infty - [-q, -q^3, -q^5, -q^7; q^{14}]_\infty = -q[-q^8, -1, -q^4, -q^2; q^{14}]_\infty.$$

This implies that

$$\begin{aligned}\sum_{n=0}^{\infty} c(2n+1)q^n &= -\frac{1}{2}(q;q)_\infty(q^7;q^7)_\infty[-q^4, -1, -q^2, -q; q^7]_\infty \\ &= -(q;q)_\infty(q^7;q^7)_\infty(-q;q)_\infty(-q^7;q^7)_\infty = -(q^2;q^2)_\infty(q^{14};q^{14})_\infty.\end{aligned}$$

It follows that $c(4n+3) = 0$ and

$$\sum_{n=0}^{\infty} c(4n+1)q^n = -(q;q)_\infty(q^7;q^7)_\infty,$$

which yields $c(4n+1) = -c(n)$. □

PROOF OF THEOREM 1.3(2). By the binomial theorem,

$$\sum_{n=0}^{\infty} B_{3,7}(n)q^n \equiv (q;q)_\infty(q^7;q^7)_\infty \pmod 3.$$

Thus $B_{3,7}(n) \equiv c(n) \pmod 3$. By Lemma 3.1, we deduce (1.17) and

$$B_{3,7}(4n+3) \equiv 0 \pmod 3. \tag{3.2}$$

Then (1.18) follows from (1.17), (3.2) and induction on α. □

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4. Modular proof of Theorem 1.3(3)

For a positive integer M, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of M. If $r \in R(M)$ and $1 = \delta_1 < \cdots < \delta_k = M$ are the positive divisors of M, we write $r = (r_\delta) = (r_{\delta_1}, \ldots, r_{\delta_k})$. Define $c_r(n)$ by

$$\prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} := \sum_{n=0}^{\infty} c_r(n) q^n.$$

In this section, we recall the approach to proving congruences for $c_r(n)$ developed by Radu [19, 20]. Compared with the classical method which uses Sturm's bound alone, this approach reduces the number of coefficients that one must check.

Let N be a positive integer and define

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \pmod{N} \right\} \quad \text{and} \quad \Gamma_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \;\middle|\; h \in \mathbb{Z} \right\}.$$

The index of $\Gamma_0(N)$ in Γ is

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}).$$

Let m be a positive integer. For any integer s, let $[s]_m$ denote the residue class of s in $\mathbb{Z}/m\mathbb{Z}$. Thus $[x]_m = [y]_m$ if and only if $x \equiv y \pmod{m}$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $\mathbb{S}_m \subseteq \mathbb{Z}_m$ be the set of all squares in $\mathbb{Z}_m^*$. For $t \in \{0, 1, \ldots, m-1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subseteq \{0, 1, \ldots, m-1\}$ by

$$P_{m,r}(t) := \left\{ t' \mid \exists [s]_{24m} \in \mathbb{S}_{24m} \text{ such that } t' \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m} \right\}.$$

Definition 4.1. Suppose m, M and N are positive integers, $r = (r_\delta) \in R(M)$ and $t \in \{0, 1, \ldots, m-1\}$. Let $\kappa = \kappa(m) := \gcd(m^2 - 1, 24)$ and write

$$\prod_{\delta|M} \delta^{|r_\delta|} = 2^s \cdot j,$$

where s and j are nonnegative integers with j odd. The set Δ^* consists of all tuples $(m, M, N, (r_\delta), t)$ satisfying these conditions and all of the following.

(1) Each prime divisor of m is also a divisor of N.
(2) $\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$.
(3) $\kappa N \sum_{\delta|M} r_\delta \, mN/\delta \equiv 0 \pmod{24}$.
(4) $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$.
(5) $24m/\gcd(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m)|N$.
(6) If $2|m$, then either $4|\kappa N$ and $8|Ns$ or $2|s$ and $8|N(1-j)$.

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Let m, M, N be positive integers. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $r \in R(M)$ and $r' \in R(N)$, set

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a+\kappa\lambda c), mc)}{\delta m}$$

and

$$p^*_{r'}(\gamma) := \frac{1}{24} \sum_{\delta|N} r'_\delta \frac{\gcd^2(\delta, c)}{\delta}.$$

Lemma 4.2 [19, Lemma 4.5]. *Let u be a positive integer, $(m, M, N, t, r = (r_\delta)) \in \Delta^*$ and $a = (a_\delta) \in R(N)$. Let $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ be a complete set of representatives of the double cosets of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p^*_{r'}(\gamma_i) \geq 0$ for all $1 \leq i \leq n$. Let $t_{\min} = \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24}\bigg(\bigg(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\bigg)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta r'_\delta\bigg) - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{\min}}{m}.$$

If the congruence $c_r(mn + t') \equiv 0 \pmod u$ holds for all $t' \in P_{m,r}(t)$ and $0 \leq n \leq \lfloor \nu \rfloor$, then it holds for all $t' \in P_{m,r}(t)$ and $n \geq 0$.

To apply this lemma, we need to find a complete set of representatives of the double coset in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$.

Lemma 4.3. *If N or $\frac{1}{2}N$ is a square-free integer, then*

$$\bigcup_{\delta|N} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \Gamma_\infty = \Gamma.$$

Proof. If $4 \nmid N$, then N is a square-free integer and the assertion follows from [20, Lemma 2.6].

Suppose $4|N$, say, $N = 4N_0$. Then $\frac{1}{2}N$ is square-free, so N_0 is an odd square-free integer. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. From the proof of [20, Lemma 2.6], it suffices to show that

$$ch \equiv d - \frac{c}{\gcd(c, N)} \pmod{N/\gcd(c, N)} \tag{4.1}$$

has a solution $h \in \mathbb{Z}$. If c is odd, then clearly $\gcd(c, N/\gcd(c, N)) = 1$ and therefore (4.1) has an integer solution. Now suppose that c is even and write $c = 2c_0$. If c_0 is even, then $N/\gcd(c, N)$ is odd and thus $\gcd(c, N/\gcd(c, N)) = 1$, which implies that (4.1) has an integer solution. Finally, if c_0 is odd, then, since $ad - bc = 1$, we see that d must be odd. Thus (4.1) is equivalent to

$$c_0 h \equiv \tfrac{1}{2}(d - c_0/\gcd(c_0, N)) \pmod{N_0/\gcd(c_0, N_0)}.$$

Since $\gcd(c_0, N_0/\gcd(c_0, N_0)) = 1$, the above equation has integer solutions. □

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TABLE 1. Parameters used in the proof of Theorem 1.3.

Equation	Gen Fn	m	M	t	u	$\lfloor\nu\rfloor$	N	$(r'_\delta)\in R(N)$
(1.20)	$1^{-2}3^{1}7^{1}$	14	21	4	2	247	42	(0, 0, 64, 0, 0, 0, 0, 0)
(1.21)	$1^{-2}3^{1}7^{1}$	16	21	1	2	582	84	(0, 0, 74, 0, 0, 0, 0, 0, 0, 0, 0, 0)
(1.22)	$1^{-2}3^{1}7^{1}$	32	21	21	2	568	42	(0, 0, 147, 0, 0, 0, 0, 0)
(1.23)	$1^{-2}5^{1}7^{1}$	7	35	6	7	22	35	(12, 0, 0, 0)
(1.24)	$1^{47}5^{1}7^{-6}$	49	35	20	49	73	35	(0, 0, 0, 0)

PROOF OF THEOREM 1.3(3). By the binomial theorem,

$$\sum_{n=0}^{\infty} B_{3,5}(n)q^n \equiv \frac{(q^3;q^3)_\infty(q^5;q^5)_\infty}{(q;q)_\infty^2}\cdot\frac{(q;q)_\infty^{25}}{(q^5;q^5)_\infty^5} \equiv \frac{(q^3;q^3)_\infty(q;q)_\infty^{23}}{(q^5;q^5)_\infty^4} \pmod{25}.$$

Let

$$\frac{(q^3;q^3)_\infty(q;q)_\infty^{23}}{(q^5;q^5)_\infty^4} = \sum_{n=0}^{\infty} b(n)q^n.$$

We choose

$$(m,M,N,r=(r_\delta),t) = (27,15,15,r=(23,1,-4,0),11).$$

It is easy to verify that $(m,M,N,r,t)\in\Delta^*$ and we compute $P_{m,r}(t)=\{11\}$.

From now on, we set $\gamma_\delta := \left(\begin{smallmatrix}1 & 0\\ \delta & 1\end{smallmatrix}\right)$. By Lemma 4.3, $\{\gamma_\delta|\delta|N\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let

$$r' = (r'_1, r'_3, r'_5, r'_{15}) = (0,0,0,0)\in R(N).$$

It is easy to verify that $p_{m,r}(\gamma_\delta)+p^*_{r'}(\gamma_\delta)\ge 0$ for each $\delta|N$. We also compute that the upper bound in Lemma 4.2 is $\lfloor\nu\rfloor = 5$. Using *Mathematica*, we have verified that $b(27n+11)\equiv 0 \pmod{25}$ for $n\le 5$. Thus, by Lemma 4.2, we conclude that $b(27n+11)\equiv 0\pmod{25}$ for any $n\ge 0$. This completes the proof of (1.19).

Congruences (1.20)–(1.24) can be proved in a similar way and we summarise the choices of variables in Table 1. Given a positive integer M, we are considering

$$\sum_{n=0}^{\infty} B_{k,\ell}(n)q^n \equiv \prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta} \pmod{u}.$$

We abbreviate the generating function in the right hand side as $\prod \delta^{r_\delta}$. In Table 1, the second column describes the generating function under consideration and the third, fifth and sixth columns specify the integers m, t and u for which we wish to show that

$$B_{k,\ell}(mn+t')\equiv 0 \pmod{u} \quad \forall t'\in P_{m,r}(t) \text{ and } \forall n\ge 0. \qquad \square$$

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LIUQUAN WANG, Department of Mathematics,
National University of Singapore, Singapore, 119076
e-mail: wangliuquan@u.nus.edu, mathlqwang@163.com