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# BOOK REVIEWS

BRIDGES, D. S. *Constructive functional Analysis* (Pitman Publishing Ltd. 1979), £7·50.

This book is a carefully written, detailed exposition of part of the theory of constructive analysis developed by E. A. Bishop and his followers. It contains accounts of metric spaces, normed spaces (with versions of the Hahn–Banach and Stone–Weierstrass theorems), measure theory on locally compact metric spaces up to Fubini's theorem, and operators on Hilbert space (with versions of the functional calculus for self-adjoint operators and the Gelfand representation theory for commutative $C^*$-algebras). In general the exposition is admirably complete, and should be accessible to final-year undergraduates with a sound grasp of conventional $\varepsilon$-$\delta$ analysis. The author's aim is to show that by accepting the disciplines of constructive analysis we are led to more informative proofs.

A description, as brief as possible, of these disciplines is perhaps appropriate. The fundamental point is that a statement of the form

$$\forall x P(x)$$

should be interpreted as: there is a systematic method of finding for any $x$, a proof of $P(x)$; while

$$\exists x P(x)$$

should be interpreted as: there is a definite programme which will yield an $x$ for which $P(x)$. In the same spirit, $p \vee q$ means "either we have a proof of $p$ or we have a proof of $q$"; the celebrated rejection of the law of the excluded middle follows naturally, since for many statements $p$ we do not have (or indeed cannot have) a proof either of $p$ or of $\sim p$.

The author accepts that constructive analysis is at present a minor offshoot of mathematics; he does not wish to challenge the validity of the propositions of ordinary analysis; indeed his whole aim seems to be to trace those results of ordinary analysis which can be proved by constructivist methods. It is not clear to me that this is a good thing. To begin with, his desire to use similar language to that of ordinary mathematics while dealing with substantially different structures leads to tiresome adjustments for the reader. For instance, he never (for good reasons) takes equivalence classes. Consequently he says that a real number "is" a suitably rapidly convergent Cauchy sequence of rationals, and when he writes "$a = b$" for real numbers he means that $a$ and $b$ are equivalent Cauchy sequences. Generally, the reader must bear constantly in mind that the symbol = almost always refers to a non-trivial equivalence relation. Similarly, the "metrics" and "norms" of this book are much closer to what most people call pseudo-metrics and seminorms. Now this amounts to a systematic abuse of language. The object which in this book is called the set of real numbers differs in so many vital respects from the objects which have gone under that name in the past that the use of traditional terminology is an actual barrier to understanding.

A second problem arises when we come to the definition of such concepts as "continuity" and "differentiability". If you take the ordinary definitions of "continuity" and "uniform continuity" and seek to interpret them in a constructivist framework, you find that "uniform continuity" behaves much as it does in ordinary analysis, but that "continuity" is very different, and (to a classical analyst) disconcertingly weak. This is not acceptable to the present author; he avoids it by an extraordinary definition of continuity specifically designed to retain the theorem that a continuous function on a complete totally bounded pseudo-metric space is uniformly continuous. Later he slices a similar knot by firmly writing down a definition of a kind of uniform differentiability and calling it "differentiability"; others of his definitions directly incompatible with normal usage are those of local compactness and equivalent metric. I should emphasise that it is only the names which he is changing; the definitions he gives are clearly recognisable as concepts which are of use in ordinary mathematics under different names—like the set of Cauchy sequences in $\mathbb{Q}$.

At a more advanced level, the discussion of measure theory is impressive for the fact that the elementary theory of the Lebesgue integral can be done at all; but it is even more apparent than elsewhere that this is paraplegic mathematics, struggling desperately to match the achievements of its un-handicapped ideal. A point which strikes me here and in the chapters on normed spaces is that the constructivist approach seems to lead to theorems similar to those which can be proved in ordinary mathematics with countable choice alone; for instance, the Hahn–Banach theorem here is restricted to separable spaces.

I am inclined to think that the author's wish to avoid the paradoxical aspects of constructivism has in fact deprived him of the only inspiration likely to come from this approach. Constructivism, as presented here, merely leads to enormous technical difficulties without shedding much light on the real questions of mathematics. Of course there are many useful flashes of ingenuity which are struck by the effort of thinking the basic theory out anew. But I should like to suggest that, if there is anything of substantial value in constructivism, it is more likely to come from a rigorous formulation, within Aristotelian logic of the acceptable rules of proof—requiring, perhaps, a proof of $\forall x P(x)$ to be a proof of $P(x)$ which is a recursive function of $x$—followed by a systematic analysis of the ways in which the theory differs from real mathematics. As an example of a topic in which such an approach has had great success I offer effective descriptive set theory.

In conclusion: No doubt it is good for constructivists to learn some functional analysis. I do not think that there is yet much reason for analysts to learn constructivism.

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In 1967 Errett Bishop's book 'Foundations of constructive analysis' appeared. Since then there has been a steady flow of work within the framework set out in Bishop's book. The book under review partially replaces Bishop's book, which has been out of print for some time. The two books do not have quite the same range of subjects but the present book contains improvements and developments since 1967, many of them due to Bridges himself.

The book gives a clear self-contained introduction to constructive analysis. Readers willing to restrict their methods of proof to meet constructive requirements should be able to pick up the perhaps unfamiliar pattern of thought without unnecessary effort. As in Bishop's book, the logic and philosophy of constructive mathematics is treated very briefly. Just as with classical analysis constructive analysis can be learnt and used without undue reflection on the fundamental notions.

The ideas motivating this book have their origin in Brouwer's intuitionistic criticism of non-constructive methods. But the rather extreme subjective aspects of Brouwer's thought have been avoided and the presentation is fairly straight forward. Bridges follows Bishop in the following respects:

(i) Mathematical objects are kept concrete, in the sense that they are always in principle arithmetically representable.
(ii) All operations on these objects are intended to be computable.
(iii) The language is kept as close as possible to the standard set theoretical one.

This entails a systematic avoidance of abstract objects obtained when taking the quotient of a set by an equivalence relation. Instead, each set has to carry with it the equivalence relation which holds between two concrete objects when they represent the same abstract object of conventional mathematics. This systematic departure from the conventional presentation can be irksome at first, but it is easy to adapt to it in practice. In fact it would be possible to give a standard set theoretical presentation of constructive analysis that *did* allow the quotient construction, but then the constructive computational character of the mathematics would no longer be explicit and it would be necessary to give a separate account of the procedure for making it explicit.

In constructive mathematics the meaning of a mathematical statement is given by specifying the mathematical constructions that are to count as proofs of the statement. The truth of a