



# The Geometry and Fundamental Group of Permutation Products and Fat Diagonals

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*Abstract.* Permutation products and their various “fat diagonal” subspaces are studied from the topological and geometric points of view. We describe in detail the stabilizer and orbit stratifications related to the permutation action, producing a sharp upper bound for its depth and then paying particular attention to the geometry of the diagonal stratum. We exhibit an expression for the fundamental group of any permutation product of a connected space  $X$  having the homotopy type of a CW complex in terms of  $\pi_1(X)$  and  $H_1(X; \mathbb{Z})$ . We then prove that the fundamental group of the configuration space of  $n$ -points on  $X$ , of which multiplicities do not exceed  $n/2$ , coincides with  $H_1(X; \mathbb{Z})$ . Further results consist in giving conditions for when fat diagonal subspaces of manifolds can be manifolds again. Various examples and homological calculations are included.

## 1 Introduction

In this paper, the first in a series, we answer several basic questions about the topology and geometry of permutation products and various analogs of configuration spaces. These are spaces of relevance to all of algebraic and geometric topology.

Let  $\Gamma$  be a subgroup of the  $n$ -th symmetric group  $\mathfrak{S}_n$ , and define the *permutation product*  $\Gamma P^n(X)$  to be the quotient of  $X^n$  by the permutation action of  $\Gamma$  on coordinates. The prototypical example being of course the  $n$ -th symmetric product  $SP^n(X)$ , which corresponds to  $\Gamma = \mathfrak{S}_n$ . The cyclic product  $CP^n(X)$  corresponds on the other hand to  $\Gamma = \mathbb{Z}_n$  the cyclic group. These spaces become interesting very quickly as is illustrated, for example, with  $CP^3(T) := T^3/\mathbb{Z}_3$ , the cyclic product of order 3 of an elliptic curve  $T$ , which has the structure of an explicit bundle over  $T$  (see Proposition 2.4).

Permutation products were of course studied early on, and a functorial description of their homology is described in [9]. The topology of the spaces  $\Gamma P^n(X)$  depends, as one might expect, on the way the group  $\Gamma$  embeds in  $\mathfrak{S}_n$ . These are stratified spaces [14]. In general, the action of a finite group  $\Gamma$  on a space  $Y$  stratifies both  $Y$  and its quotient  $Y/\Gamma$  according to *stabilizer* and *orbit* types respectively, and the bulk of this work is understanding these stratifications in the case of  $\Gamma$  acting on  $X^n$  by permutations.

In Section 3 we discuss the notion of “depth” of a group  $\Gamma \subset \mathfrak{S}_n$  acting by permutation on  $X^n$  and then compare it to the length  $\ell(\Gamma)$  of the group, which we recall is the length of the longest chain of subgroups in  $\Gamma$ . The depth is bounded above by  $\ell(\Gamma)$ , and the interesting part is to observe that this upper-bound is attained for permutation products derived from the permutation representation of  $\Gamma$  (Theorem 3.7).

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In the case of a permutation product of a manifold, we can refine our understanding of the associated strata. We give in Section 4 a handy description of the tubular neighborhood of the diagonal  $X$  in  $\Gamma P^n X$  as a fiberwise cone on a fiberwise quotient bundle associated with a sphere bundle over  $X$  (Proposition 4.1).

We next turn to computing the fundamental group. We will assume throughout that  $X$  is of the homotopy type of a CW complex. We recall that a permutation subgroup  $\Gamma \subset \mathfrak{S}_n := \text{Aut}\{1, 2, \dots, n\}$  is called transitive if any  $i$  is mapped to any  $j$  via the action. The following prototypical result is derived in Section 5:

If  $\Gamma$  is a transitive subgroup of  $\mathfrak{S}_n$  acting on  $X^n$  by permutation of coordinates and  $n \geq 2$ , then  $\pi_1(\Gamma P^n X)$  is abelian and there is an isomorphism  $\pi_1(\Gamma P^n X) \cong H_1(X; \mathbb{Z})$ .

Very early on, P. A. Smith in [23] showed that  $\pi_1(\text{SP}^2 X)$  is abelian, and his simple geometric argument extends in an elementary and self-contained manner to cover the more general case (see Section 5). Moreover, since any non transitive subgroup of  $\mathfrak{S}_n$  is up to conjugation contained in a standard Young subgroup  $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$ , permutation products of non transitive subgroups can be reduced to transitive ones of smaller size and a general statement for  $\pi_1(\Gamma P^n X)$  can be given as follows.<sup>1</sup> We write  $\Omega := \{1, \dots, n\}$  and again  $\mathfrak{S}_n = \text{Aut}(\Omega)$ . We say an orbit is *non-trivial* if it is not reduced to a single point.

**Theorem 1.1** *If  $\Gamma \subset \mathfrak{S}_n$ , then*

$$\pi_1(\Gamma P^n X) \cong \pi_1(X)^{n - \sum_i k_i} \times H_1(X; \mathbb{Z})^r$$

where  $r$  is the number of non-trivial transitive orbits under the action of  $\Gamma$  on  $\Omega$  and  $k_1, k_2, \dots, k_r$  are the sizes of these non-trivial orbits.

In the case  $\Gamma = \mathfrak{S}_n$ , we can further compare the higher homotopy groups  $\pi_i(\text{SP}^n X)$  to the corresponding homology groups, and these coincide provided  $0 \leq i \leq r + 2n - 1$ , where  $r \geq 1$  is the connectivity of  $X$  and  $n > 1$  (Theorem 5.9). This recovers an original result of Dold and Puppe [10].

The second part of this paper deals with the fat diagonal subspaces and their complements. For a given positive integer  $d$ , define  $B_d(X, n)$  to be the subspace of  $\text{SP}^n X$  of all unordered tuples  $[x_1, \dots, x_n]$ , or *configurations*, such that at least  $d$  entries are equal. These are the spaces we refer to as *fat diagonals*. We have a decreasing filtration

$$B_1(X, n) = \text{SP}^n(X) \supset B_2(X, n) \supset \dots \supset B_n(X, n) = X,$$

that interpolates between the symmetric product and the thin diagonal. For example,  $B_2(X, n)$  is the standard fat diagonal in  $\text{SP}^n X$  consisting of configurations with at least two points coinciding. In Section 6 we characterize  $B_d(-, n)$  as homotopy functors with certain properties, and then give precise conditions for  $B_d(X, n)$  to be a manifold if  $X$  is (Theorem 6.7).

As in the case of permutation products, it turns out that the fundamental group of fat diagonals abelianizes “whenever it can”. The following is proved in Section 7 using an adaptation of Smith’s argument.

<sup>1</sup>This result is motivated by a question of Mathai Varghese to the first author.

**Theorem 1.2** *Let  $X$  be a based connected topological space. Then  $\pi_1(B_d(X, n)) \cong H_1(X; \mathbb{Z})$ , provided that  $1 \leq d \leq \frac{n}{2}$ .*

Note that the case  $d > n/2$  presents no particular interest, since in this case  $B_d(X, n) \cong X \times \text{SP}^{n-d}X$ . More generally one shows that if  $\Gamma$  is a “ $d$ -transitive” subgroup of  $\mathfrak{S}_n$ , then  $\pi_1(F_d(X, n)/\Gamma)$  is abelian for  $n \geq 2$  and  $1 \leq d \leq \frac{n}{2}$  (Corollary 7.3). Here  $F_d(X, n) \subset X^n$  is the so-called “ $d$ -equal subspace” consisting of all tuples  $(x_1, \dots, x_n)$  such that  $x_{i_1} = \dots = x_{i_d}$  for some choice of sequence  $i_1 < \dots < i_d$ .

Euler characteristics of the spaces  $B_d(X, n)$  and their complements  $B^d(X, n) := \text{SP}^n(X) - B_{d+1}(X, n)$  will be considered in a later work. The fundamental group of  $B^d(X, n)$  is determined in upcoming work by the first author and Inès Saihi.

## 2 A Geometric Sample

All spaces in this paper are assumed to be first countable, locally compact, Hausdorff, and path-connected. Elements of  $\Gamma\text{P}^n(X) = X^n/\Gamma$ , with  $\Gamma \hookrightarrow \mathfrak{S}_n$ , are written as equivalence classes  $[x_1, \dots, x_n] := \pi(x_1, \dots, x_n)$ , where  $\pi: X^n \rightarrow \Gamma\text{P}^n(X)$  is the quotient map. It will be convenient to call an element in  $\Gamma\text{P}^n(X)$  a *configuration*. When  $n = 2$ , the only permutation product of interest is the symmetric square  $\text{SP}^2(X)$ .

The homotopy type of a permutation product  $\Gamma\text{P}^nX$  depends on the way  $\Gamma$  embeds in  $\mathfrak{S}_n$ . Take, for example,  $\Gamma = \mathbb{Z}_2$  and embed it in  $\mathfrak{S}_4$  by sending the generator to either (12) or to (13)(24). Then the permutation products corresponding to these embeddings are respectively

$$\Gamma_1\text{P}^4(X) = \text{SP}^2(X) \times X \times X \quad \text{and} \quad \Gamma_2\text{P}^4(X) = \text{SP}^2(X \times X).$$

Of course conjugate embeddings yield homeomorphic quotients.

**Example 2.1** Permutation products of the circle are particularly interesting. They are special cases of what are called *toroidal orbifolds* (see [1]). Following an argument of Morton [21], we can see that for  $\Gamma \subset \mathfrak{S}_n$ , there is a bundle projection  $\Gamma\text{P}^n(S^1) \rightarrow S^1$  with fiber identified with the quotient  $V/G_0$ , where  $V$  and  $G_0$  are defined as follows. First let  $\Gamma$  and  $\mathbb{Z}^n$  act on  $\mathbb{R}^n$  by permutations and translations respectively. Define  $G$  to be the affine subgroup generated by  $\Gamma$  and  $\mathbb{Z}^n$ . Then  $V$  is the hyperplane  $\{x_1 + \dots + x_n = 0\} \subset \mathbb{R}^n$  and  $G_0 \subset G$  is the stabilizer of  $V$ ,  $G/G_0 \cong \mathbb{Z}$ . In the case of symmetric products, Morton shows that the quotient  $V/G_0$  is an  $(n - 1)$ -simplex, and so the product map  $\text{SP}^n(S^1) \rightarrow S^1$  is a disk bundle that is furthermore orientable if and only if  $n$  is odd [21]. For example  $\text{SP}^2(S^1)$  is the Möbius band, and the diagonal copy of  $S^1 \subset \text{SP}^2(S^1)$  is precisely the boundary of the band. In the case of the cyclic product  $\text{CP}^3(S^1)$  for example, the fiber of  $\text{CP}^3(S^1) \rightarrow S^1$  is a copy of  $S^2$  (Proposition 2.3 gives a complete treatment).

**Example 2.2** Let  $T = S^1 \times S^1$  be a torus,  $S^1 \subset \mathbb{C}$ . Here too we have an (abelian) multiplication  $m: \text{SP}^2T \rightarrow T$ , which is a bundle projection and the fiber at 1 is the quotient  $T/\mathfrak{S}_2$ , where  $\mathfrak{S}_2$  acts by taking  $(x, y) \mapsto (\bar{x}, \bar{y}) \in T$ . This action corresponds to the hyperelliptic involution on the torus with 4 fixed points, and the

quotient is a copy of the Riemann sphere. This is, of course, a special case of the well-known fact that multiplication  $\mu: \mathbb{S}P^n(T) \rightarrow T$  is an analytic fiber bundle over the “Jacobian”  $T$  with fiber  $\mu^{-1}(1) \cong \mathbb{P}^{n-1}$ .

When  $n \geq 3$ , the cyclic and symmetric products start to differ. We can, for example, use Morton’s description of  $\mathbb{S}P^n(S^1)$  discussed in Example 2.1 to deduce the following nice characterization [25].

**Proposition 2.3** *There is a homeomorphism  $CP^3(S^1) = S^1 \times S^2$ .*

**Proof** The key point is the existence of two distinct sections to the projection  $\pi: CP^3(S^1) \rightarrow SP^3(S^1)$ . Indeed the clockwise (or counterclockwise) order of any three points on the circle is not changed by a cyclic  $\mathbb{Z}_3$ -permutation, which means that, given  $\{x_1, x_2, x_3\} \subset S^1$ , any choice of writing these points in clockwise (resp. counterclockwise) orientation gives a well-defined element in  $CP^3(S^1)$ . The images of these two sections give two copies of  $SP^3(S^1)$  in  $CP^3(S^1)$ . Since any triple of points in  $S^1$  has either a clockwise or counterclockwise orientation, we have the decomposition

$$CP^3(S^1) = SP^3(S^1) \cup SP^3(S^1),$$

and the union is over the fat diagonal  $B_2(S^1, 3)$  of points of the form  $[x, x, y]$ . But  $SP^3(S^1)$  is the product bundle over  $S^1$  with fiber the two-disk  $D^2$  according to Morton’s description (Example 2.1), and the configurations with repeated points consist of its sphere bundle  $\partial D^2 \times S^1 = S^1 \times S^1$ . This means that  $CP^3(S^1)$  is a union of a pair of disk bundles joined along their common circle bundle; *i.e.*, this is the *fiberwise pushout* of two trivial disk bundles

$$S^2 = D^2 \cup_{S^1} D^2 \longrightarrow CP^3(S^1) \longrightarrow S^1,$$

and it is trivial as a bundle. ■

It is slightly harder to determine  $CP^3(T)$  when  $T = S^1 \times S^1$  is the torus. There is a branched degree two covering,  $\pi: CP^3(T) \rightarrow SP^3(T)$ , but there is no obvious section this time. The following characterizes  $CP^3(T)$  algebraically and is to be compared to [4, Corollary 3.4]. We write the group structure on  $T$  additively.

**Proposition 2.4** *Let  $T$  be an elliptic curve embedded in  $\mathbb{P}^2$ . Addition,*

$$\mu_c: CP^3(T) \longrightarrow T, \quad [x_1, x_2, x_3] \longmapsto x_1 + x_2 + x_3,$$

*is a bundle projection with fiber a simply-connected algebraic surface ramified over  $\mathbb{P}^2$  along a sextic curve. The cohomology groups of  $CP^3(T)$  are given according to*

$H^0$	$H^1$	$H^2$	$H^3$	$H^4$	$H^5$	$H^6$
$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^5$	$\mathbb{Z}^8$	$\mathbb{Z}^5$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}$

**Proof** The map  $\mu_c$  makes  $CP^3(T)$  into a bundle over  $T$ . The preimage  $\mu_c^{-1}(0)$  consists of all equivalence classes of triple of points  $[x, y, z] \in CP^3(T)$  with  $x + y + z = 0$ .

This can be easily identified with the quotient  $\mu_c^{-1}(0) = (T \times T)/\mathbb{Z}_3$ , where  $\mathbb{Z}_3$  acts via its generator  $\tau$  of order three as follows:

$$\tau: (x, y) \mapsto (-x - y, x).$$

For simplicity we write  $\mathbb{T} := T \times T$ . We then have the map of bundles

$$\begin{array}{ccccc} \mathbb{T}/\mathbb{Z}_3 & \longrightarrow & \mathbb{C}P^3(T) & \xrightarrow{\mu_c} & T \\ \downarrow 2:1 & & \downarrow \pi & & \downarrow = \\ \mathbb{T}/\mathfrak{S}_3 & \longrightarrow & \mathbb{S}P^3(T) & \xrightarrow{\mu_s} & T \end{array} .$$

The symmetric group  $\mathfrak{S}_3$  acts on  $\mathbb{T}$  via its generators  $\tau$  of order 3 and  $J: (x, y) \mapsto (y, x)$  of order 2. Of course  $\mu_c^{-1}(0) \rightarrow \mu_s^{-1}(0)$  is a branched degree two cover (denoted by the usual notation 2:1). The branching locus is the image of the diagonal  $C := \{[x, x] \mid x \in T\}$  in  $\mathbb{T}/\mathfrak{S}_3$ . This locus can now be explicitly identified. To that end we start by identifying the quotient  $\mu_s^{-1}(0) = \mathbb{T}/\mathfrak{S}_3$  with  $\mathbb{P}^2$ . We already know this from Remark 2.2, but we need an explicit correspondence. Let  $(\mathbb{P}^2)^*$  be the dual of  $\mathbb{P}^2$  consisting of projective lines in  $\mathbb{P}^2$ , and this is again a copy of  $\mathbb{P}^2$ . Since  $T \subset \mathbb{P}^2$ , consider the map

$$\mathbb{T} = T \times T \longrightarrow (\mathbb{P}^2)^*, \quad (x, y) \mapsto \begin{cases} \text{unique line in } \mathbb{P}^2 \text{ through } x \text{ and } y & \text{if } x \neq y, \\ \text{tangent to } T \text{ at } x & \text{if } x = y. \end{cases}$$

This map is surjective, since any line in  $\mathbb{P}^2$  must intersect  $T$ , and it descends to the quotient  $\mathbb{T}/\mathfrak{S}_3$  simply because, by the very definition of the group law on  $T$ , the line through  $(x, y)$  meets the curve again at the point  $-x-y$ , so that  $(x, y)$  and  $(-x-y, x)$  define the same line in  $\mathbb{P}^2$ . This leads to the identification  $\mu_s^{-1}(0) \cong (\mathbb{P}^2)^*$ . The branch locus curve  $C$  maps isomorphically to the dual curve  $T^*$  made out of the tangent lines to  $T$ . But the dual curve to a smooth plane cubic has 9 singular points (cusps) that correspond under the duality to the 9 flexes of  $T$  (i.e., the inflection points of  $T$  or equivalently the points of order 3 in the group law for  $T$ ). Note that this dual curve is of course of genus 1 and hence must have degree 6 (i.e., is a sextic) by the degree-genus formula.

To summarize, the map  $\mu_c^{-1}(0) \rightarrow \mu_s^{-1}(0)$  is identified with a map  $\mu_c^{-1}(0) \rightarrow \mathbb{P}^2$  that ramifies over the curve  $T^*$ . Over the cusps in  $T^*$  lie the 9 singular points of  $\mu_c^{-1}(0)$ . Note that  $\mu_c^{-1}(0)$  is simply connected because of the short exact sequence

$$0 = \pi_2(T) \longrightarrow \pi_1(\mu_c^{-1}(0)) \longrightarrow \pi_1(\mathbb{C}P^3 T) \xrightarrow{\mu_{c*}} \pi_1(T) \longrightarrow 0$$

associated with the bundle  $\mu_c: \mathbb{C}P^3 T \rightarrow T$  and the fact that according to Theorem 1.1 and the existence of a section,  $\mu_{c*}: \pi_1(\mathbb{C}P^3 T) \rightarrow \pi_1(T)$  must be an isomorphism between two copies of  $\mathbb{Z} \oplus \mathbb{Z}$ .

Finally, for any prime  $p$ , the cohomology of  $CP^p(T)$  can be deduced from [1]. Here one thinks of  $CP^3(T)$  as a *toroidal orbifold*, that is, as a quotient of  $(\mathbb{R}^2/\mathbb{Z}^2)^3$  by an action of  $\mathbb{Z}_3$  preserving the lattice. This action is the sum of two regular representations  $\mathbb{Z}_3 \rightarrow GL_3(\mathbb{R})$ , since any such representation corresponds to cyclically permuting the basis vectors. The calculation above for  $H^*(CP^3(S^1 \times S^1))$  is then deduced from [1, Theorem 1]. Details can be found in [24]. ■

### 3 Stabilizer and Orbit Stratifications

The category of  $G$ -stratified spaces is a suitable category in which to consider permutation products. As in [3], we assume  $G$  to act discontinuously,<sup>2</sup> and we define, for a  $G$ -space  $X$ , a stratum for each conjugacy class ( $H$ ) of a subgroup  $H \subset G$  as follows:

$$X_{(H)} := \{x \in X \mid G_x \sim H\} = \bigcup_{K \sim H} X_K,$$

where  $\sim$  means “is conjugate to”,  $G_x := \{g \in G \mid gx = x\}$  the isotropy group of  $x$ , and

$$X_K := \{x \in X \mid G_x = K\}.$$

In other words,  $X_K$  consists of those points  $x \in X$  that are fixed by  $K$  and by no larger subgroup of  $G$ . This space is contained in the fixed point set of  $K$  and it also depends strongly on  $G$ . We can see that  $X_{(G)} = X_G$  consists of the fixed points of the action. On the other hand  $X_{(Id)} = X_{Id}$  is the subset of points of  $X$  on which  $G$  acts freely; here  $Id$  is the trivial subgroup. As  $H$  ranges over all subgroups of  $G$  we get a stratification of  $X$  (i.e., a partition of  $X$ , see [14]) called the *stabilizer stratification*.<sup>3</sup> The *strata* are the various  $X_{(H)}$ ,  $H \subset G$ . One checks that  $X_{(H)}$  are  $G$ -invariant subspaces, since  $G_{gx} = g \cdot G_x \cdot g^{-1}$ , and that the corresponding quotients  $X_{(H)}/G$  stratify  $X/G$  as well, making up the so-called *orbit stratification*.

**Example 3.1** The stabilizer stratification of  $\mathbb{Z}_2$  acting on  $S^n$  by reflection with respect to the plane of an equator has two strata: the two open hemispheres form one stratum (the “free part”) and the equator forms another (the “fixed point” part).

**Example 3.2** Strata corresponding to proper subgroups can be empty. For example, if  $G$  is a non-trivial group acting trivially on a space  $X$ , i.e.,  $gx = x$  for all  $g \in G$ ,  $x \in X$ , then  $X_{(H)} = \emptyset$  for all proper subgroups  $H \subset G$ . In this case there is only one (non-empty) stratum consisting of the fixed points  $X_{(G)} = X$ .

*General Properties:* Assume  $G$  is acting on  $X$  and  $H \subset G$  is a subgroup. We will write  $\bar{A} \subset X$  throughout for the closure of  $A$  in  $X$ .

- If we set  $H^g = g^{-1}Hg$ , then  $X_{H^g} = g^{-1}X_H$ , so we can write  $X_{(H)} = \bigcup_{g \in G} gX_H$ .

<sup>2</sup>See [6, Definition 1.3]: A group  $G$  acts discontinuously on a space  $X$  if the stabilizer of each point of  $X$  is finite, and each point  $x \in X$  has a neighborhood  $V_x$  such that any element  $g$  of  $G$  not in the stabilizer of  $x$  satisfies  $V_x \cap g \cdot V_x = \emptyset$ .

<sup>3</sup>See [3]. We depart from the terminology of Beshears who calls this stratification the “orbit type” stratification.

- If  $x \in \overline{X}_H$ , then  $H \subset G_x$  the stabilizer of  $x$  in  $G$ . Being in the closure of  $X_H$ , there is a sequence of points  $x_i \in X_H$  converging to  $x$  (all spaces are first countable as mentioned). But for any  $h \in H$ ,  $hx_i = x_i$ , so  $hx_i$  converges to  $hx = x$ , and  $H \subset G_x$ .
- If  $X_K \cap \overline{X}_H \neq \emptyset$ , then  $H \subset K$ . This is a direct consequence of the preceding property. The reciprocal statement is not always true. Let  $\mathfrak{S}_4$  act on  $X^4$  by permuting coordinates,  $H = \mathbb{Z}_4 \subset \mathfrak{S}_4 = K$ . Then  $X_{(\mathbb{Z}_4)}^4$  is empty, but  $X_{(\mathfrak{S}_4)}^4 \cong X$  is not.

For a stratified space  $X$  one defines the relation  $\leq$  on the index set  $\mathcal{J}$  of a stratification by setting  $i \leq j$  if and only if  $X_i \subset \overline{X}_j$ , the closure of  $X_j$  in  $X$ . A stratification  $\{X_i\}_{i \in \mathcal{J}}$  satisfies the *frontier condition* if for every  $i, j \in \mathcal{J}$ ,  $X_i \cap \overline{X}_j \neq \emptyset$  implies  $X_i \subseteq \overline{X}_j$  (and  $i < j$  if  $i \neq j$ ). For example, a locally finite simplicial complex is such a stratified space with strata the open simplices. Similarly if  $N$  is a submanifold of  $M$ , then  $N$  and  $M - N$  form a stratification of  $M$  satisfying the frontier condition.

It is not hard to come up with group actions where both the orbit and stabilizer stratifications do not satisfy the frontier condition. For example let  $X$  be a fork with 3 tines, and  $G = \mathbb{Z}_2$ , acting by spinning the fork around its axis through angle  $\pi$ . The following lemma gives a sufficient condition for when such stratifications have the frontier condition.

**Lemma 3.3** *Let  $\Gamma$  be a finite group acting on  $X$ , and suppose that for any pair of subgroups  $H \subset K$  of  $\Gamma$  such that  $X_H \neq \emptyset$ , we have  $X_K \subset \overline{X}_H$ . Then the stabilizer stratification satisfies the frontier condition. Moreover,*

$$(3.1) \quad \overline{X}_{(H)} = \bigcup_{K \supset H} X_{(K)}.$$

**Proof** Suppose that  $X_{(K)} \cap \overline{X}_{(H)} \neq \emptyset$ . Then  $X_K \cap \overline{X}_{H^\sigma} \neq \emptyset$  for some conjugate subgroup  $H^\sigma$  of  $H$ . By one of the general properties mentioned above,  $H^\sigma \subset K$ . Since  $X_H \neq \emptyset$ ,  $X_{H^\sigma} = \sigma^{-1}X_H$  is also non-empty, and hence by our hypothesis  $X_K \subset \overline{X}_{H^\sigma}$  so that  $X_K \subset \overline{X}_{(H)}$  and then obviously  $X_{(K)} \subset \overline{X}_{(H)}$ . This shows that the frontier condition is satisfied. To get the claimed description of  $\overline{X}_{(H)}$ , one needs to see first that  $\bigcup_{K \supset H} X_K \subset \overline{X}_H$ , and this is an immediate consequence of the hypothesis. The reverse inclusion  $\overline{X}_H \subset \bigcup_{K \supset H} X_K$  follows from one of properties listed earlier that if  $x \in \overline{X}_H$ , then  $H \subset K$  where  $K$  is the stabilizer of  $x$ . We then have that  $\overline{X}_H = \bigcup_{K \supset H} X_K$ . Since, on the other hand,  $X_{(H)} = \bigcup_{H' \sim H} X_{H'}$ , we can pass to the closure, and equality (3.1) follows. ■

### 3.1 Permutation Products

In the case of permutation products, the orbit and stabilizer stratification turn out to be well behaved. The first point to make is that for path-connected  $X$ , the stabilizer of a point  $(x_1, \dots, x_n) \in X^n$  only depends on the multiplicities of the  $x_i$ 's and not on their location in  $X$ . Let  $\sigma \in \Gamma \subset \mathfrak{S}_n$  be a permutation. It has a *cycle structure*, i.e., can be expressed as a product of pairwise disjoint cycles

$$(3.2) \quad \sigma = (i_1^1 \cdots i_{d_1}^1)(i_1^2 \cdots i_{d_2}^2) \cdots (i_1^k \cdots i_{d_k}^k)$$

Fixed points of  $\sigma \in \Gamma$  with such a cycle structure consist of all  $n$ -tuples having equal entries at  $i_1^1, \dots, i_{d_1}^1$ , then equal entries at  $i_1^2, \dots, i_{d_2}^2$ , etc.

Consider the case of  $\mathfrak{S}_n$  acting on  $X^n$ . Throughout, we will think of  $\mathfrak{S}_n = \text{Aut}(\Omega)$  as the permutation group of  $\Omega := \{1, \dots, n\}$ . Let  $\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k}$  be the subgroup where  $\mathfrak{S}_{d_1} = \text{Aut}\{1, \dots, d_1\}$ ,  $\mathfrak{S}_{d_2} = \text{Aut}\{d_1 + 1, \dots, d_1 + d_2\}$ , etc, with  $d_i \geq 1$  and  $\sum d_i = n$ . We refer to any subgroup  $H \subset \mathfrak{S}_n$  conjugate to such a product as a *Young subgroup*. The stratum  $X^n_{(\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k})}$  corresponds to all  $n$ -tuples of points that after permutation can be brought to the form

$$\underbrace{(x_1, \dots, x_1)}_{d_1}, \underbrace{(x_2, \dots, x_2)}_{d_2}, \dots, \underbrace{(x_k, \dots, x_k)}_{d_k}$$

with  $x_i \neq x_j$  for  $i \neq j$  and  $\sum d_i = n$ . From this description, it is easy to see that  $X^n_K$  is trivial if  $K$  is not a Young subgroup of  $\mathfrak{S}_n$ . Moreover, each cycle structure (3.2) corresponds to a unique conjugacy class of a Young subgroup of  $\mathfrak{S}_n$ . This implies in particular that for various choices of Young subgroups, the subspaces  $X^n_{(\text{young})}$  give rise to a stratification of  $X^n$ , which is, moreover,  $\mathfrak{S}_n$ -invariant. We refer to the subspace  $X^n_{(\text{young})}$  as a *Young stratum*. The closure of the Young stratum  $X^n_{(\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k})}$  is made up of all Young strata obtained by merging  $\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k}$  into Young subgroups with smaller factors, so, for example,  $X^n_{(\mathfrak{S}_{d_1+d_2} \times \mathfrak{S}_{d_3} \times \dots \times \mathfrak{S}_{d_k})}$  is in the closure. In fact this closure corresponds to the fixed point sets, since

$$\overline{X^n_{\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k}}} = \text{Fix}(\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k}).$$

To sum up,  $X^n$ , under the action of  $\mathfrak{S}_n$ , is stratified by the Young strata, and when  $X$  is path connected, this stratification satisfies the frontier condition. The latter fact turns out to be true for any finite group acting by permutation on a path-connected space as we now explain. Before we do so, we need the following useful remark.

**Remark 3.4** We give a complete description of  $\text{Fix}(H) \subset X^n$  for any subgroup  $H \subset \mathfrak{S}_n$ . View again  $\mathfrak{S}_n = \text{Aut}(\Omega)$ ,  $\Omega = \{1, 2, \dots, n\}$ . The action of  $H$  on  $\Omega$  breaks it into orbits  $\Omega = \Omega_1 \sqcup \dots \sqcup \Omega_d$  of sizes  $n_1, \dots, n_d$  respectively, with  $\sum n_i = n$ . Then  $(x_1, \dots, x_n) \in X^n$  is fixed by  $H$  if and only if  $x_i = x_j$  whenever  $i, j \in \Omega_r$  for some  $r$ . To each orbit  $\Omega_j$  corresponds a subtuple of  $(x_1, \dots, x_n)$ , or a *bloc*, having equal entries. For example, if  $H = \mathbb{Z}_2 \times \mathbb{Z}_3 \subset \mathfrak{S}_5$ ,  $\mathbb{Z}_2$  generated by (12) and  $\mathbb{Z}_3$  generated by (345), then

$$\text{Fix}(H) = \{(x, x, y, y, y) \in X^5\}.$$

The orbits are  $\Omega_1 = \{1, 2\}$  and  $\Omega_2 = \{3, 4, 5\}$ . To  $\Omega_1$  corresponds the bloc of  $x$ 's and to  $\Omega_2$  the bloc of  $y$ 's.

**Proposition 3.5** Suppose  $X$  is path-connected and let  $\Gamma \subset \mathfrak{S}_n$  act on  $X^n$  by permutation. Then both the associated stabilizer and orbit stratifications satisfy the frontier condition.

**Proof** Let  $H \subset \mathfrak{S}_n$  and let  $\mathcal{P}_H$  be the partition of orbits  $\Omega_1, \dots, \Omega_d$  associated with  $H$  as in Remark 3.4. Then a tuple  $(x_1, \dots, x_n) \in X^n$  is fixed by  $H$  if it decomposes



into blocs of equal entries, each bloc determined by some  $\Omega_j$ . We will say in this case that  $(x_1, \dots, x_n)$  is subordinate to the partition  $\mathcal{P}_H$ . Notice that a tuple  $(x_1, \dots, x_n)$  is in the closure  $\overline{X^n_H}$  if  $x_i = x_j$  whenever  $i$  and  $j$  are in some union of the  $\Omega_r$ 's, that is, whenever it is subordinate to a coarser partition.

Assume that  $H \subset K$  two subgroups of  $\Gamma$ . Then the partition of orbits of  $H$  is finer than that defined by  $K$ . For path-connected  $X$ , it is easy to arrange that any  $(x_1, \dots, x_n)$  subordinate to a partition  $\mathcal{P}$  is the limit of tuples subordinate to a finer partition. Assume  $\overline{X^n_H} \neq \emptyset$ . Then any point in  $X^n_K$  (if non-empty) must be a limit point in  $X^n_H$  and so  $X^n_K \subset \overline{X^n_H}$ . According to Lemma 3.3, the stabilizer stratification satisfies the frontier condition. ■

### 3.2 Depth

We assume here that all groups are finite. Given a finite stratification  $\{X_i\}_{i \in \mathcal{J}}$  of a space  $X$  satisfying the frontier condition, we define the depth of a stratum  $X_s$  to be the maximal length  $k$  of a sequence  $s = s_0 < s_1 < \dots < s_k$  in  $\mathcal{J}$ . We recall that  $s_i < s_j$  means  $X_{s_i} \subset \overline{X_{s_j}}$ . The depth of a (finite) stratification as a whole is the maximum over the depths of its strata. In this section we assume that stabilizer stratifications associated with group actions satisfy the frontier condition.

**Definition 3.6** Represent an action of  $\Gamma$  on a space  $X$  by a homomorphism  $\phi: \Gamma \rightarrow \text{Homeo}(X)$ . Denote by  $d_\phi(\Gamma, X)$  the depth of the stabilizer stratification associated with this action. As is clear, this depth depends on both  $\phi$  and  $X$ .

For given  $\Gamma$ , the various depths  $d_\phi(\Gamma, X)$  can be compared to the length of  $\Gamma$ . Recall that the length  $\ell(\Gamma)$  of a finite group  $\Gamma$  is the length  $\ell$  of a longest chain of subgroups  $\{1\} = \Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma_\ell = \Gamma$ .

**Theorem 3.7** If  $\Gamma$  is a finite group, then  $0 \leq d_\phi(\Gamma, X) \leq \ell(\Gamma)$  and these equalities are sharp.

**Proof** To see that  $d_\phi(\Gamma, X)$  can be 0, simply choose  $X$  to be a point. To show the right inequality, recall by one of our earlier properties that in any stabilizer stratification,  $\emptyset \neq X_{(K)} \subset \overline{X_{(H)}}$  implies that  $H \subset K^\sigma$  for some conjugate subgroup  $K^\sigma \sim K$ . It follows that any chain of strata in the stratification must correspond to a chain of subgroups and hence the depth of the stratification is at most  $\ell(\Gamma)$ . To see that this inequality can be sharp, we will consider the permutation action of  $\Gamma$  on the set of its elements  $\Omega = \{1, \dots, |\Gamma|\}$ . This action is free and transitive. It gives an embedding  $\psi: \Gamma \hookrightarrow \mathfrak{S}_n, n = |\Gamma|$  (the so-called permutation representation of  $\Gamma$ ) and an induced permutation action on  $X^n$

$$\phi: \Gamma \longrightarrow \text{Homeo}(X^n), \quad n = |\Gamma|$$

We claim that  $d_\phi(\Gamma, X^n) = \ell(\Gamma)$  for path-connected  $X$ . The main step is to show that any subgroup  $H \subset \Gamma$  gives rise to a non-empty stratum  $X^n_{(H)}$ . We proceed by exhibiting an element in  $\text{Fix}(H)$  that cannot be fixed by any larger subgroup. For ease we write  $\psi(H)$  to refer to  $H$  acting on  $\Omega$ , and we write  $H$  when we wish to indicate that  $H$  acts on  $X^n$ .

As before,  $(x_1, \dots, x_n) \in \text{Fix}(H)$  if it can be decomposed into  $d$ -blobs corresponding to  $\Omega_1, \dots, \Omega_d$ , the orbits of the action of  $\psi(H)$  on  $\Omega = \{1, \dots, n\}$ . Each bloc is a subtuple characterized by the fact that its entries are equal. Moreover,  $H$  acts on each bloc transitively and does not permute elements from different blocs. We will choose  $\zeta = (x_1, \dots, x_n) \in \text{Fix}(H)$  such that  $x_i \neq x_j$  if they are not in the same bloc, *i.e.*, if  $i$  and  $j$  are not in the same  $\Omega_r$  (of course this is possible if  $X$  is not reduced to a point). We claim that  $\Gamma_\zeta = H$ . This is where the permutation representation intervenes.

Let  $g \in \Gamma_\zeta$ . Then  $g$  leaves the bloc corresponding to  $\Omega_1$  with entries equal to  $x_1$ , say, invariant. All orbits  $\Omega_i$  have cardinality at least two, since  $\psi(H)$  acts freely on  $\Omega$ . Choose  $i, j \in \Omega_1$  such that  $\psi(g)(i) = j$ . Since  $H$  acts transitively on that bloc, there is  $\psi(h) \in \psi(H)$  such that  $\psi(h)(i) = j$  as well. This means that at the level of the group  $\Gamma$ , there are generators  $g_i, g_j \in \Gamma$  such that  $gg_i = g_j$  and  $hg_i = g_j$ , which means that  $h = g$  and  $g \in H$ . This shows therefore that  $\zeta \in X^n_{(H)}$  and the stratum  $X^n_{(H)}$  is non-empty.

The previous argument asserts that for every  $H \subset \Gamma$ , we have a non-empty stratum  $X^n_{(H)}$ . Another argument of limit using the path-connectedness of  $X$  readily shows that if  $H \subset K$ , then  $X^n_K \cap \overline{X^n_H} \neq \emptyset$ . It follows by Proposition 3.5 that  $X^n_{(K)} \subset \overline{X^n_{(H)}}$ . Any chain of subgroups gives, therefore, a non-empty chain of strata and  $d_\phi(\Gamma, X^n) \geq \ell(\Gamma)$ , so that  $d_\phi(\Gamma, X^n) = \ell(\Gamma)$ , concluding the proof. ■

**Definition 3.8** For a given embedding  $\phi : \Gamma \hookrightarrow \mathfrak{S}_n$ , we can write  $d_\phi(\Gamma)$  the depth of the orbit stratification of  $\Gamma$  acting on  $X^n$  where  $X$  is path-connected and not reduced to point. This does not depend on  $X$  as we pointed out.

**Example 3.9** For  $\phi : \Gamma \hookrightarrow \mathfrak{S}_n$ ,  $d_\phi(\Gamma) \leq \ell(\Gamma) \leq \ell(\mathfrak{S}_n)$ , the length of  $\mathfrak{S}_n$  (Theorem 3.7). By a beautiful theorem of Cameron, Solomon, and Turull [7], this is

$$\ell(\mathfrak{S}_n) = [(3n - 1)/2] - b(n),$$

where  $b(n)$  is the number of ones in the dyadic representation of  $n$ .

**Example 3.10** Let  $\phi = \text{Id}$  be the identity of  $\mathfrak{S}_n$  with associated quotient space  $\text{SP}^n(X)$ . The strata of the orbit type stratification are in one to one correspondence with integer partitions of the form

$$P = [p_1, p_2, \dots, p_k], \quad p_i \leq p_{i+1}, \quad \sum p_i = n$$

To the partition  $[p_1, p_2, \dots, p_k]$  corresponds the stratum of points of the form

$$[x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_k, \dots, x_k],$$

where  $x_i \neq x_j$  if  $i \neq j$ , and each  $x_i$  is repeated  $p_i$ -times. This is the stratum  $X^n_{(\mathfrak{S}_{p_1} \times \dots \times \mathfrak{S}_{p_k})} / \mathfrak{S}_n$ . The generic (open and dense) stratum is the configuration space of distinct points  $B(X, n) = F(X, n) / \mathfrak{S}_n = X^n_{(1)} / \mathfrak{S}_n$ , and this corresponds to the partition  $[1, \dots, 1]$ . The diagonal in  $\text{SP}^n X$  corresponds to the partition  $[n]$ . The depth of this stratification is  $d_{\text{Id}}(\mathfrak{S}_n) = n - 1$ . The poset of strata here is the set of partitions partially ordered according to  $P \leq Q$  if  $Q$  is a refinement of  $P$ , that is, if  $Q$  is obtained by further partitioning the parts of  $P$ .

**Example 3.11** The depth of  $\mathbb{Z}_n$  acting on  $X^n$  is the length of  $\mathbb{Z}_n$ . This is because the inclusion  $\mathbb{Z}_n \hookrightarrow \mathfrak{S}_n$ , sending the generator to the  $n$ -cycle  $(12 \cdots n)$ , corresponds to the permutation representation of  $\mathbb{Z}_n$  (Theorem 3.7). Note that if  $p|n$ , then  $X^n_{(\mathbb{Z}_p)} = X^n_{\mathbb{Z}_p}$  in the stabilizer stratification associated with the action of  $\mathbb{Z}_n$  on  $X^n$ , consists, up to cyclic permutation, of all points of the form

$$(x_1, x_2, \dots, x_{n/p}, x_1, x_2, \dots, x_{n/p}, \dots, x_1, x_2, \dots, x_{n/p})$$

with  $x_i \neq x_j$  if  $i \neq j$ , each  $x_i$  repeating  $p$  times.

### 4 Permutation Products of Manifolds

The fact that  $SP^n(S)$  is a complex manifold when  $S$  is a smooth algebraic curve is an exceptional property that is no longer true for general  $\Gamma P^n(S)$ . For example, here is a standard argument as to why  $CP^3(\mathbb{C})$  cannot be homeomorphic to  $\mathbb{R}^6$ . Decompose  $\mathbb{C}^3 = L \times H$ , where  $L$  is the diagonal line in  $\mathbb{C}^3$  and  $H$  its orthogonal hyperplane. The linear action of  $\mathbb{Z}_3$  takes place entirely in  $H$ . This action restricts to a fixed point free action on the unit sphere  $S^3 \subset H$  (since we have factored out the diagonal that consists of the fixed points). It follows that  $\mathbb{R}^6/\mathbb{Z}_3 \cong \mathbb{R}^2 \times cM$ , where  $M = S^3/\mathbb{Z}_3$  is a 3-manifold with  $H_1(M) = \mathbb{Z}_3$  and  $cM$  is the cone on  $M$  (with apex at the origin and the cone extending to  $\infty$ ). This quotient cannot be homeomorphic to  $\mathbb{R}^6$ .

In general, if  $\Gamma$  is a finite group acting on a smooth manifold, then its associated stabilizer stratification is *Whitney stratified* (see [22, Theorem 1.3.7]). In particular, submanifold strata do have tubular neighborhoods whose boundaries are not sphere bundles. We analyze here in detail a neighborhood of the diagonal stratum  $M \hookrightarrow \Gamma P^n M, x \mapsto [x, \dots, x]$  for  $n \geq 2$ .

Let  $V$  be a (real) vector space and let  $\mathfrak{S}_n$  act on  $V^n$  by permutation. The orthogonal complement of the diagonal subspace is  $H = \{(v_1, \dots, v_n) \in V^n, \sum v_i = 0\}$  and has by restriction an action of  $\mathfrak{S}_n$ . Consider the map

$$\Phi: V^{n-1} \longrightarrow V^n, \quad (v_1, \dots, v_{n-1}) \longmapsto (-v_1, \dots, -v_{n-1}, v_1 + \dots + v_{n-1}).$$

If we require that this map be  $\mathfrak{S}_n$ -equivariant, then the action on the left factor  $V^{n-1}$  is determined implicitly, and we refer to it as the *induced* action. For example for  $n = 2$  the induced action on  $V$  is generated by the involution  $v \mapsto -v$ . If  $n = 3$ , then the generating 3-cycle  $\sigma \in \mathfrak{S}_3$  acts on  $V^3$  via  $\sigma(-v_1, -v_2, v_1 + v_2) = (v_1 + v_2, -v_1, -v_2)$  and this dictates the induced action  $\sigma(v_1, v_2) = (-v_1 - v_2, v_1)$  in  $V^2$  (as in Proposition 2.4). The point is that the map  $\Phi$  maps  $V^{n-1}$  isomorphically and equivariantly onto  $H$ .

Now let  $TM^{\oplus n-1}$  be the  $(n - 1)$ -fold Whitney sum of the tangent bundle  $TM$  of  $M$ ,  $\dim M = m$ , and let  $\Gamma \subset \mathfrak{S}_n$  act on each fiber  $V^{n-1} \cong (\mathbb{R}^m)^{n-1}$  via the induced action. The fiberwise quotient is a bundle with fiber  $V^{n-1}/\Gamma$  homeomorphic to  $H/\Gamma$ . But  $\Gamma$  acts linearly on  $H \subset V^n$ , so that the quotient is the open cone  $c(S/\Gamma)$ , where  $S \cong S^{(n-1)m-1}$  is the unit sphere in  $H$ . The action being fiberwise, we obtain, therefore, a new bundle  $S_\Gamma M$  over  $M$  with fiber homeomorphic to  $c(S/\Gamma)$ .

**Proposition 4.1** *For  $M$  a smooth closed manifold, a neighborhood deformation retract  $V$  of the diagonal  $M$  in  $\Gamma P^n M$  is homeomorphic to the total space of  $S_\Gamma M$ . The section corresponding to the cone points is the inclusion of  $M$  into  $V$ .*

**Proof** The normal bundle of the diagonal copy of  $M$  in  $M^n$  is isomorphic to the Whitney sum  $TM^{\oplus n-1}$ . The reason for this is that if a point  $x$  is sufficiently close to a point  $y$ , then it determines a unique vector  $v \in T_x M$  such that  $\exp_x(v) = y$ . So if  $(x, y_1, \dots, y_{n-1})$  is a point in  $M^n$  with  $y_i$  sufficiently close to  $x$ , then it determines a tuple of vectors  $(v_1, \dots, v_{n-1})$  with  $\exp_x(v_i) = y_i$ . For each  $x \in M$ , we write  $T_x M^{\oplus n-1}$  for the fiber at  $x$  of  $TM^{\oplus n-1} \rightarrow M$ .

Choose a metric on  $M$  with injectivity radius greater than 2 and let  $DM \subset TM$  be the unit disc bundle of the tangent bundle of  $M$ . We will write  $\exp$  for the exponential map  $DM \rightarrow M$  and  $\exp_x$  for its restriction to the fiber  $D_x M$  at  $x \in M$ . Let  $DTM^{\oplus n-1}$  be the unit disk bundle in the Whitney sum. We get a map

$$DTM^{\oplus n-1} \xrightarrow{\phi} M^n$$

defined on each fiber by

$$(x, (v_1, \dots, v_{n-1})) \mapsto (\exp_x(-v_1), \dots, \exp_x(-v_{n-1}), \exp_x(v_1 + \dots + v_{n-1})).$$

This map is  $\mathfrak{S}_n$ -equivariant by the very definition of the induced action; it is one-to-one and onto its image  $V$ . Since  $\exp_x(0) = x$ , this image produces a  $\mathfrak{S}_n$ -invariant neighborhood of the diagonal in  $M^n$ , which is then homeomorphic to the total space of  $DTM^{\oplus n-1}$ . A tubular neighborhood of  $M$  in  $\Gamma P^n$  is therefore homeomorphic to the fiberwise quotient of  $DTM^{\oplus n-1}$  by the action of  $\Gamma$ , and this is in turn homeomorphic to  $S_\Gamma M$ . ■

**Corollary 4.2** *Assume  $M$  is a closed smooth manifold. A neighborhood deformation retract  $V$  of the diagonal  $M$  in  $SP^2 M$  is homeomorphic to the fiberwise cone on the projectivized tangent bundle of  $M$ .*

**Proof** According to Proposition 4.1, the fiber at  $x \in M$  of  $S_{\mathbb{Z}_2} M$  is the cone on  $S^{m-1}/\mathbb{Z}_2$ , where  $S^{m-1}$  is the unit tangent sphere at  $x$  and  $\mathbb{Z}_2$  acts via the antipodal action  $v \mapsto -v$ . ■

**Remark 4.3** In the case of the sphere, the complement of the diagonal in  $SP^2 S^n$  is  $B(S^n, 2)$ , and this is known to be a deformation retract onto  $\mathbb{R}P^n$ .

## 5 The Fundamental Group of Permutation Products

View  $\mathfrak{S}_n$  as the permutation group of  $\Omega = \{1, 2, \dots, n\}$ . The set  $\Omega$  breaks into orbits under the action of  $\Gamma$  viewed as a subgroup of  $\mathfrak{S}_n$ , and we will say that an orbit is non-trivial if it contains at least two elements. In this section we prove main Theorem 1.1 in the introduction. Let  $r$  be the number of non-trivial transitive orbits under this permutation action of  $\Gamma$  on  $\Omega$  and let  $k_1, k_2, \dots, k_r$  be the sizes of the non-trivial orbits. If  $\pi_1(X)$  is based at  $* \in X$ , then  $\pi_1(\Gamma P^n X)$  is based at  $[\ast, \dots, \ast]$ .

**Theorem 5.1** *There is an isomorphism  $\pi_1(\Gamma P^n X) \cong \pi_1(X)^{n-\sum_i k_i} \times H_1(X; \mathbb{Z})^r$ .*

We are aware that this theorem can be derived using the methods of Brown and Higgins [6]. Our approach, however, is geometrically appealing and totally self-contained. We start by proving that if  $\Gamma$  is a transitive subgroup of  $\mathfrak{S}_n$  acting by permutations on  $X^n$  and  $n \geq 2$ , then  $\pi_1(\Gamma P^n X) \cong H_1(X; \mathbb{Z})^{(4)}$ . A universal example of transitive action is when  $H \subset \Gamma$  is a subgroup and  $\Gamma$  acts transitively on the right cosets of  $H$  [15].

**Example 5.2** A special example of a transitive action is when  $\Gamma = \mathfrak{S}_\ell \wr \mathfrak{S}_k$  is a wreath product (i.e., these form the maximal, transitive, and imprimitive subgroups of  $\mathfrak{S}_{\ell k}$ ,  $\ell, k > 1$ ). In this case  $\Gamma P^{\ell k}(X) \cong \text{SP}^k(\text{SP}^\ell X)$  and  $\pi_1(\Gamma P^{\ell k}(X)) \cong H_1(\text{SP}^\ell X; \mathbb{Z}) \cong H_1(X; \mathbb{Z})$ , consistently with the fact that  $\Gamma$  has only one transitive orbit.

In order to prove Theorem 5.1 we need some useful facts about path-lifting of maps. We say a map  $\pi: Y \rightarrow X$  has the *path-lifting property* if we can lift any path  $[0, 1] \rightarrow X$  to a path in  $Y$  knowing where it starts in  $Y$ . The following is [6, Proposition 1.5].

**Proposition 5.3** *If  $G$  is a group acting discontinuously on a Hausdorff space  $X$ , then  $\pi: X \rightarrow X/G$  has the path lifting property.*

**Corollary 5.4** *If a finite group  $G$  acts on a Hausdorff connected space  $X$  with a fixed point  $x_0$ , then  $\pi_1(X, x_0) \rightarrow \pi_1(X/G, x_0)$  is surjective. In particular  $\pi_1(X^n) \rightarrow \pi_1(\Gamma P^n X)$  is surjective for any  $\Gamma \subset \mathfrak{S}_n$ .*

**Proof** A loop  $\gamma$  based at  $x_0$  in  $X/G$  must lift to a loop  $\tilde{\gamma}$ , and the homotopy class  $[\tilde{\gamma}]$  maps to  $[\gamma]$ . This establishes the main statement. In the case of  $X^n/\Gamma$ , one chooses as basepoint a diagonal element  $[*, \dots, *], * \in X$ . ■

**Remark 5.5** Note that the surjection  $\pi_1(X^n) \rightarrow \pi_1(\Gamma P^n X)$  is equivalent to the following intuitive fact: if  $\mathbf{x} = [x_1, \dots, x_n]$ ,  $\mathbf{y} = [y_1, \dots, y_n]$  are two configurations in  $\Gamma P^n(X)$ , and  $\gamma$  a path between them, then there are paths  $\gamma_1, \dots, \gamma_n$  in  $X$  such that  $\gamma(t) = [\gamma_1(t), \dots, \gamma_n(t)]$  with  $\gamma_i(0) = x_i$  and  $\gamma_i(1) = y_{\sigma(i)}$  for some permutation  $\sigma \in \Gamma$ .

We are now in a position to prove Theorem 5.1. We first consider the case when  $\Gamma$  is a transitive subgroup of  $\mathfrak{S}_n$ . Write  $\pi: X^n \rightarrow \Gamma P^n X$  for the projection. We will proceed in two steps: (i)  $\pi_1(\Gamma P^n X)$  is abelian if  $\Gamma$  is transitive and (ii)  $\pi_1(\Gamma P^n X) = H_1(X, \mathbb{Z})$ .

To prove (i), we will use the trick of Smith that consists of *lifting and commuting* loops. First of all notice that by Corollary 5.4,  $\pi$  is surjective on fundamental groups. Write  $\eta_j, 1 \leq j \leq k$  for the  $j$ -th generator of  $\pi_1(X)$ , and write a presentation of the

<sup>4</sup>We know of only one place where this result is mentioned ([10, 12.15]), and it is said there that this isomorphism in the case of a transitive action would follow from earlier results of Dold, without further details.

product as

$$\pi_1(X)^n = \langle \eta_{1,1}, \dots, \eta_{1,k}, \dots, \eta_{n,1}, \dots, \eta_{n,k} \mid \text{relations} \rangle$$

where  $\eta_{i,j}$  is the  $j$ -th generator of the  $i$ -th copy of  $\pi_1(X)$  in  $\pi_1(X)^n$ . Let  $\alpha$  and  $\beta$  be two homotopy classes in  $\pi_1(\Gamma P^n(X))$ . We will show that they commute. Choose  $\tilde{\alpha}$  and  $\tilde{\beta}$  in  $\pi_1(X^n) = \pi_1(X)^n$  that project respectively onto  $\alpha$  and  $\beta$ . Write these two classes additively as

$$\tilde{\alpha} = \sum_{i,j} a_{i,j} \eta_{i,j}, \quad \tilde{\beta} = \sum_{i,j} b_{i,j} \eta_{i,j}.$$

Observe that since  $\pi_*(\eta_{i,j}) = \pi_*(g\eta_{i,j}) = \pi_*(\eta_{g(i),j})$ ,  $g \in \Gamma$ , the transitivity of the action guarantees that

$$\pi_*(\eta_{r,j}) = \pi_*(\eta_{s,j}), \quad 1 \leq r, s \leq n.$$

If we define

$$\tilde{\alpha}_1 = \sum_{i,j} a_{i,j} \eta_{1,j}, \quad \tilde{\beta}_2 = \sum_{i,j} b_{i,j} \eta_{2,j},$$

then

$$\pi_*(\tilde{\alpha}_1) = \pi_*(\tilde{\alpha}) = \alpha, \quad \pi_*(\tilde{\beta}_2) = \pi_*(\tilde{\beta}) = \beta.$$

The point of this manipulation is that now  $\tilde{\alpha}_1$  and  $\tilde{\beta}_2$  commute in  $\pi_1(X^n)$ , since they are made out of generators that live in different factors of  $\pi_1(X)^n$ . Since these classes commute, so do their images  $\alpha$  and  $\beta$  in  $\pi_1(\Gamma P^n(X))$ . This “lifting and commuting” argument proves, therefore, that  $\pi_1(\Gamma P^n X)$  is abelian. In fact we have proven a little more.

**Corollary 5.6** *In the case of a transitive action  $\Gamma \subset \text{Aut}\{1, \dots, n\}$ , the composite*

$$\tau: \pi_1(X) \hookrightarrow \pi_1(X)^n \longrightarrow \pi_1(\Gamma P^n X) = H_1(\Gamma P^n X)$$

*is surjective, where the first map is inclusion as the first factor.*

**Proof** This is restating what we previously constructed. If we refer to the first factor of  $\pi_1(X)^n$  by  $\pi_1(X)$  which is generated by the  $\eta_{1,j}$ ,  $1 \leq j \leq k$ , then for  $\alpha \in \pi_1(\Gamma P^n X)$ , with preimage  $\tilde{\alpha} \in \pi_1(X)^n$  and  $\tilde{\alpha} = \sum a_{i,j} \eta_{i,j}$ , the class  $\tilde{\alpha}_1 := \sum a_{i,j} \eta_{1,j}$  is also a preimage. ■

Finally, we verify that if  $n \geq 2$ ,  $\pi_1(\Gamma P^n X) = H_1(\Gamma P^n X)$  coincides with  $H_1(X; \mathbb{Z})$ . The composite  $\tau$  in Corollary 5.6 factors necessarily through the abelianization  $\pi_*: H_1(X; \mathbb{Z}) \rightarrow H_1(\Gamma P^n X)$ , and this map, as we indicated, is surjective. It then remains to show that it is also injective, and this can be done through a neat little argument.

**Lemma 5.7** *Let  $i: X \hookrightarrow \Gamma P^n X$  be the basepoint inclusion  $x \mapsto [x, *, \dots, *]$ ,  $n \geq 2$ . Then for all  $k \geq 0$ ,  $i_*: H_k(X; \mathbb{Z}) \rightarrow H_k(\Gamma P^n X)$  is a monomorphism.*

**Proof** We show that on cohomology, the induced map  $i^*: H^k(\Gamma P^n X) \rightarrow H^k(X)$  is surjective. Let  $\alpha \in H^k(X; \mathbb{Z})$ . Since  $X$  is of the homotopy type of a CW-complex, the class  $\alpha$  can be represented by the (homotopy class) of a map  $f: X \rightarrow K(\mathbb{Z}, k)$ ; that is  $\alpha = f^*(\iota)$  where  $\iota \in H^k(K(\mathbb{Z}, k)) \cong \mathbb{Z}$  is a generator. A model for the Eilenberg–MacLane space  $K(\mathbb{Z}, k)$  is  $SP^\infty(S^k)$ , and this is an abelian topological monoid (which we write multiplicatively). We can then extend  $f$  to a map  $\widehat{f}: \Gamma P^n X \rightarrow K(\mathbb{Z}, k)$ , sending  $[x_1, \dots, x_n] \mapsto f(x_1) \cdots f(x_n)$ . The diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & K(\mathbb{Z}, k) \\ \downarrow i & \nearrow \widehat{f} & \\ \Gamma P^n X & & \end{array}$$

defines a class  $\widehat{\alpha} := \widehat{f}^*(\iota)$  such that  $i^*(\widehat{\alpha}) = \alpha$ . Since  $i^*$  is surjective,  $i_*$  is injective. ■

The next lemma (and its proof) shows what happens when we go from a single orbit to multiple ones.

**Lemma 5.8** *Let  $\Gamma \subset \text{Aut}\{1, \dots, n\}$  and let  $m$  be the total number of orbits under this action. Then*

$$H_1(\Gamma P^n X; \mathbb{Z}) \cong (H_1(X; \mathbb{Z})^{\oplus n})^\Gamma \cong H_1(X; \mathbb{Z})^m,$$

where the middle term is the submodule of invariants under the action of  $\Gamma$  permuting the factors.

**Proof** We break the action of  $\Gamma$  on  $\Omega = \{1, \dots, n\}$  into orbits  $\Omega_1, \dots, \Omega_m$  of respective sizes  $k_i \geq 1$ . We can change the action of  $\Gamma$  up to conjugation so that those orbits are

$$\begin{aligned} \Omega_1 &= \{1, \dots, k_1\}, & \Omega_2 &= \{k_1 + 1, \dots, k_1 + k_2\}, \dots, \\ \Omega_m &= \{k_1 + \dots + k_{m-1} + 1, \dots, \sum k_i\}. \end{aligned}$$

Set  $Y_i := \text{Aut}(\Omega_i) \cong \mathfrak{S}_{k_i}$ . Then the  $Y_i$  commute among each other and their product in  $\mathfrak{S}_n$  is a subgroup  $Y_1 \cdots Y_m$  that is isomorphic to the Young subgroup  $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_m}$  (notation as discussed at the start of Section 3.1). We can assume from now on and without loss of generality that  $\Gamma \subset \mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_m}$ , so that  $\Gamma$  permutes  $\{1, \dots, k_1\}$ , then  $\{k_1 + 1, \dots, k_1 + k_2\}$ , etc. We decompose  $X^n = X^{k_1} \times X^{k_2} \times \cdots \times X^{k_m}$  so that we have the composite

$$X^m \xrightarrow{i_m} X^n \xrightarrow{\pi} \Gamma P^n(X) \xrightarrow{q_m} SP^{k_1}(X) \times \cdots \times SP^{k_m}(X),$$

where  $\pi$  and  $q_m$  are quotient maps and  $i_m$  is the map that identifies the  $i$ -th factor of  $X^m$  with the first factor of  $X$  in  $X^{k_i}$ . It is immediate to see that this composite induces an isomorphism on  $H_1$  (because each composite  $X \rightarrow X^{k_i} \rightarrow SP^{k_i}(X)$  does

by Lemma 5.6), and so  $H_1(\Gamma P^n; \mathbb{Z})$  contains  $H_1(X)^{\oplus m}$  as a subgroup. In fact we show this is an isomorphism. Indeed we know that  $\pi_*: H_1(X^n) \rightarrow H_1(\Gamma P^n(X))$  is surjective by Corollary 5.4 and by the fact that the fundamental group of  $\Gamma P^n(X)$  is abelian. Given  $\alpha \in H_1(\Gamma P^n(X))$ , we choose a lift  $\tilde{\alpha} = \sum a_{ij} \eta_{ij} \in H_1(X^n) \cong H_1(X)^{\oplus n}$ , where  $\eta_{ij}$  is the  $j$ -th generator in the  $i$ -th copy of  $H_1(X) \hookrightarrow H_1(X)^{\oplus n}$ . Suppose  $i$  is in the  $t$ -th orbit  $\Omega_t$ , then, since the action of  $\Gamma$  on  $\Omega_t$  is transitive, we have that

$$(5.1) \quad \pi_*(\eta_{ij}) = \pi_*(\eta_{k_1+\dots+k_{t-1}+1,j}).$$

This means that a lift of  $\alpha$  can be chosen to be in the form  $\sum b_{ij} \eta_{k_1+\dots+k_i+1,j}$  and hence is in the image of  $i_m: H_1(X)^{\oplus m} \hookrightarrow H_1(X^n)$ . The composite  $\pi \circ i_m: H_1(X)^{\oplus m} \rightarrow H_1(\Gamma P^n X)$  is therefore surjective, and hence  $H_1(\Gamma P^n X) \cong H_1(X)^{\oplus m}$ . It remains to see that this is also isomorphic to  $H_1(X^n)^\Gamma \cong (H_1(X)^{\oplus n})^\Gamma$ , but the arguments are the same, and we leave them to the reader. ■

**Proof of Theorem 5.1** As in the proof of Lemma 5.8, we suppose that  $\Gamma$  acts with non-trivial orbits  $\Omega_1, \dots, \Omega_r$ , of lengths  $k_1, \dots, k_r \geq 2$ , and the remaining orbits  $n - \sum k_i$  are trivial. We can also assume that  $\Omega_1 = \{1, \dots, k_1\}, \Omega_2 = \{k_1+1, \dots, k_1+k_2\}$ , etc. The quotient of  $X^n$  by  $\Gamma$  splits as  $(X^{\sum_1^r k_i} / \Gamma) \times X^{n-\sum_1^r k_i}$ , and we therefore have to determine the fundamental group of the first factor. By lifting homotopy classes as in Lemma 5.8 (see (5.1)) and using the lifting and commuting argument explained earlier, we show that  $\pi_1(X^{\sum k_i} / \Gamma)$  is abelian. By Lemma 5.8 we can then write

$$\pi_1(X^{\sum k_i} / \Gamma) \cong H_1(X^{\sum_1^r k_i} / \Gamma; \mathbb{Z}) \cong H_1(X; \mathbb{Z})^r,$$

and the theorem follows. ■

### 5.1 Higher Homotopy Groups

We can ask generally how  $\pi_i(\Gamma P^n X)$  compares to  $H_i(X, \mathbb{Z})$  for finite  $n, i > 1$ . We say a simply connected  $X$  is  $r$ -connected if its homology vanishes up to and including degree  $r$ . When  $\Gamma = \mathfrak{S}_n$  and using simplicial techniques, Dold and Puppe show the following theorem [10].

**Theorem 5.9** *Let  $X$  be  $r$ -connected,  $r \geq 1$ , then for  $n > 1$*

$$\pi_i(SP^n X) \cong \tilde{H}_i(X; \mathbb{Z})$$

*provided that  $0 \leq i \leq r + 2n - 1$ .*

We give a short and novel proof of this theorem based on results in [16].

**Proof** If  $X$  is simply connected, then so are its symmetric products. It follows that  $\pi_1(SP^{n+1} X, SP^n X) = 0$ . The relative Hurewicz theorem (see [13], §4.2) then says that the first non-zero  $\pi_*(SP^{n+1} X, SP^n X)$  is isomorphic to the first non-zero  $H_*(SP^{n+1} X, SP^n X)$ . Since  $H_i(SP^{n+1} X, SP^n X) \cong \tilde{H}_i(\overline{SP}^{n+1} X)$  is zero for  $i \leq r + 2n$



by [16, Theorem 1.3], it follows that  $\pi_i(\mathrm{SP}^{n+1}X, \mathrm{SP}^nX)$  is zero within the same range. From the long exact sequence of the pair  $(\mathrm{SP}^nX, \mathrm{SP}^{n-1}X)$  we find that

$$\pi_i(\mathrm{SP}^nX) \xrightarrow{\cong} \pi_i(\mathrm{SP}^{n+1}X) \quad \text{for } i \leq r + 2n - 1,$$

and hence by induction  $\pi_i(\mathrm{SP}^nX) \cong \pi_i(\mathrm{SP}^\infty X) \cong \tilde{H}_i(X, \mathbb{Z})$  for  $0 \leq i \leq r + 2n - 1$ . ■

**Remark 5.10** Theorem 5.9 is no longer true if  $X$  is not simply connected. It can be shown for example that for  $C$  a closed topological surface of genus  $g = 2$  there is a Laurent polynomial description<sup>5</sup>

$$\pi_2(\mathrm{SP}^2C) = \mathbb{Z}[t_i, t_i^{-1}], \quad 1 \leq i \leq 4.$$

## 6 Bounded Multiplicity Configurations: Constructions and Examples

There are naturally defined  $\mathfrak{S}_n$ -invariant subspaces in  $X^n$  (for  $n \geq 2$  a fixed integer) that we list below. Throughout,  $*$   $\in X$  denotes the basepoint.

- The  $d$ -th fat diagonal  $F_d(X, n)$  is the subspace of  $X^n$  of all tuples  $(x_1, \dots, x_n)$  such that  $x_{i_1} = x_{i_2} = \dots = x_{i_d}$  for some  $d$ -set of indices  $1 \leq i_1 < i_2 < \dots < i_d \leq n$ . These are the  $d$ -equal subspaces considered, for example, in [5] in the case  $X = \mathbb{R}^2$ .
- $F^d(X, n) = \{(x_1, \dots, x_n) \in X^n \mid \text{no entry } x_i \text{ appears more than } d\text{-times}\}$ . We have

$$F^d(X, n) = X^n - F_{d+1}(X, n).$$

- The symmetric group  $\mathfrak{S}_n$  acts by permuting coordinates and preserves these subspaces, thus we can define the unordered analogs of the above constructions

$$B_d(X, n) = F_d(X, n)/\mathfrak{S}_n \quad \text{and} \quad B^d(X, n) = F^d(X, n)/\mathfrak{S}_n.$$

Clearly  $B_n(X, n) = X$  while  $B_2(X, n)$  is the image of the “fat diagonal” in  $\mathrm{SP}^nX$ . These are the main objects of study in this paper. A widespread notation in the literature is to write

$$F(X, n) := F^1(X, n), \quad B(X, n) = B^1(X, n)$$

for the configuration spaces of ordered (resp. unordered) pairwise distinct points of  $X$ .

Note that  $B^d(X, n)$  is not a functorial construction, since the constant map  $X \rightarrow X, x \mapsto *$ , does not induce a self-morphism of  $B^d(X, n)$ . On the other hand the fat diagonal  $B_d(X, n)$  is functorial and is an invariant of homotopy type.

**Example 6.1** There is a homeomorphism

$$B_d(X, n) = X \times \mathrm{SP}^{n-d}(X) \quad \text{when } d > [n/2] \text{ and } n \geq 3.$$

This is true because there can be only one entry with multiplicity  $\lfloor \frac{n}{2} \rfloor + 1$  or higher, so it can be singled out.

<sup>5</sup>See *Remarks on symmetric products of curves*, by the first author, [arxiv:math.AT/0402267](https://arxiv.org/abs/math/0402267).

**Example 6.2** Suppose  $[n/3] < d \leq [n/2]$ . Then there is a pushout diagram

$$\begin{CD} (X \times X) \times \mathrm{SP}^{n-2d}(X) @>f>> X \times \mathrm{SP}^{n-d}(X) \\ @V{\pi \times 1}VV @VV{\beta}V \\ \mathrm{SP}^2(X) \times \mathrm{SP}^{n-2d}(X) @>\alpha>> B_d(X, n), \end{CD}$$

where  $f(x, y, z) = (x, y^d z)$ ,  $\beta(a, b) = a^d b$  and  $\alpha(xy, z) = (x^d y^d z)$ . Here, for simplicity, we use the abelian product  $x^r y^s$  to denote the point  $[x, \dots, x, y, \dots, y]$  ( $x$  repeating  $r$ -times and  $y$  repeating  $s$  times). The justification behind this pushout is that when  $[n/3] < d \leq [n/2]$ , there are at most two points having multiplicity at least  $d$ , and when both occur they are indistinguishable.

The following has been discussed in Section 3.1. We recall that ‘‘Young’’ refers to a subgroup of  $\mathfrak{S}_n$  conjugate to a standard embedding of  $\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k}$  in  $\mathfrak{S}_n$ , with  $\sum d_i = n$ . Write  $\mathfrak{S}_d \subset \mathfrak{S}_n$  for the standard inclusion consisting of the subgroup permuting the first  $d$  letters in a set of  $n$  letters,  $d \leq n$ , leaving the others fixed. Then we have the following lemma.

**Lemma 6.3** We have the identification  $F_d(X, n) = \overline{X^n}_{(\mathfrak{S}_d)}$ .

Our next statement is a general observation that fits in naturally in this section. We shall give a characterization in terms of Young strata  $X^n_{(\text{young})}$  of some naturally occurring constructions. Below  $n \geq 1$  is a fixed integer and all spaces are path-connected. A space  $X$  is called *locally contractible* if any  $x \in X$  has a neighborhood deformation retract.

**Proposition 6.4** For each locally contractible  $X$ , suppose we can associate a  $\mathfrak{S}_n$ -invariant subspace  $F(X) \subset X^n$  so that any map  $f: X \rightarrow Y$  induces maps

$$(6.1) \quad \begin{CD} F(X) @<< \hookrightarrow X^n \\ @VVV @VV{f^n}V \\ F(Y) @<< \hookrightarrow Y^n. \end{CD}$$

Then  $F(X)$  must be a union of skeleta of the form  $\overline{X^n}_{(H)}$  for some choices of Young-subgroups  $H \subset \mathfrak{S}_n$ .

**Proof** What this says for  $n = 1$  is that  $F(X) = X$ , and for  $n = 2$  there are two possibilities; either  $F(X) = X^2$  or  $F(X)$  is the diagonal subspace. Note that by choosing  $f: \{x\} \rightarrow X, x \mapsto x$ , in diagram (6.1), we see that  $(x, \dots, x) \in F(X)$  and  $F(X)$  always contains the diagonal.

We say that a space  $X$  is *k-homogeneous* if for all  $(x_1, \dots, x_k) \in X^k$  satisfying  $x_i \neq x_j$  when  $i \neq j$  and all  $(y_1, \dots, y_k)$ , there exists a continuous<sup>6</sup> map  $h: X \rightarrow X$  such

<sup>6</sup>Customarily the definition of homogeneity involves the existence of homeomorphisms. This is too strong for our purpose.

that  $h(x_i) = y_i$ . We show that any locally contractible space is  $k$ -homogeneous for any  $k$ . Pick  $(x_1, \dots, x_k)$  and  $(y_1, \dots, y_k)$  as above. Remember that a pair of spaces  $(X, A)$  has the homotopy extension property (HEP) if for every map  $f: X \rightarrow Y$  and every homotopy  $h: A \times I \rightarrow Y$  with  $h(-, 0) = f|_A$ , there is a homotopy  $H: X \times I \rightarrow Y$  extending  $h$ . An NDR pair has the HEP, in particular, if  $A$  is a finite set of points in a locally contractible  $X$ , then  $(X, A)$  has the HEP. Set  $A = \{x_1, \dots, x_k\} \subset X$  and consider the constant map  $f: X \rightarrow X$  at  $x_0 \in X$ . The restriction of  $f$  to  $A$  is homotopic to the map  $h_1: A \rightarrow X$  sending  $x_i \mapsto y_i$  (this is because  $X$  is path-connected and any choice of paths from  $y_i$  to  $x_0$  gives a null homotopy for  $h_1$ ). The homotopy  $h_t: A \rightarrow X$  from  $h_0 = f$  to  $h_1$  extends to  $H_t: X \rightarrow X$ , and so the map  $H_1: X \rightarrow X$  is an extension of  $h_1$  with the desired properties.

Returning to the proof, let us pick an element  $(x_1, \dots, x_n) \in F(X)$ . It must belong to some stratum  $X^n_{(H)}$ ,  $H \cong \mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k}$ , since the Young strata form a stratification. The transitivity property established above shows that there are selfmaps of  $X$  that send  $(x_1, \dots, x_n)$  to any tuple  $(y_1, \dots, y_n)$  provided that  $y_{i_1} = \dots = y_{i_d}$  if  $x_{i_1} = \dots = x_{i_d}$ . This immediately implies that not only the whole stratum is in  $F(X)$ , but so is its closure  $\overline{X^n_{(H)}}$ . From there we deduce that  $F(X)$  must be a union of strata  $\overline{X^n_{(H)}}$  as asserted. ■

### 6.1 Cell and Manifold Structures

When  $X$  is a simplicial complex (i.e., a triangulated space), then  $\Gamma P^n X$  has the structure of a CW complex. For the case of symmetric products, this cell structure is discussed in [9, 18], for example. These descriptions lead to a more precise statement.

**Proposition 6.5** *For a simplicial complex  $X$ ,  $\Gamma P^n X$  is a CW complex such that the skeleta of its associated orbit stratification are subcomplexes.*

The point is the existence of a  $\mathfrak{S}_n$ -invariant (thus  $\Gamma$ -invariant) simplicial structure on  $X^n$ , induced from one on  $X$ , such that the fat diagonal is a CW-subcomplex. Because this simplicial structure is invariant, it induces a cellular (not necessarily simplicial) structure on the quotient  $\Gamma P^n X$  and such that the quotient map  $\pi: X^n \rightarrow \Gamma P^n X$  is cellular.

We recall that a space  $X$  is  $r$ -connected,  $r \geq 0$ , if the homotopy groups  $\pi_i(X)$  vanish for  $i \leq r$ .

**Corollary 6.6** *If  $X$  is an  $r$ -connected simplicial complex, then  $B_d(X, n)$  is also  $r$ -connected.*

**Proof** The proof of [17, Corollary 3.5] applies verbatim to this situation. It uses the CW decomposition on  $B_d(X, n)$  as a subcomplex of  $SP^n X$  (Proposition 6.5). ■

Even if  $X$  is a manifold, the spaces  $B_d(X, n)$  are rarely manifolds themselves. If  $\dim X = 1$ , then two cases occur:  $X = I$  is an open interval or  $X = S^1$ . When  $X$  is homeomorphic to an open interval  $I^\circ = ]0, 1[$ , then  $B_d(I^\circ, n)$  is the obvious subspace of the  $n$ -simplex

$$SP^n(I^\circ) = \{(t_1, \dots, t_n), 0 < t_1 \leq \dots \leq t_n < 1\} = \Delta_n$$

consisting of part of the boundary where  $t_{i_1} = \dots = t_{i_d}$  for some subset  $\{i_1, \dots, i_d\}$ . When  $X = S^1$ , then by a refined version of the result of Morton (Remark 2.1),  $B_d(S^1, n)$  is a bundle over  $S^1$  with fiber some faces of  $\Delta_{n-1}$ . For certain values of  $d$  this is a manifold. For example,  $B_2(S^1, n) \subset \text{SP}^n(S^1)$  is an  $n - 2$ -sphere bundle over  $S^1$ , and this is a manifold.

When  $\dim X \geq 2$ , the following theorem gives the complete conditions for  $B_d(X, n)$  to be manifolds.

**Theorem 6.7** *Let  $X$  be a manifold of dimension  $m \geq 2$  and suppose  $n \geq 3, d \geq 2$ . Then  $B_d(X, n)$  is a manifold again if and only if one of the following conditions is satisfied:*

- (i)  $d = n$  or  $d = n - 1$ ,
- (ii)  $d > [n/2]$  and  $X$  is a topological surface.

**Proof** The “if” part is clear, since  $B_n(X, n) = X, B_{n-1}(X, n) \cong X \times X$ , and  $B_d(X, n) \cong X \times \text{SP}^{n-d}(X)$  when  $d > [n/2]$ , and the symmetric products of topological surfaces are manifolds. We need to see that aside from these special cases,  $B_d(X, n)$  cannot be a manifold. Let us assume at the start that  $m \geq 2$  and that  $d \geq 3$ . Here we can set  $n \geq d + 2$  (the cases  $n = d$  and  $n = d + 1$  having been settled). Pick a point  $\zeta = [x, \dots, x, y, y, z_1, \dots, z_{n-d-2}] \in B_d(X, n)$ ,  $x$  repeating  $d$ -times. We assume that  $x \in V, y \in U$  and  $z_i \in U_i$  all lie in neighborhoods  $V, U$ , and  $U_i$  that are pairwise disjoint and homeomorphic to  $\mathbb{R}^m, m = \dim X$ . Then an open neighborhood of  $\zeta \in B_d(X, n)$  is  $V \times \text{SP}^2(U) \times U_1 \times \dots \times U_{n-d-2}$ . This can be given a Euclidean structure if and only if  $\text{SP}^2(U) \cong \text{SP}^2(\mathbb{R}^m)$  can be given a Euclidean structure. Unless  $m = 1$  or  $m = 2$ , it is not possible to find a neighborhood of the origin  $[0, 0] \in \text{SP}^2(\mathbb{R}^m)$  that is homeomorphic to  $\mathbb{R}^{2m}$ . The argument here was given at the start of Section 4 and is based on showing that  $\text{SP}^2(\mathbb{R}^m) \cong \mathbb{R}^m \times c\mathbb{R}P^{m-1}$ . To see this one decomposes  $W = \mathbb{R}^m \oplus \mathbb{R}^m$  as  $D \oplus V$  where  $D$  is the diagonal subspace fixed by the transposition  $A(x, y) = (y, x)$  acting on  $W$ , and where  $V = \{(x, -x) \mid x \in \mathbb{R}^m\}$  is the orthogonal subspace. Now  $V$  is a copy of  $\mathbb{R}^m$  on which  $A$  acts via  $x \mapsto -x$ . The quotient of  $D \oplus V$  by this action  $\mathbb{Z}_2$  (generated by  $A$ ) is clearly  $\mathbb{R}^m \times c\mathbb{R}P^{m-1}$ . Since no neighborhood of the cone point  $[0, 0]$  can be homeomorphic to  $\mathbb{R}^{2m}$  if  $m \geq 3$ , the only possibility is therefore  $m = 2$  (or  $m = 1$ , but this is not relevant as  $m \geq 2$ ) and  $X$  is a topological surface.

We are then left to discuss the case  $d = 2, m \geq 2$ . Assume without loss of generality that  $n = 2d = 4$ . Let  $x, y$  be two points of  $X$  lying in disjoint Euclidean neighborhoods  $U$  and  $V$  respectively. A neighborhood of  $x^2y^2 \in B_2(X, 4)$  can be seen to be the pushout

$$(6.2) \quad U \times \text{SP}^2(V) \cup_{U \times V} \text{SP}^2(U) \times V,$$

where  $U \times V$  includes in  $B_2(X, 4)$  via the map  $(x, y) \mapsto x^2y^2$ . This neighborhood has an embedded copy of  $U \times V \cong \mathbb{R}^m \times \mathbb{R}^m$ . This embedded copy is of codimension  $m \geq 2$ . Therefore if the neighborhood in (6.2) were homeomorphic to Euclidean space, removing  $U \times V$  from this neighborhood would not disconnect it. But this is not the case since the complement of  $U \times V$  in this pushout is disconnected. ■

### 6.2 Homology

We make some useful remarks about the homological structure of  $B_d(X, n)$ , relevant to Section 7. First of all, using the known fact that  $H_*(\mathbb{S}P^n X; \mathbb{Z})$  embeds in  $H_*(\mathbb{S}P^{n+d} X; \mathbb{Z})$  for  $d \geq 1$  via the *basepoint embedding*  $\mathbb{S}P^n(X) \hookrightarrow \mathbb{S}P^{n+d}(X)$  sending  $[x_1, \dots, x_n] \mapsto [x_1, \dots, x_n, *, \dots, *]$ , where  $*$  is a basepoint, we immediately deduce the following lemma.

**Lemma 6.8** *The inclusion*

$$\mathbb{S}P^n X \hookrightarrow B_d(X, n + d), [x_1, \dots, x_n] \longmapsto [x_1, \dots, x_n, *, \dots, *]$$

*induces a monomorphism in homology.*

Generally, however,  $H_*(B_d(X, n))$  does not embed in  $H_*(B_d(X, n + 1))$ . This is illustrated in the case  $d = 2$  by the following calculation for example. Here we recall that  $B_2(S^n, 3) \cong S^n \times S^n$ , and we have the following proposition.

**Proposition 6.9** ([24])  *$B_2(S^n, 4)$  has the integral homology of*

$$S^n \vee \Sigma^{n+1} \mathbb{R}P^{n-1} \vee \Sigma^{n+1} \mathbb{R}P^{n-1} \vee \Sigma^{2n+1} \mathbb{R}P^{n-1} \vee \Sigma^{2n-1} \mathbb{R}P^2.$$

**Proof** There is an identification due to Steenrod of the mapping cone of the basepoint inclusion  $S^n \hookrightarrow \mathbb{S}P^2(S^n)$  with the suspension  $\Sigma^{n+1} \mathbb{R}P^{n-1}$  (see [13, example 4K.5]). This gives that  $H_*(\mathbb{S}P^2(S^n))$  has a torsion free class in degree  $n$  and another in degree  $2n$  only if  $n$  is even, and a copy of  $\mathbb{Z}_2$  in degrees  $r = n + 2k$  for  $1 \leq k \leq [(n - 1)/2]$ . The rest of the argument follows from a careful analysis of the Mayer–Vietoris exact sequence associated with the pushout diagram in Example 6.2 (details in [24]). ■

Note that  $H_*(B_2(S^n, 4))$  does not satisfy Poincaré duality consistently with Theorem 6.7. The following more general calculation for fat diagonals of spheres is valid with rational coefficients.

**Proposition 6.10** ([11]) *The rational homotopy type of the fat diagonal of an odd sphere is*

$$B_2(S^{2k+1}, n) \simeq_{\mathbb{Q}} \begin{cases} S^{2k+1}, & n \text{ even,} \\ S^{2k+1} \times S^{m(2k+1)-1}, & n = 2m + 1 \text{ odd.} \end{cases}$$

Note how this is naturally consistent with Proposition 6.9 when  $n = 4$ , since  $\mathbb{R}P^{2k}$  has the rational homology of a point.

## 7 Fat Diagonals and their Fundamental Groups

In this last section we prove Theorem 1.2. We will be using the van Kampen Theorem in the following form: let  $G = \langle g_1, \dots, g_n \mid r_G \rangle$  and  $H = \langle h_1, \dots, h_m \mid r_H \rangle$  be finitely presented groups, where  $r_G$  and  $r_H$  stand for the corresponding ideals of relations.

Let  $K$  have presentation  $K = \langle k_1, \dots, k_t \mid r_K \rangle$ , mapping to  $H$  and  $G$  via  $\alpha$  and  $\beta$  respectively. The amalgamated product  $H *_K G$  has presentation

$$\langle g_1, \dots, g_n, h_1, \dots, h_m \mid r_G, r_H, \alpha(k_i)\beta(k_i)^{-1} \rangle.$$

Note that the relations  $r_K$  do not matter. Note also that if  $\alpha$  is surjective, then the “universal” map  $G \rightarrow H *_K G$  is surjective as well.

Here is the main assertion of this section.

**Lemma 7.1** For  $n \geq 2$  and  $1 \leq d \leq \frac{n}{2}$ ,  $\pi_1(B_d(X, n))$  is abelian.

**Proof** We can assume that  $d \geq 2$ , since when  $d = 1$ ,  $B_1(X, n) = \text{SP}^n X$ . As in the proof of the commutativity of  $\pi_1(\text{FP}^n X)$  in Section 5, the idea here is to start with two elements in  $\pi_1(B_d(X, n))$ , lift them to  $\pi_1(F_d(X, n))$  via the surjection  $\pi_* : \pi_1(F_d(X, n)) \twoheadrightarrow \pi_1(B_d(X, n))$  (see Corollary 5.4), then, using the action of  $\mathfrak{S}_n$ , separate them into elements that commute. Write

$$\pi_1(X) = G = \langle g_i, i \in J \mid \text{relations} \rangle, \quad \text{with } J \text{ some finite indexing set.}$$

For  $I = \{i_1, \dots, i_d\}$ , define

$$\Delta_I(X, n) = \Delta_{i_1, \dots, i_d}(X, n) := \{(x_1, \dots, x_n) \mid x_{i_1} = \dots = x_{i_d}\}$$

Then  $\pi_1(\Delta_{i_1, \dots, i_d}(X, n)) \subset G^n$  has generators

$$g_i^I \mid g_{a,j}^I, \quad i, j \in J, a \notin I,$$

where  $g_i^I = (e, \dots, g_i, \dots, g_i, \dots, e)$  is the element of  $G^n$  with  $g_i$  in the  $i_1, i_2, \dots, i_d$  entries and the identity  $e$  in the remaining entries, while  $g_{a,j}^I = (e, \dots, g_j, e, \dots, e)$  is the element of  $G^n$  with  $g_j$  in the  $a$ -th entry and the identity  $e$  in the remaining entries. Under the isomorphism  $\pi_1(\Delta_{i_1, \dots, i_d}(X, n)) \cong G \times G^{n-d}$ , the  $g_i^I$  generate the first factor  $G$  and the  $g_{a,j}^I$  generate the second factor  $G^{n-d}$ . For example,  $\pi_1(\Delta_{1,2}(X, 3))$  has generators

$$g_i^{12}, g_{3,j}^{12}, \quad i, j \in J.$$

Since  $F_d(X, n) = \bigcup \Delta_I(X, n)$ , with  $I$  a sequence of the form  $1 \leq i_1 < \dots < i_d \leq n$ , an iterated use of the van Kampen theorem shows that  $\pi_1(F_d(X, n))$  is generated by the  $g_i^I$  and  $g_{a,j}^I$  with  $I$  varying over all such sequences,  $a \notin I$  and  $i, j \in J$ , subject to various relations. Among these relations are those coming from  $\pi_1(\Delta_I(X, n))$ , which are

- $g_i^I$  and  $g_{a,j}^I$ ,  $a \notin I$ , commute since they come from different factors of  $G^n$ ;
- similarly  $g_{a,r}^I$  and  $g_{b,s}^I$  commute if  $a \neq b$ .

Write  $I_d = \{1, 2, \dots, d\}$  and  $I^d = \{n-d+1, n-d+2, \dots, n\}$ . These are disjoint subsets, since  $n \geq 2d$ . Consider the pushout diagram

$$\begin{array}{ccc} X \times X \times X^{n-2d} & \xrightarrow{f} & \Delta_{I_d}(X, n) \\ \downarrow g & & \downarrow \\ \Delta_{I^d}(X, n) & \longrightarrow & \Delta_{I^d}(X, n) \cup \Delta_{I_d}(X, n), \end{array}$$

where the image of  $(x, y, (z_1, \dots, z_{n-2d}))$  under both inclusions  $f$  and  $g$  is

$$(x, x, \dots, x, z_1, \dots, z_{n-2d}, y, y, \dots, y).$$

If we write  $\pi_1(X \times X \times X^{n-2d})$  as  $G \times G \times G^{n-2d}$ , then the generators  $g_{1,i}$  coming from the first copy of  $G$  map to

$$f_*(g_{1,i}) = g_i^{I^d}, \quad g_*(g_{1,i}) = g_{1,i}^{I^d} + g_{2,i}^{I^d} + \dots + g_{d,i}^{I^d}.$$

By the van Kampen theorem, this implies that in  $\pi_1(F_d(X, n))$ , the generators of the same name  $g_i^{I^d}$  and  $g_{a,i}^{I^d}$  satisfy the relation for given  $i \in J$

$$(7.1) \quad g_i^{I^d} = g_{1,i}^{I^d} + g_{2,i}^{I^d} + \dots + g_{d,i}^{I^d}.$$

Now let  $\alpha, \beta$  be two elements in  $\pi_1(B_d(X, n))$  and write  $\tilde{\alpha}, \tilde{\beta}$  in  $\pi_1(F_d(X, n))$  such that  $\pi_*(\tilde{\alpha}) = \alpha$  and  $\pi_*(\tilde{\beta}) = \beta$ . In terms of generators we can express  $\tilde{\alpha}$  as a finite sum

$$\tilde{\alpha} = \sum n_i^I g_i^I + \sum m_j^{I'} g_{a,j}^{I'}, \quad n^I, m^{I'} \in \mathbb{Z}$$

for some choice of  $I, I', a, i, j$ . We know that under the  $\mathfrak{S}_n$  quotient,

$$(7.2) \quad \pi_*(g_i^I) = \pi_*(g_i^{\sigma(I)}), \quad \pi_*(g_{a,j}^{I'}) = \pi_*(g_{\sigma(a),j}^{\sigma(I')}), \quad a \notin I.$$

The first equality follows from the fact that  $\mathfrak{S}_n$  acts  $d$ -transitively on  $\{1, \dots, n\}$  meaning that any subset of cardinality  $d$  is mapped to any other by the action. From the relations (7.2), another choice of a lift for  $\alpha$  would then be

$$\tilde{\alpha}_1 = \sum n_i g_i^{I^d} + \sum m_j g_{1,j}^{I^d},$$

and this lives in  $\pi_1(\Delta_{I^d}(X, n))$ . We replace  $g_i^{I^d}$  by  $g_i^{I^d}$ , and this in turn can be replaced by  $g_{1,i}^{I^d} + \dots + g_{d,i}^{I^d}$  according to (7.1). Similarly we can replace each  $g_{a,i}^{I^d}$  by  $g_{1,i}^{I^d}$ , since one is mapped into the other by the transposition  $(1, a)$ . At the end and for  $n \geq 2d$  there is a choice of lift

$$(7.3) \quad \tilde{\alpha}_2 = \sum t_k g_{1,k}^{I^d} \quad t_k \in \mathbb{Z}$$

such that  $\pi_*(\tilde{\alpha}_2) = \alpha$ . Similarly there is a choice of a lift for  $\beta$  of the form

$$(7.4) \quad \tilde{\beta}_2 = \sum \ell_k g_{2,k}^{I^d} \quad \ell_k \in \mathbb{Z}.$$

This choice is possible, since  $d \geq 2$ . The expressions in (7.3) and (7.4) commute in  $\pi_1(F_d(X, n))$ , and hence  $\alpha$  commutes with  $\beta$  in  $\pi_1(B_d(X, n))$ . This shows that  $\pi_1(B_d(X, n))$  is abelian. ■

The following corollary implies Theorem 1.2.

**Corollary 7.2** Assume  $2 \leq d \leq n/2$ . Then the map

$$X \hookrightarrow B_d(X, n), \quad x \mapsto [x, *, *, \dots, *]$$

induces an isomorphism on  $H_1$ .

**Proof** Equation (7.3) shows that if  $X \hookrightarrow \Delta_{I^d}(X, n)$  is the inclusion of the first factor (basepoint everywhere else), then the composite  $X \rightarrow \Delta_{I^d}(X, n) \rightarrow B_d(X, n)$  is surjective on  $\pi_1$  and thus on  $H_1$ . It remains to see that  $H_1(X) \rightarrow H_1(B_d(X, n))$  is injective but this is the content of Lemma 6.8. ■

A subgroup of  $\mathfrak{S}_n$  is  $d$ -transitive if it takes any ordered  $d$  points in  $\{1, \dots, n\}$  to any other ordered  $d$ -points. The following more general result follows from an adaptation of the proof of Theorem 7.1.

**Corollary 7.3** If  $\Gamma$  is  $d$ -transitive and  $1 \leq d \leq n/2$ , then  $\pi_1(F_d(X, n)/\Gamma)$  is abelian.

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