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PRENEX NORMAL FORM THEOREMS IN SEMI-CLASSICAL ARITHMETIC

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Abstract. Akama et al. [1] systematically studied an arithmetical hierarchy of the law of excluded middle and related principles in the context of first-order arithmetic. In that paper, they first provide a prenex normal form theorem as a justification of their semi-classical principles restricted to prenex formulas. However, there are some errors in their proof. In this paper, we provide a simple counterexample of their prenex normal form theorem [1, Theorem 2.7], then modify it in an appropriate way which still serves to largely justify the arithmetical hierarchy. In addition, we characterize a variety of prenex normal form theorems by logical principles in the arithmetical hierarchy. The characterization results reveal that our prenex normal form theorems are optimal. For the characterization results, we establish a new conservation theorem on semi-classical arithmetic. The theorem generalizes a well-known fact that classical arithmetic is Π_2-conservative over intuitionistic arithmetic.

§1. Introduction. Prenex normal form theorem is one of the most basic theorems on theories based on classical first-order predicate logic. In contrast, it does not hold for intuitionistic theories in general. Therefore it does not make sense to consider an arithmetical hierarchy in an intuitionistic theory. On the other hand, if one reasons in some semi-classical arithmetic which lies in-between classical arithmetic and intuitionistic arithmetic, one can take an equivalent formula in prenex normal form for any formula with low complexity. Akama et al. [1] introduces the classes of formulas E_k and U_k which correspond to the classes of classical Σ_k and Π_k formulas respectively, and showed that the former is equivalent to the class of formulas of Σ_k form and the latter is so for Π_k over some semi-classical arithmetic respectively. This prenex normal form theorem justifies their investigation on the arithmetical hierarchy in the context of intuitionistic first-order arithmetic. Unfortunately, however, there are some crucial errors in their proof of the prenex normal form theorem [1, Theorem 2.7]. In this paper, we revisit their formulation and modify their prenex normal form theorem in an appropriate way.

In §2, we recall the definitions and basic properties. In §3, we provide a simple counterexample of [1, Theorem 2.7]. In §5, we show the corrected version of the prenex normal form theorem (see Theorem 5.3). In addition, we present a simplified version of the prenex normal form theorem for formulas which do not contain the disjunction (see Theorem 5.7). In §6, we establish a new conservation theorem on semi-classical arithmetic. The theorem generalizes a well-known fact

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that classical arithmetic is Π_2-conservative over intuitionistic arithmetic. In §7, using the generalized conservation theorem in §6, we characterize several prenex normal form theorems with respect to semi-classical arithmetic. In particular, among other things, we show that for any theory T in-between intuitionistic arithmetic and classical arithmetic, T proves a semi-classical principle $(\Pi_k \vee \Pi_k)$-DNE if and only if T satisfies the prenex normal form theorem for $\mathrm{U}_{k'}$ and $\Pi_{k'}$ for all $k' \leq k$ (see Theorem 7.3).

Throughout this paper, we work basically over intuitionistic arithmetic. When we use some principle (including induction hypothesis [I.H.]) which is not available in intuitionistic arithmetic, it will be exhibited explicitly. As regards basic reasoning over intuitionistic first-order logic, we refer the reader to [10, §6.2].

§2. Preparation. Throughout this paper, we work with a standard formulation of intuitionistic arithmetic HA described e.g., in [8, §1.3], which has function symbols for all primitive recursive functions. We work in the language containing all the logical constants $\forall, \exists, \rightarrow, \wedge, \vee, \perp$. Let T denote a theory (e.g., HA), and P and Q denote schemata (e.g., logical principles). Then $T + \mathrm{P}$ denotes the theory obtained from T by adding P into the axioms. In particular, the classical variant PA is defined as $\mathsf{HA} + \mathrm{LEM}$, where LEM is the axiom scheme of the law of excluded middle. We write $T \vdash \mathrm{Q}$ (or T proves Q) if any instance of Q is provable in T. We write $T \vdash \mathrm{P} + \mathrm{Q}$ if $T \vdash \mathrm{P}$ and $T \vdash \mathrm{Q}$.

NOTATION 1. For a formula φ, $\mathsf{FV}(\varphi)$ denotes the set of free variables in φ. Quantifier-free formulas are denoted with subscript "qf" as φ_{qf}. In addition, a list of variables is denoted with an over-line as $\overline{x}$. In particular, a list of quantifiers of the same kind is denoted as $\exists \overline{x}$ and $\forall \overline{x}$ respectively.

DEFINITION 2.1. The classes Σ_k and Π_k of formulas are defined as follows:

- Σ_0, as well as Π_0, is the class of all quantifier-free formulas;
- Π_{k+1} is the class of all formulas of form $Q_1\overline{x_1} \cdots Q_{k+1}\overline{x_{k+1}}\, \varphi_{\mathrm{qf}}$;
- Σ_{k+1} is the class of all formulas of form $Q'_1\overline{x_1} \cdots Q'_{k+1}\overline{x_{k+1}}\, \varphi_{\mathrm{qf}}$;

where Q_i represents $\forall$ for odd i and $\exists$ for even i and Q'_i represents $\exists$ for odd i and $\forall$ for even i. Following [1], we define the classes Σ_k and Π_k in the non-cumulative manner (namely, each $Q_i\overline{x_i}$ and $Q'_i\overline{x_i}$ must not be empty). A formula φ is of *prenex normal form* if $\varphi \in \Sigma_k \cup \Pi_k$ for some k.

REMARK 2.2. Since the list of variables can be contracted into one variable in HA by using a fixed primitive recursive pairing function (see e.g., [8, §1.3.9]), one may assume that for each natural number $k > 0$, a formula in Σ_k is of form $\exists x \varphi(x)$ with some $\varphi(x) \in \Pi_{k-1}$ and a formula in Π_k is of form $\forall x \psi(x)$ with some $\psi(x) \in \Sigma_{k-1}$ without loss of generality.

LEMMA 2.3. *Let k be a natural number. Let φ be in Π_k and ψ be in Σ_k. Then, for all natural numbers i* and *j, there exist $\varphi', \psi' \in \Pi_{k+i}$ and $\varphi'', \psi'' \in \Sigma_{k+j}$ such that* $\mathsf{FV}(\varphi) = \mathsf{FV}(\varphi') = \mathsf{FV}(\varphi'')$, $\mathsf{FV}(\psi) = \mathsf{FV}(\psi') = \mathsf{FV}(\psi'')$, $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi' \leftrightarrow \varphi''$ *and* $\mathsf{HA} \vdash \psi \leftrightarrow \psi' \leftrightarrow \psi''$.

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PROOF. Straightforward by the fact that

$$\mathsf{HA} \vdash \xi \leftrightarrow \forall z \xi \leftrightarrow \exists z \xi \tag{1}$$

for any $z \notin \mathrm{FV}(\xi)$. ⊣

DEFINITION 2.4. For a class Γ of formulas, $\Gamma(\overline{x})$ denotes the class of formulas φ in Γ such that $\mathrm{FV}(\varphi) \subseteq \{\overline{x}\}$.

REMARK 2.5. In the light of Lemma 2.3, throughout this paper, we identify the classes Σ_k and Π_k with the classes defined as in Definition 2.1 with allowing the quantifiers Q_i and Q'_i to be empty. Under this identification, for all k and k' such that $k < k'$, $\Pi_k(\overline{x})$ and $\Sigma_k(\overline{x})$ are considered to be sub-classes of $\Sigma_{k'}(\overline{x}) \cap \Pi_{k'}(\overline{x})$. We frequently use this property in what follows.

Recall the logical principles from [1] and related principles:

DEFINITION 2.6. Let Γ and Γ' be classes of formulas.

- Γ-LEM : $\forall x\, (\varphi(x) \vee \neg\varphi(x))$ where $\varphi(x) \in \Gamma(x)$.
- Γ-DML : $\forall x\, (\neg(\varphi(x) \wedge \psi(x)) \to \neg\varphi(x) \vee \neg\psi(x))$ where $\varphi(x), \psi(x) \in \Gamma(x)$.
- Γ-DNE : $\forall x\, (\neg\neg\varphi(x) \to \varphi(x))$ where $\varphi(x) \in \Gamma(x)$.
- $(\Gamma \vee \Gamma')$-DNE : $\forall x\, (\neg\neg(\varphi(x) \vee \psi(x)) \to \varphi(x) \vee \psi(x))$ where $\varphi(x) \in \Gamma(x)$ and $\psi(x) \in \Gamma'(x)$.
- Γ-DNS : $\forall x\, (\forall y \neg\neg\varphi(x,y) \to \neg\neg\forall y \varphi(x,y))$ where $\varphi(x,y) \in \Gamma(x,y)$.
- Let $\mathrm{P} \in \{\Gamma\text{-LEM}, \Gamma\text{-DML}, \Gamma\text{-DNE}, (\Gamma \vee \Gamma')\text{-DNE}, \Gamma\text{-DNS}\}$. $\neg\neg\mathrm{P}$: $\neg\neg\xi$ where ξ is an instance of P.

Note that our logical principles are equivalent also to those defined with lists of quantifiers of the same kind (cf. Remark 2.2).

REMARK 2.7. One has to care about the formulation of the double negated variants. That is, one has to take the double negations of the universal closure of the original logical principles as in Definition 2.6. The double negated variants defined as such are not provable in HA, which has been overlooked in the proof of [1, Theorem 2.7] (see also §3). In fact, one may think of the double negated versions as variants of the double negation shift principle (see [3]). In addition, our double negated versions are equivalent to (the universal closures of) those with allowing free variables (cf. [3, Remark 2.5]).

REMARK 2.8. For any class Γ of formulas, Γ-DNS is intuitionistically equivalent to $\neg\neg\Gamma$-DNS since

$$\begin{aligned} & \forall x\, (\forall y \neg\neg\varphi \to \neg\neg\forall y \varphi) \\ \longleftrightarrow\ & \forall x \neg\neg\, (\forall y \neg\neg\varphi \to \neg\neg\forall y \varphi) \\ \longleftrightarrow\ & \neg\neg\forall x \neg\neg\, (\forall y \neg\neg\varphi \to \neg\neg\forall y \varphi) \\ \longleftrightarrow\ & \neg\neg\forall x\, (\forall y \neg\neg\varphi \to \neg\neg\forall y \varphi)\,. \end{aligned}$$

Akama et al. [1] introduces the classes F_k, U_k and E_k of formulas. In the following, we reformulate them and introduce two additional classes U_k^+ and E_k^+ in a formal manner.

DEFINITION 2.9. An alternation path is a finite sequence of + and – in which + and – appear alternatively. For an alternation path s, let $i(s)$ denote the first symbol

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of s if $s \not\equiv \langle\rangle$ (empty sequence); $\times$ if $s \equiv \langle\rangle$. Let $s^{\perp}$ denote the alternation path which is obtained by switching $+$ and $-$ in s, and let $l(s)$ denote the length of s.

DEFINITION 2.10. For a formula φ, the set of alternation paths $Alt(\varphi)$ of φ is defined as follows:

- If φ is quantifier-free, then $Alt(\varphi) := \{\langle\rangle\}$;
- Otherwise, $Alt(\varphi)$ is defined inductively by the following rule:
  - If $\varphi \equiv \varphi_1 \wedge \varphi_2$ or $\varphi \equiv \varphi_1 \vee \varphi_2$, then $Alt(\varphi) := Alt(\varphi_1) \cup Alt(\varphi_2)$;
  - If $\varphi \equiv \varphi_1 \to \varphi_2$, then $Alt(\varphi) := \{s^{\perp} \mid s \in Alt(\varphi_1)\} \cup Alt(\varphi_2)$;
  - If $\varphi \equiv \forall x \varphi_1$, then $Alt(\varphi) := \{s \mid s \in Alt(\varphi_1) \text{ and } i(s) \equiv -\} \cup \{-s \mid s \in Alt(\varphi_1) \text{ and } i(s) \not\equiv -\}$;
  - If $\varphi \equiv \exists x \varphi_1$, then $Alt(\varphi) := \{s \mid s \in Alt(\varphi_1) \text{ and } i(s) \equiv +\} \cup \{+s \mid s \in Alt(\varphi_1) \text{ and } i(s) \not\equiv +\}$.

In addition, for a formula φ, the degree $deg(\varphi)$ of φ is defined as

$$deg(\varphi) := \max\{l(s) \mid s \in Alt(\varphi)\}.$$

DEFINITION 2.11. The classes $\mathrm{F}_k, \mathrm{U}_k, \mathrm{E}_k$ (from [1, Definition 2.4]), U_k^+ and E_k^+ of formulas are defined as follows:

- $\mathrm{F}_k := \{\varphi \mid deg(\varphi) = k\}$;
- $\mathrm{U}_0 := \mathrm{E}_0 := \mathrm{F}_0$;
- $\mathrm{U}_{k+1} := \{\varphi \in \mathrm{F}_{k+1} \mid i(s) \equiv - \text{ for all } s \in Alt(\varphi) \text{ such that } l(s) = k+1\}$;
- $\mathrm{E}_{k+1} := \{\varphi \in \mathrm{F}_{k+1} \mid i(s) \equiv + \text{ for all } s \in Alt(\varphi) \text{ such that } l(s) = k+1\}$;
- $\mathrm{U}_k^+ := \mathrm{U}_k \cup \bigcup_{i<k} \mathrm{F}_i$;
- $\mathrm{E}_k^+ := \mathrm{E}_k \cup \bigcup_{i<k} \mathrm{F}_i$.

REMARK 2.12. From the perspective of Proposition 4.6 below, the introduction of U_k^+ and E_k^+ in addition to U_k and E_k is mathematically superfluous. However, we introduce these auxiliary classes for facilitating our arguments below.

Note $\mathrm{F}_0 = \Sigma_0 = \Pi_0$. For each formula $\varphi \in \mathrm{E}_k$ (resp. $\psi \in \mathrm{U}_k$) of PA, one can take a formula $\varphi' \in \Sigma_k$ (resp. $\psi' \in \Pi_k$) of PA which is equivalent to φ (resp. ψ) over PA. On the other hand, this is not the case for HA. In what follows, we study what kind of semi-classical arithmetic in-between PA and HA captures this property for each k. In fact, Akama et al. [1] has already undertaken this. In particular, [1, Theorem 2.7] asserts the following:

1. For any $\varphi \in \mathrm{E}_k$, there exists $\varphi' \in \Sigma_k$ such that
$$\mathsf{HA} + \Sigma_k\text{-DNE} \vdash \varphi \leftrightarrow \varphi'.$$
2. For any $\varphi \in \mathrm{U}_k$, there exists $\varphi' \in \Pi_k$ such that
$$\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \varphi \leftrightarrow \varphi'.$$

However, the first assertion is wrong as we show in §3. In fact, a weak variant U_k-DNS of the double negation shift principle is missing in the verification theory, which will be revealed by our modified version of the prenex normal form theorem (Theorem 5.3) below. On the other hand, the second assertion is correct. This will be revealed also by Theorem 5.3.

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§3. A counter example. Recall that [1, Theorem 2.7] asserts that for any $\varphi \in \mathrm{E}_k$, there exists $\varphi' \in \Sigma_k$ such that

$$\mathsf{HA} + \Sigma_k\text{-DNE} \vdash \varphi \leftrightarrow \varphi'.$$

However, there are some errors in the proof. In particular, in [1, p. 5, lines 15–17], it is written that "Since the double negations of DNE is intuitionistically provable, $\vdash_{\mathsf{HA}} \neg\neg A_0 \leftrightarrow \neg\neg\exists x_0.C_0$ (which means $\mathsf{HA} \vdash \neg\neg A_0 \leftrightarrow \neg\neg\exists x_0 C_0$ in our notation)". As studied in [3], however, the double negations of (the universal closure of) DNE is not provable in HA, and hence, their proof actually uses some double negated logical principles in the sense of Definition 2.6. Our counterexample below shows that such a use of some additional principle is unavoidable.

Recall the arithmetical form of Church's thesis from [8, §3.2.14]:

$$\mathrm{CT}_0 : \forall x \exists y\, \varphi(x, y) \to \exists e \forall x \exists v \left(\mathrm{T}(e, x, v) \wedge \varphi(x, \mathrm{U}(v))\right),$$

where T and U are the standard primitive recursive predicate and function from the Kleene normal form theorem. Note that CT_0 is a sort of combination of so-called Church's thesis stating that every function is recursive and the countable choice principle (see [9, §4.3.2]).

Proposition 3.1. *The following sentence*

$$\varphi_0 :\equiv \neg\forall x \left(\neg\exists u \left(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) = 0\right) \vee \neg\exists u \left(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) \neq 0\right)\right)$$

is not equivalent to any sentence $\varphi_0' \in \Sigma_1$ over $\mathsf{HA} + \Sigma_1$-DNE.

Proof. We first claim that $\mathsf{HA} + \mathrm{CT}_0$ proves φ_0. For the sake of contradiction, assume

$$\forall x \left(\neg\exists u \left(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) = 0\right) \vee \neg\exists u \left(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) \neq 0\right)\right) \tag{2}$$

and reason in $\mathsf{HA} + \mathrm{CT}_0$. Since $\varphi_1 \vee \varphi_2 \leftrightarrow \exists k \left((k = 0 \to \varphi_1) \wedge (k \neq 0 \to \varphi_2)\right)$ (see [8, §1.3.7]), by CT_0, there exists e such that

$$\forall x \exists v \begin{pmatrix} \mathrm{T}(e, x, v) \\ \wedge \left(\mathrm{U}(v) = 0 \to \neg\exists u \left(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) = 0\right)\right) \\ \wedge \left(\mathrm{U}(v) \neq 0 \to \neg\exists u \left(\mathrm{T}(x, x, u) \wedge \mathrm{U}(u) \neq 0\right)\right) \end{pmatrix}.$$

In particular, for that e, there exists v_e such that $\mathrm{T}(e, e, v_e)$,

$$\mathrm{U}(v_e) = 0 \to \neg\exists u \left(\mathrm{T}(e, e, u) \wedge \mathrm{U}(u) = 0\right)$$

and

$$\mathrm{U}(v_e) \neq 0 \to \neg\exists u \left(\mathrm{T}(e, e, u) \wedge \mathrm{U}(u) \neq 0\right).$$

Since $\mathrm{U}(v_e) = 0 \vee \mathrm{U}(v_e) \neq 0$, we obtain a contradiction straightforwardly.

If φ_0 is equivalent to some sentence $\varphi_0' \in \Sigma_1$ over $\mathsf{HA} + \Sigma_1$-DNE, we have $\mathsf{HA} + \Sigma_1\text{-DNE} + \mathrm{CT}_0 \vdash \varphi_0'$ from the above claim. Since $\varphi_0' \in \Sigma_1$, by the soundness of Kleene realizability (see [8, §3.2.22]), we have that

$$\mathsf{HA} + \Sigma_1\text{-DNE} \vdash \varphi_0',$$

and hence, $\mathsf{HA} + \Sigma_1\text{-DNE} \vdash \varphi_0$. On the other hand, since

$$\forall x \neg \left(\exists u \left(\mathrm{T}(x,x,u) \wedge \mathrm{U}(u) = 0\right) \wedge \exists u \left(\mathrm{T}(x,x,u) \wedge \mathrm{U}(u) \neq 0\right)\right)$$

is provable in HA, we have $\mathsf{HA} + \Sigma_1\text{-DML} \vdash (2)$. Therefore we have

$$\mathsf{HA} + \Sigma_1\text{-DNE} + \Sigma_1\text{-DML} \vdash \perp,$$

and hence, $\mathsf{PA} \vdash \perp$, which is a contradiction. ⊣

REMARK 3.2. One can easily see that φ_0 in Proposition 3.1 is in E_1. Thus Proposition 3.1 shows that φ_0 is a counterexample of [1, Theorem 2.7] for $k = 1$.

§4. Basic lemmata. In this section, we show several lemmata which we use in the proofs of our prenex normal form theorems.

LEMMA 4.1. *For any logical principle* P *in Definition* 2.6 *and any formula* φ (*possibly containing free variables*), *if* $\mathsf{HA} + \mathrm{P} \vdash \varphi$, *then* $\mathsf{HA} + \neg\neg\mathrm{P} \vdash \neg\neg\varphi$.

PROOF. Assume $\mathsf{HA} + \mathrm{P} \vdash \varphi$. Then there exists finite instances $\psi_1, \dots, \psi_k$ of P such that $\mathsf{HA} + \psi_1 + \cdots + \psi_k \vdash \varphi$. Since HA satisfies the deduction theorem, we have that HA proves $\psi_1 \wedge \cdots \wedge \psi_k \to \varphi$, and hence, $\neg\neg(\psi_1 \wedge \cdots \wedge \psi_k \to \varphi)$, which is equivalent to $\neg\neg\psi_1 \wedge \cdots \wedge \neg\neg\psi_k \to \neg\neg\varphi$. Then we have $\mathsf{HA} + \neg\neg\mathrm{P} \vdash \neg\neg\varphi$. ⊣

COROLLARY 4.2. *For any logical principle* P *in Definition* 2.6 *and any formulas* φ_1 *and* φ_2 (*possibly containing free variables*), *if* $\mathsf{HA} + \mathrm{P} \vdash \varphi_1 \leftrightarrow \varphi_2$, *then* $\mathsf{HA} + \neg\neg\mathrm{P} \vdash \neg\neg\varphi_1 \leftrightarrow \neg\neg\varphi_2$.

PROOF. Immediate from Lemma 4.1 and the fact that $\neg\neg(\varphi_1 \leftrightarrow \varphi_2)$ is intuitionistically equivalent to $\neg\neg\varphi_1 \leftrightarrow \neg\neg\varphi_2$. ⊣

LEMMA 4.3. *Let k be a natural number. Let φ_1 and φ_2 be formulas in Σ_k, and let φ_3 and φ_4 be formulas in Π_k. Then the following hold*:

1. *There exists a formula* $\varphi \in \Sigma_k$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi_1 \wedge \varphi_2)$ *and* $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi_1 \wedge \varphi_2$.
2. *There exists a formula* $\varphi' \in \Pi_k$ *such that* $\mathrm{FV}(\varphi') = \mathrm{FV}(\varphi_3 \wedge \varphi_4)$ *and* $\mathsf{HA} \vdash \varphi' \leftrightarrow \varphi_3 \wedge \varphi_4$.

PROOF. Straightforward by simultaneous induction on k. ⊣

LEMMA 4.4. *For any formulas φ_1 and φ_2 in Σ_k, there exists a formula $\varphi \in \Sigma_k$ such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi_1 \vee \varphi_2)$ *and* $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi_1 \vee \varphi_2$.

PROOF. Note that $\varphi_1 \vee \varphi_2$ is equivalent to

$$\exists k\left((k = 0 \to \varphi_1) \wedge (k \neq 0 \to \varphi_2)\right)$$

over HA (see [8, §1.3.7]). Since $\varphi_{\mathrm{qf}} \to \exists x \psi(x)$ and $\varphi_{\mathrm{qf}} \to \forall x \psi(x)$ are equivalent to $\exists x \left(\varphi_{\mathrm{qf}} \to \psi(x)\right)$ and $\forall x \left(\varphi_{\mathrm{qf}} \to \psi(x)\right)$ respectively over HA when $x \notin \mathrm{FV}\left(\varphi_{\mathrm{qf}}\right)$, our assertion follows from Lemma 4.3 straightforwardly. ⊣

LEMMA 4.5. *Let k be a natural number greater than* 0.

1. $\varphi_1 \wedge \varphi_2$ *is in* U_k^+ (*resp.* E_k^+) *if and only if both of* φ_1 *and* φ_2 *are in* U_k^+ (*resp.* E_k^+).
2. $\varphi_1 \vee \varphi_2$ *is in* U_k^+ (*resp.* E_k^+) *if and only if both of* φ_1 *and* φ_2 *are in* U_k^+ (*resp.* E_k^+).

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3. $\varphi_1 \to \varphi_2$ *is in* U_k^+ (*resp.* E_k^+) *if and only if* φ_1 *is in* E_k^+ (*resp.* U_k^+) *and* φ_2 *is in* U_k^+ (*resp.* E_k^+).
4. $\forall x\varphi_1$ *is in* U_k^+ *if and only if* φ_1 *is in* U_k^+.
5. $\exists x\varphi_1$ *is in* E_k^+ *if and only if* φ_1 *is in* E_k^+.
6. $\forall x\varphi_1$ *is in* E_{k+1}^+ *if and only if it is in* U_k^+.
7. $\exists x\varphi_1$ *is in* U_{k+1}^+ *if and only if it is in* E_k^+.

PROOF. (1): Assume $\varphi_1 \wedge \varphi_2 \in \mathrm{U}_k^+$. Then $l(s) \leq k$ for all $s \in Alt(\varphi_1 \wedge \varphi_2) = Alt(\varphi_1) \cup Alt(\varphi_2)$.

- If $l(s) < k$ for all $s \in Alt(\varphi_1)$, then φ_1 is in $\bigcup_{i<k} \mathrm{F}_i \subseteq \mathrm{U}_k^+$.
- Otherwise, there is $s_0 \in Alt(\varphi_1)$ such that $l(s_0) = k$. Then, since $\varphi_1 \wedge \varphi_2 \notin \bigcup_{i<k} \mathrm{F}_i$, we have $\varphi_1 \wedge \varphi_2 \in \mathrm{U}_k$. Then, for each $s \in Alt(\varphi_1)$ such that $l(s) = k$, we have $i(s) \equiv -$ since $s \in Alt(\varphi_1 \wedge \varphi_2)$. Thus $\varphi_1 \in \mathrm{U}_k \subseteq \mathrm{U}_k^+$.

We also have $\varphi_2 \in \mathrm{U}_k^+$ in the same manner.

For the converse direction, assume that φ_1 and φ_2 are in U_k^+. Then, for all $s \in Alt(\varphi_1 \wedge \varphi_2)$, since $s \in Alt(\varphi_1)$ or $s \in Alt(\varphi_2)$, we have $l(s) \leq k$, in particular, $i(s) \equiv -$ if $l(s) = k$. Thus $\varphi_1 \wedge \varphi_2$ is in U_k^+.

As for the case of E_k^+, an analogous proof works.

(2): Analogous to (1).

(3): Assume $\varphi_1 \to \varphi_2 \in \mathrm{U}_k^+$. Let s be in $Alt(\varphi_1)$. By the definition of $Alt(\varphi_1 \to \varphi_2)$, we have $s^\perp \in Alt(\varphi_1 \to \varphi_2)$ and $l(s) \leq k$.

- If $l(s) < k$ for all $s \in Alt(\varphi_1)$, then φ_1 is in $\bigcup_{i<k} \mathrm{F}_i \subseteq \mathrm{E}_k^+$.
- Otherwise, there is $s_0 \in Alt(\varphi_1)$ such that $l(s_0) = k$. Since ${s_0}^\perp \in Alt(\varphi_1 \to \varphi_2)$, we have $\varphi_1 \to \varphi_2 \in \mathrm{U}_k$. Then, for each $s \in Alt(\varphi_1)$ such that $l(s) = k$, we have $i(s^\perp) \equiv -$, and hence, $i(s) \equiv +$. Thus $\varphi_1 \in \mathrm{E}_k \subseteq \mathrm{E}_k^+$.

We also have $\varphi_2 \in \mathrm{U}_k^+$ in the same manner.

For the converse direction, assume $\varphi_1 \in \mathrm{E}_k^+$ and $\varphi_2 \in \mathrm{U}_k^+$. Since $deg(\varphi_1) \leq k$ and $deg(\varphi_2) \leq k$, we have $deg(\varphi_1 \to \varphi_2) \leq k$.

- If $deg(\varphi_1 \to \varphi_2) < k$, then $\varphi_1 \to \varphi_2 \in \bigcup_{i<k} \mathrm{F}_i \subseteq \mathrm{U}_k^+$.
- If $deg(\varphi_1 \to \varphi_2) = k$, for all $s \in Alt(\varphi_1 \to \varphi_2)$ such that $l(s) = k$, we have $s \in Alt(\varphi_2)$ or $s \equiv {s_0}^\perp$ for some $s_0 \in Alt(\varphi_1)$. In the former case, we have $i(s) \equiv -$ by $\varphi_2 \in \mathrm{U}_k^+$. In the latter case, we have $i(s_0) \equiv +$ by $\varphi_1 \in \mathrm{E}_k^+$, and hence, $i(s) \equiv -$.

One can also show that $\varphi_1 \to \varphi_2$ is in E_k^+ if and only if φ_1 is in U_k^+ and φ_2 is in E_k^+ analogously.

(4): Assume $\forall x\varphi_1 \in \mathrm{U}_k^+$.

- If $\forall x\varphi_1 \notin \mathrm{U}_k$, then $\forall x\varphi_1 \in \bigcup_{i<k} \mathrm{F}_i$. Since $deg(\varphi_1) \leq deg(\forall x\varphi_1) < k$, we have $\varphi_1 \in \bigcup_{i<k} \mathrm{F}_i \subseteq \mathrm{U}_k^+$.

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- Otherwise, $deg(\varphi_1) \leq deg(\forall x\varphi_1) = k$. If $deg(\varphi_1) < k$, then we have $\varphi_1 \in \bigcup_{i<k} \mathrm{F}_i \subseteq \mathrm{U}_k^+$. Assume $deg(\varphi_1) = k$. Let s be an alternation path of φ_1 such that $l(s) = k$. If $i(s) \not\equiv -$, by the definition of $Alt(\forall x\varphi_1)$, we have $-s \in Alt(\forall x\varphi_1)$, which contradicts $deg(\forall x\varphi_1) = k$ since $l(-s) = k+1$. Then we have $i(s) \equiv -$. Thus we have $\varphi_1 \in \mathrm{U}_k \subseteq \mathrm{U}_k^+$.

For the converse direction, assume $\varphi_1 \in \mathrm{U}_k^+$.

- If $\varphi_1 \notin \mathrm{U}_k$, then $\varphi_1 \in \bigcup_{i<k} \mathrm{F}_i$. Thus $deg(\varphi_1) < k$, and hence, $deg(\forall x\varphi_1) \leq k$. If $deg(\forall x\varphi_1) < k$, then $\forall x\varphi_1 \in \bigcup_{i<k} \mathrm{F}_i \subseteq \mathrm{U}_k^+$. If $deg(\forall x\varphi_1) = k$, since $i(s) \equiv -$ for all $s \in Alt(\forall x\varphi_1)$, we have $\forall x\varphi_1 \in \mathrm{U}_k \subseteq \mathrm{U}_k^+$.
- Otherwise, $deg(\varphi_1) = k$ and $i(s) \equiv -$ for all $s \in Alt(\varphi_1)$ such that $l(s) = k$. By the definition of $Alt(\forall x\varphi_1)$, for all $s \in Alt(\forall x\varphi_1)$, we have $l(s) \leq k$, and hence, $deg(\forall x\varphi_1) = k$. In addition, again by the definition of $Alt(\forall x\varphi_1)$, we have $i(s) \equiv -$ for all $s \in Alt(\forall x\varphi_1)$ such that $l(s) = k$. Thus $\forall x\varphi_1 \in \mathrm{U}_k \subseteq \mathrm{U}_k^+$.

(5): Analogous to (4).

(6): Assume $\forall x\varphi_1 \in \mathrm{E}_{k+1}^+$. Since $i(s) \equiv -$ for all $s \in Alt(\forall x\varphi_1)$, $\forall x\varphi_1$ is not in E_{k+1}. Then $\forall x\varphi_1 \in \bigcup_{i\leq k} \mathrm{F}_i$, and hence, $deg(\forall x\varphi_1) \leq k$.

- If $deg(\forall x\varphi_1) < k$, then $\forall x\varphi_1 \in \bigcup_{i<k} \mathrm{F}_i \subseteq \mathrm{U}_k^+$.
- If $deg(\forall x\varphi_1) = k$, since $i(s) \equiv -$ for all $s \in Alt(\forall x\varphi_1)$, we have $\forall x\varphi_1 \in \mathrm{U}_k \subseteq \mathrm{U}_k^+$.

The converse direction is trivial since $\mathrm{U}_k^+ \subseteq \bigcup_{i<k+1} \mathrm{F}_i \subseteq \mathrm{E}_{k+1}^+$.

(7): Analogous to (6). ⊣

As mentioned in Lemma 2.3, our non-cumulative definition of the classes Σ_k and Π_k does not cause any trouble. In a similar sense, the following proposition allows us to think of E_k and U_k in the cumulative manner:

PROPOSITION 4.6. *Let k be a natural number. Then the following hold*:

1. *If $\varphi \in \mathrm{U}_k^+$, then there exist $\varphi' \in \mathrm{U}_k$ such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and* $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi'$.
2. *If $\varphi \in \mathrm{E}_k^+$, then there exist $\varphi' \in \mathrm{E}_k$ such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and* $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi'$.

PROOF. By simultaneous induction on k. The base case is trivial.

For the induction step, assume that the assertion holds for k. We show the assertion for $k+1$ by induction on the structure of formulas.

The case of that φ is prime: Suppose $\varphi \in \mathrm{U}_{k+1}^+ \cup \mathrm{E}_{k+1}^+$. Since φ is prime, the assertion for k ensures that there exist $\varphi' \in \mathrm{U}_k$ and $\varphi'' \in \mathrm{E}_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi') = \mathrm{FV}(\varphi'')$ and $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi' \leftrightarrow \varphi''$. Put $\psi' :\equiv \exists x\varphi'$ and $\psi'' :\equiv \forall x\varphi''$ where $x \notin \mathrm{FV}(\varphi') \cup \mathrm{FV}(\varphi'')$. By the definition, it is straightforward to show that

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$\psi' \in \mathrm{E}_{k+1}$, $\psi'' \in \mathrm{U}_{k+1}$ and $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi') = \mathrm{FV}(\psi'')$. In addition, by (1) in the proof of Lemma 2.3, we have $\mathsf{HA} \vdash \varphi \leftrightarrow \psi' \leftrightarrow \psi''$.

The case of $\varphi :\equiv \varphi_1 \wedge \varphi_2$: Suppose $\varphi \in \mathrm{U}^+_{k+1}$. By Lemma 4.5.(1), we have $\varphi_1 \in \mathrm{U}^+_{k+1}$ and $\varphi_2 \in \mathrm{U}^+_{k+1}$. By the induction hypothesis, there exist $\varphi_1', \varphi_2' \in \mathrm{U}_k$ such that $\mathrm{FV}(\varphi_1) = \mathrm{FV}(\varphi_1')$, $\mathrm{FV}(\varphi_2) = \mathrm{FV}(\varphi_2')$, $\mathsf{HA} \vdash \varphi_1 \leftrightarrow \varphi_1'$ and $\mathsf{HA} \vdash \varphi_2 \leftrightarrow \varphi_2'$. Then it is straightforward to show that $\varphi_1' \wedge \varphi_2' \in \mathrm{U}_k$, $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi_1' \wedge \varphi_2')$ and $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi_1' \wedge \varphi_2'$. In the same manner, if $\varphi \in \mathrm{E}^+_{k+1}$, there exists $\varphi' \in \mathrm{E}_{k+1}$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi'$.

The case of $\varphi :\equiv \varphi_1 \vee \varphi_2$ is similar to the case of $\varphi :\equiv \varphi_1 \wedge \varphi_2$ (use Lemma 4.5.(2) instead of Lemma 4.5.(1)).

The case of $\varphi :\equiv \varphi_1 \to \varphi_2$: Suppose $\varphi \in \mathrm{U}^+_{k+1}$. By Lemma 4.5.(3), we have $\varphi_1 \in \mathrm{E}^+_{k+1}$ and $\varphi_2 \in \mathrm{U}^+_{k+1}$. By the induction hypothesis, there exist $\varphi_1' \in \mathrm{E}_{k+1}$ and $\varphi_2' \in \mathrm{U}_{k+1}$ such that $\mathrm{FV}(\varphi_1) = \mathrm{FV}(\varphi_1')$, $\mathrm{FV}(\varphi_2) = \mathrm{FV}(\varphi_2')$, $\mathsf{HA} \vdash \varphi_1 \leftrightarrow \varphi_1'$ and $\mathsf{HA} \vdash \varphi_2 \leftrightarrow \varphi_2'$. Then it is straightforward to show that $\varphi_1' \to \varphi_2' \in \mathrm{U}_{k+1}$, $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi_1' \to \varphi_2')$ and $\mathsf{HA} \vdash \varphi \leftrightarrow (\varphi_1' \to \varphi_2')$. In the same manner, if $\varphi \in \mathrm{E}^+_{k+1}$, there exists $\varphi' \in \mathrm{E}_{k+1}$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi'$.

The case of $\varphi :\equiv \forall x \varphi_1$: First, suppose $\varphi \in \mathrm{U}^+_{k+1}$. By Lemma 4.5.(4), we have $\varphi_1 \in \mathrm{U}^+_{k+1}$. By the induction hypothesis, there exists $\varphi_1' \in \mathrm{U}_{k+1}$ such that $\mathrm{FV}(\varphi_1) = \mathrm{FV}(\varphi_1')$ and $\mathsf{HA} \vdash \varphi_1 \leftrightarrow \varphi_1'$. Then it is straightforward to show that $\forall x \varphi_1' \in \mathrm{U}_{k+1}$, $\mathrm{FV}(\varphi) = \mathrm{FV}(\forall x \varphi_1')$ and $\mathsf{HA} \vdash \varphi \leftrightarrow \forall x \varphi_1'$. Next, suppose $\varphi \in \mathrm{E}^+_{k+1}$. By Lemma 4.5.(6), we have $\varphi \in \mathrm{U}^+_k$. The assertion for k ensures that there exists $\varphi' \in \mathrm{U}_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and $\mathsf{HA} \vdash \varphi \leftrightarrow \varphi'$. Put $\psi :\equiv \exists y \varphi'$ where $y \notin \mathrm{FV}(\varphi')$. By the definition, it is straightforward to show that $\psi \in \mathrm{E}_{k+1}$, $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi)$ and $\mathsf{HA} \vdash \varphi \leftrightarrow \psi$.

The case of $\varphi :\equiv \exists x \varphi_1$ is similar to the case of $\varphi :\equiv \forall x \varphi_1$ (use Lemma 4.5.(7) and Lemma 4.5.(5)). $\dashv$

In what follows, based on Proposition 4.6, we often identify U_k-DNS with U^+_k-DNS (especially in the assertions of our theorems). The latter will play a crucial role in the arguments below.

LEMMA 4.7. *Let k be a natural number. For all φ_1 and φ_2 in Π_k, there exists $\varphi \in \Pi_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi_1 \vee \varphi_2)$ and $\mathsf{HA} + \neg\neg\Sigma_{k-1}$-DNE ($\mathsf{HA}$ if $k = 0$) proves $\neg\neg(\varphi_1 \vee \varphi_2) \leftrightarrow \neg\neg\varphi$.*

PROOF. Without loss of generality, assume $k > 0$, $\varphi_1 :\equiv \forall x \rho_1(x)$ and $\varphi_2 :\equiv \forall y \rho_2(y)$ where $\rho_1(x), \rho_2(y) \in \Sigma_{k-1}$ (see Remark 2.2). By Lemma 4.4, it suffices to show

$$\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-DNE} \vdash \neg\neg\left(\forall x \rho_1(x) \vee \forall y \rho_2(y)\right) \leftrightarrow \neg\neg\forall x, y\left(\rho_1(x) \vee \rho_2(y)\right).$$

The implication from the left to the right is straightforward. The converse implication is shown as follows:

$$\begin{array}{cl} & \neg\neg\forall x, y\left(\rho_1(x) \vee \rho_2(y)\right) \\ \underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} & \neg\neg\forall x, y\left(\neg\neg\rho_1(x) \vee \neg\neg\rho_2(y)\right) \\ \longrightarrow & \forall x, y\neg\neg\left(\neg\neg\rho_1(x) \vee \neg\neg\rho_2(y)\right) \\ \longleftrightarrow & \neg\exists x, y\neg\left(\neg\neg\rho_1(x) \vee \neg\neg\rho_2(y)\right) \\ \longleftrightarrow & \neg\exists x, y\left(\neg\rho_1(x) \wedge \neg\rho_2(y)\right) \end{array}$$

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$$
\begin{aligned}
&\longleftrightarrow && \neg\left(\neg\neg\exists x\neg\rho_1(x)\wedge\neg\neg\exists y\neg\rho_2(y)\right)\\
&\longleftrightarrow && \neg\left(\neg\forall x\neg\neg\rho_1(x)\wedge\neg\forall y\neg\neg\rho_2(y)\right)\\
&\longleftrightarrow && \neg\neg\left(\forall x\neg\neg\rho_1(x)\vee\forall y\neg\neg\rho_2(y)\right)\\
&\underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} && \neg\neg\left(\forall x\rho_1(x)\vee\forall y\rho_2(y)\right).
\end{aligned}
$$
⊣

Lemma 4.8. *Let k be a natural number.*

1. *For all $\varphi\in\Pi_k$, there exists $\psi\in\Sigma_k$ such that* $\mathrm{FV}(\varphi)=\mathrm{FV}(\psi)$ *and* $\mathsf{HA}+\Sigma_k$-DNE *proves* $\neg\varphi\leftrightarrow\psi$.
2. *For all $\varphi\in\Sigma_k$, there exists $\psi\in\Pi_k$ such that* $\mathrm{FV}(\varphi)=\mathrm{FV}(\psi)$ *and* $\mathsf{HA}+\Sigma_{k-1}$-DNE (HA *if* $k=0$) *proves* $\neg\varphi\leftrightarrow\psi$.

Proof. By simultaneous induction on k. The base case is trivial. In what follows, we show the induction step for $k+1$.

Let $\varphi:\equiv\forall x\rho(x)$ where $\rho(x)\in\Sigma_k$. By induction hypothesis, there exists $\rho'(x)\in\Pi_k$ such that $\mathrm{FV}(\rho(x))=\mathrm{FV}(\rho'(x))$ and

$$\mathsf{HA}+\Sigma_{k-1}\text{-DNE}\vdash\neg\rho(x)\leftrightarrow\rho'(x).$$

Then $\mathsf{HA}+\Sigma_{k+1}$-DNE proves

$$
\begin{aligned}
& && \neg\forall x\rho(x)\\
&\underset{\Sigma_k\text{-DNE}}{\longleftrightarrow} && \neg\forall x\neg\neg\rho(x)\\
&\longleftrightarrow && \neg\neg\exists x\neg\rho(x)\\
&\underset{\text{[I.H.] }\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} && \neg\neg\exists x\rho'(x)\\
&\underset{\Sigma_{k+1}\text{-DNE}}{\longleftrightarrow} && \exists x\rho'(x),
\end{aligned}
$$

which is in Σ_{k+1}.

Next, let $\varphi:\equiv\exists x\rho(x)$ where $\rho(x)\in\Pi_k$. By induction hypothesis, there exists $\rho'(x)\in\Sigma_k$ such that $\mathrm{FV}(\rho(x))=\mathrm{FV}(\rho'(x))$ and

$$\mathsf{HA}+\Sigma_k\text{-DNE}\vdash\neg\rho(x)\leftrightarrow\rho'(x).$$

Then $\mathsf{HA}+\Sigma_k$-DNE proves

$$\neg\exists x\rho(x)\leftrightarrow\forall x\neg\rho(x)\underset{\text{[I.H.] }\Sigma_k\text{-DNE}}{\longleftrightarrow}\forall x\rho'(x),$$

which is in Π_{k+1}. ⊣

Lemma 4.9. *Let k be a natural number. For all $\varphi\in\Pi_k$, there exists $\psi\in\Sigma_k$ such that* $\mathrm{FV}(\varphi)=\mathrm{FV}(\psi)$ *and* $\mathsf{HA}+\neg\neg\Sigma_k$-DNE $\vdash\neg\varphi\leftrightarrow\neg\neg\psi$.

Proof. Let $\varphi\in\Pi_k$. By Lemma 4.8.(1), there exists $\psi\in\Sigma_k$ such that $\mathrm{FV}(\varphi)=\mathrm{FV}(\psi)$ and $\mathsf{HA}+\Sigma_k$-DNE $\vdash\neg\varphi\leftrightarrow\psi$. By Corollary 4.2, we have $\mathsf{HA}+\neg\neg\Sigma_k$-DNE $\vdash\neg\varphi\leftrightarrow\neg\neg\psi$. ⊣

Lemma 4.10. $\mathsf{HA}+\mathrm{U}_k$-DNS $\vdash\neg\neg\Sigma_{k-1}$-LEM *for each natural number* $k>0$.

Proof. Fix an instance of $\neg\neg\Sigma_{k-1}$-LEM

$$\varphi:\equiv\neg\neg\forall x\left(\varphi_1(x)\vee\neg\varphi_1(x)\right),$$

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where $\varphi_1(x) \in \Sigma_{k-1}$. Note $(\varphi_1(x) \vee \neg\varphi_1(x)) \in \mathrm{F}_{k-1} \subseteq \mathrm{U}_k^+$. Since HA proves $\forall x \neg\neg(\varphi_1(x) \vee \neg\varphi_1(x))$, we have that $\mathsf{HA} + \mathrm{U}_k$-DNS proves $\neg\neg\forall x(\varphi_1(x) \vee \neg\varphi_1(x))$, namely, φ. ⊣

Corollary 4.11. $\mathsf{HA} + \mathrm{U}_k\text{-DNS} \vdash \neg\neg\Sigma_{k-1}\text{-DNE}$ *for each natural number* $k > 0$.

Proof. Immediate from Lemma 4.10 and the fact that Σ_{k-1}-LEM implies Σ_{k-1}-DNE. ⊣

§5. Prenex normal form theorems.

In this section, we show the modified version of [1, Theorem 2.7]. Prior to that, we first show a variant of the prenex normal form theorem:

Lemma 5.1. *For each natural number k and a formula φ, if $\varphi \in \mathrm{U}_k^+$, then there exists $\varphi' \in \Pi_k$ such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and*

$$\mathsf{HA} + \mathrm{U}_k\text{-DNS} \vdash \neg\neg\varphi \leftrightarrow \neg\neg\varphi'.$$

Proof. By simultaneous induction on k, we show the following two statements (which are in fact equivalent):

1. If $\varphi \in \mathrm{E}_k^+$, then there exists $\varphi' \in \Pi_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and
$$\mathsf{HA} + \mathrm{U}_k^+\text{-DNS} \vdash \neg\varphi \leftrightarrow \neg\neg\varphi'.$$
2. If $\varphi \in \mathrm{U}_k^+$, then there exists $\varphi' \in \Pi_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and
$$\mathsf{HA} + \mathrm{U}_k^+\text{-DNS} \vdash \neg\neg\varphi \leftrightarrow \neg\neg\varphi'.$$

The base case is trivial (one can take φ' as φ itself). In what follows, we show the induction step.

For the induction step, assume the items 1 and 2 for $k-1$. We show the items 1 and 2 for k simultaneously by induction on the structure of formulas. When φ is a prime formula, by Lemma 2.3, we have φ' which satisfies the requirement. For the induction step, assume that the items 1 and 2 hold for φ_1 and φ_2. When it is clear from the context, we suppress the argument on free variables.

The case o $\varphi_1 \wedge \varphi_2$: First, assume $\varphi_1 \wedge \varphi_2 \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1, \varphi_2 \in \mathrm{E}_k^+$. By induction hypothesis, there exist $\varphi_1', \varphi_2' \in \Pi_k$ such that $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves $\neg\varphi_1 \leftrightarrow \neg\neg\varphi_1'$ and $\neg\varphi_2 \leftrightarrow \neg\neg\varphi_2'$. By Lemma 4.7, there exists $\varphi' \in \Pi_k$ such that

$$\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-DNE} \vdash \neg\neg(\varphi_1' \vee \varphi_2') \leftrightarrow \neg\neg\varphi'.$$

By Corollary 4.11, $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\begin{aligned}
& \neg(\varphi_1 \wedge \varphi_2) \\
\longleftrightarrow\; & \neg(\neg\neg\varphi_1 \wedge \neg\neg\varphi_2) \\
\underset{\text{[I.H.] } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow}\; & \neg(\neg\varphi_1' \wedge \neg\varphi_2') \\
\longleftrightarrow\; & \neg\neg(\varphi_1' \vee \neg\varphi_2') \\
\underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow}\; & \neg\neg\varphi'.
\end{aligned}$$

Next, assume $\varphi_1 \wedge \varphi_2 \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\varphi_1, \varphi_2 \in \mathrm{U}_k^+$. By induction hypothesis, there exist $\varphi_1', \varphi_2' \in \Pi_k$ such that $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves $\neg\neg\varphi_1 \leftrightarrow \neg\neg\varphi_1'$

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and $\neg\neg\varphi_2 \leftrightarrow \neg\neg\varphi_2'$. By Lemma 4.3, there exists $\varphi' \in \Pi_k$ such that $\mathsf{HA} \vdash \varphi' \leftrightarrow \varphi_1' \wedge \varphi_2'$. Then $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\neg\neg(\varphi_1 \wedge \varphi_2) \leftrightarrow \neg\neg\varphi_1 \wedge \neg\neg\varphi_2 \underset{\text{[I.H.] } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} \neg\neg\varphi_1' \wedge \neg\neg\varphi_2' \leftrightarrow \neg\neg(\varphi_1' \wedge \varphi_2') \leftrightarrow \neg\neg\varphi'.$$

The case of $\varphi_1 \vee \varphi_2$: First, assume $\varphi_1 \vee \varphi_2 \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1, \varphi_2 \in \mathrm{E}_k^+$. By induction hypothesis, there exist $\varphi_1', \varphi_2' \in \Pi_k$ such that $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves $\neg\varphi_1 \leftrightarrow \neg\neg\varphi_1'$ and $\neg\varphi_2 \leftrightarrow \neg\neg\varphi_2'$. By Lemma 4.3, there exists $\varphi' \in \Pi_k$ such that $\mathsf{HA} \vdash \varphi' \leftrightarrow \varphi_1' \wedge \varphi_2'$. Then $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\begin{array}{cl} & \neg(\varphi_1 \vee \varphi_2) \\ \longleftrightarrow & \neg\varphi_1 \wedge \neg\varphi_2 \\ \underset{\text{[I.H.] } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} & \neg\neg\varphi_1' \wedge \neg\neg\varphi_2' \\ \longleftrightarrow & \neg\neg(\varphi_1' \wedge \varphi_2') \\ \longleftrightarrow & \neg\neg\varphi'. \end{array}$$

Next, assume $\varphi_1 \vee \varphi_2 \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\varphi_1, \varphi_2 \in \mathrm{U}_k^+$. By induction hypothesis, there exist $\varphi_1', \varphi_2' \in \Pi_k$ such that $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves $\neg\neg\varphi_1 \leftrightarrow \neg\neg\varphi_1'$ and $\neg\neg\varphi_2 \leftrightarrow \neg\neg\varphi_2'$. By Lemma 4.7, there exists $\varphi' \in \Pi_k$ such that

$$\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-DNE} \vdash \neg\neg(\varphi_1' \vee \varphi_2') \leftrightarrow \neg\neg\varphi'.$$

By Corollary 4.11, $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\begin{array}{cl} & \neg\neg(\varphi_1 \vee \varphi_2) \\ \longleftrightarrow & \neg\neg(\neg\neg\varphi_1 \vee \neg\neg\varphi_2) \\ \underset{\text{[I.H.] } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} & \neg\neg(\neg\neg\varphi_1' \vee \neg\neg\varphi_2') \\ \longleftrightarrow & \neg\neg(\varphi_1' \vee \varphi_2') \\ \underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} & \neg\neg\varphi'. \end{array}$$

The case of $\varphi_1 \to \varphi_2$: First, assume $\varphi_1 \to \varphi_2 \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1 \in \mathrm{U}_k^+$ and $\varphi_2 \in \mathrm{E}_k^+$. By induction hypothesis, there exist $\varphi_1', \varphi_2' \in \Pi_k$ such that $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves $\neg\neg\varphi_1 \leftrightarrow \neg\neg\varphi_1'$ and $\neg\varphi_2 \leftrightarrow \neg\neg\varphi_2'$. By Lemma 4.3, there exists $\varphi' \in \Pi_k$ such that $\mathsf{HA} \vdash \varphi' \leftrightarrow \varphi_1' \wedge \varphi_2'$. Then $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\begin{array}{cl} & \neg(\varphi_1 \to \varphi_2) \\ \longleftrightarrow & \neg\neg\varphi_1 \wedge \neg\varphi_2 \\ \underset{\text{[I.H.] } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} & \neg\neg\varphi_1' \wedge \neg\neg\varphi_2' \\ \longleftrightarrow & \neg\neg(\varphi_1' \wedge \varphi_2') \\ \longleftrightarrow & \neg\neg\varphi'. \end{array}$$

Next, assume $\varphi_1 \to \varphi_2 \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\varphi_1 \in \mathrm{E}_k^+$ and $\varphi_2 \in \mathrm{U}_k^+$. By induction hypothesis, there exist $\varphi_1', \varphi_2' \in \Pi_k$ such that $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves $\neg\varphi_1 \leftrightarrow \neg\neg\varphi_1'$ and $\neg\neg\varphi_2 \leftrightarrow \neg\neg\varphi_2'$. By Lemma 4.7, there exists $\varphi' \in \Pi_k$ such that

$$\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-DNE} \vdash \neg\neg(\varphi_1' \vee \varphi_2') \leftrightarrow \neg\neg\varphi'.$$

By Corollary 4.11, $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\begin{array}{rl} & \neg\neg(\varphi_1 \to \varphi_2) \\ \longleftrightarrow & \neg(\neg\neg\varphi_1 \wedge \neg\varphi_2) \\ \underset{\text{[I.H.] } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} & \neg(\neg\varphi_1' \wedge \neg\varphi_2') \\ \longleftrightarrow & \neg\neg(\varphi_1' \vee \varphi_2') \\ \underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} & \neg\neg\varphi'. \end{array}$$

The case of $\forall x \varphi_1(x)$: First, assume $\forall x \varphi_1(x) \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\forall x \varphi_1(x) \in \mathrm{U}_{k-1}^+$. By the item 2 for $k-1$, there exists $\varphi' \in \Pi_{k-1}$ such that

$$\mathsf{HA} + \mathrm{U}_{k-1}^+\text{-DNS} \vdash \neg\neg\forall x \varphi_1(x) \leftrightarrow \neg\neg\varphi'.$$

By Lemma 4.9, there exists $\varphi'' \in \Sigma_{k-1} \subseteq \Pi_k$ (see Remark 2.5) such that

$$\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-DNE} \vdash \neg\varphi' \leftrightarrow \neg\neg\varphi''.$$

By Corollary 4.11, $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\neg\forall x \varphi_1(x) \underset{\text{[I.H.] } \mathrm{U}_{k-1}^+\text{-DNS}}{\longleftrightarrow} \neg\varphi' \underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} \neg\neg\varphi''.$$

Next, assume $\forall x \varphi_1(x) \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\varphi_1(x) \in \mathrm{U}_k^+$. By induction hypothesis, there exists $\varphi_1'(x) \in \Pi_k$ such that

$$\mathsf{HA} + \mathrm{U}_k^+\text{-DNS} \vdash \neg\neg\varphi_1(x) \leftrightarrow \neg\neg\varphi_1'(x).$$

Then $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\neg\neg\forall x \varphi_1(x) \underset{\mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} \forall x \neg\neg\varphi_1(x) \underset{\text{[I.H.] } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} \forall x \neg\neg\varphi_1'(x) \underset{\mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} \neg\neg\forall x \varphi_1'(x).$$

The case of $\exists x \varphi_1(x)$: First, assume $\exists x \varphi_1(x) \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1(x) \in \mathrm{E}_k^+$. By induction hypothesis, there exists $\varphi_1'(x) \in \Pi_k$ such that

$$\mathsf{HA} + \mathrm{U}_k^+\text{-DNS} \vdash \neg\varphi_1(x) \leftrightarrow \neg\neg\varphi_1'(x).$$

Then $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\neg\exists x \varphi_1(x) \leftrightarrow \forall x \neg\varphi_1(x) \underset{\text{[I.H.] } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} \forall x \neg\neg\varphi_1'(x) \underset{\mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} \neg\neg\forall x \varphi_1'(x).$$

Next, assume that $\exists x \varphi_1(x) \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\exists x \varphi_1(x) \in \mathrm{E}_{k-1}^+$. By the item 1 for $k-1$, there exists $\varphi' \in \Pi_{k-1}$ such that

$$\mathsf{HA} + \mathrm{U}_{k-1}^+\text{-DNS} \vdash \neg\exists x \varphi_1(x) \leftrightarrow \neg\neg\varphi'.$$

By Lemma 4.9, there exists $\varphi'' \in \Sigma_{k-1} \subseteq \Pi_k$ (see Remark 2.5) such that

$$\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-DNE} \vdash \neg\varphi' \leftrightarrow \neg\neg\varphi''.$$

By Corollary 4.11, $\mathsf{HA} + \mathrm{U}_k^+$-DNS proves

$$\neg\neg\exists x \varphi_1(x) \underset{\text{[I.H.] } \mathrm{U}_{k-1}^+\text{-DNS}}{\longleftrightarrow} \neg\varphi' \underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} \neg\neg\varphi''. \qquad \dashv$$

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The following lemma is used a lot of times implicitly in the proof of our prenex normal form theorem (Theorem 5.3).

LEMMA 5.2 (cf. Fact 2.2 in [1]). *Let k be a natural number.*

1. $\mathsf{HA} + \Sigma_{k+1}$-DNE $\vdash (\Pi_k \vee \Pi_k)$-DNE.
2. $\mathsf{HA} + (\Pi_{k+1} \vee \Pi_{k+1})$-DNE $\vdash \Sigma_k$-DNE.
3. $\mathsf{HA} + \neg\neg(\Pi_{k+1} \vee \Pi_{k+1})$-DNE $\vdash \neg\neg\Sigma_k$-DNE.
4. $\mathsf{HA} + \Sigma_k$-DNE $\vdash \Pi_{k+1}$-DNE.

PROOF. (1): For formulas φ_1 and φ_2 in Π_k, $\varphi_1 \vee \varphi_2$ is equivalent (over HA) to

$$\exists k\, ((k = 0 \to \varphi_1) \wedge (k \neq 0 \to \varphi_2))\,,$$

which is equivalent to some $\varphi \in \Sigma_{k+1}$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi_1 \vee \varphi_2)$ over HA by Lemma 4.3.(2). Therefore any instance of $(\Pi_k \vee \Pi_k)$-DNE is derived from some instance of Σ_{k+1}-DNE.

(2): Any instance of Σ_k-DNE is derived from some instance of $(\Pi_{k+1} \vee \Pi_{k+1})$-DNE since $\varphi \in \Sigma_k$ is equivalent to $\forall y \varphi \in \Pi_{k+1}$ with a variable y not occurring freely in φ (cf. Lemma 2.3).

(3): Immediate from (2) and Corollary 4.2.

(4): Note that $\neg\neg\forall x \varphi(x)$ implies $\neg\neg\forall x \neg\neg\varphi(x)$, which is intuitionistically equivalent to $\forall x \neg\neg\varphi(x)$. Then any instance of Π_{k+1}-DNE is derived from some instance of Σ_k-DNE. ⊣

We are now ready to show the modified version of [1, Theorem 2.7].

THEOREM 5.3. *For each natural number k and a formula φ, the following hold:*

1. *If $\varphi \in \mathrm{E}_k^+$, then there exists $\varphi' \in \Sigma_k$ such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and*

$$\mathsf{HA} + \Sigma_k\text{-DNE} + \mathrm{U}_k\text{-DNS} \vdash \varphi \leftrightarrow \varphi';$$

2. *If $\varphi \in \mathrm{U}_k^+$, then there exists $\varphi' \in \Pi_k$ such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and*

$$\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \varphi \leftrightarrow \varphi'.$$

PROOF. For the proof, we prepare the following auxiliary assertion (which is in fact a consequence of the item 2):

3. If $\varphi \in \mathrm{E}_k^+$, then there exists $\varphi' \in \Pi_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and

$$\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE} \vdash \neg\varphi \leftrightarrow \neg\neg\varphi'.$$

We show the items 1–3 by induction on k simultaneously. The base case is trivial (one can take φ' as φ itself). In what follows, we show the induction step.

Assume the items 1–3 for $k-1$. Since $\mathsf{HA} + \Pi_{k-1}$-DNE $\vdash \Pi_{k-1}$-DNS, by the item 2 for $k-1$, we have

$$\mathsf{HA} + (\Pi_{k-1} \vee \Pi_{k-1})\text{-DNE} \vdash \mathrm{U}_{k-1}^+\text{-DNS}. \tag{3}$$

We show the items 1–3 simultaneously by induction on the structure of formulas. When φ is a prime formula, by Lemma 2.3, we have φ' which satisfies the requirement. For the induction step, assume that the items 1–3 hold for φ_1 and φ_2. When it is clear from the context, we suppress the argument on free variables.

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The case of $\varphi_1 \wedge \varphi_2$: For the second item, assume $\varphi_1 \wedge \varphi_2 \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\varphi_1, \varphi_2 \in \mathrm{U}_k^+$. By induction hypothesis, there exist $\varphi_1', \varphi_2' \in \Pi_k$ such that $\mathsf{HA} + (\Pi_k \vee \Pi_k)$-DNE proves $\varphi_1 \leftrightarrow \varphi_1'$ and $\varphi_2 \leftrightarrow \varphi_2'$. By Lemma 4.3, there exists $\varphi' \in \Pi_k$ such that $\mathsf{HA} \vdash \varphi' \leftrightarrow \varphi_1' \wedge \varphi_2'$. Then $\mathsf{HA} + (\Pi_k \vee \Pi_k)$-DNE proves

$$\varphi_1 \wedge \varphi_2 \underset{\text{[I.H.] } (\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} \varphi_1' \wedge \varphi_2' \leftrightarrow \varphi'.$$

For the first and third items, assume $\varphi_1 \wedge \varphi_2 \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1, \varphi_2 \in \mathrm{E}_k^+$. Then we have $\varphi' \in \Sigma_k$ such that $\mathsf{HA} + \Sigma_k$-DNE $+ \mathrm{U}_k^+$-DNS $\vdash \varphi_1 \wedge \varphi_2 \leftrightarrow \varphi'$ as in the second item. On the other hand, by induction hypothesis, there exist $\varphi_1'', \varphi_2'' \in \Pi_k$ such that $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)$-DNE proves $\neg\varphi_1 \leftrightarrow \neg\neg\varphi_1''$ and $\neg\varphi_2 \leftrightarrow \neg\neg\varphi_2''$. In addition, by Lemma 4.7, there exists $\varphi'' \in \Pi_k$ such that $\mathsf{HA} + \neg\neg\Sigma_{k-1}$-DNE $\vdash \neg\neg\varphi'' \leftrightarrow \neg\neg(\varphi_1'' \vee \varphi_2'')$. Then, by Lemma 5.2, we have that $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)$-DNE proves

$$\begin{aligned}
& \neg(\varphi_1 \wedge \varphi_2) \\
\longleftrightarrow \quad & \neg(\neg\neg\varphi_1 \wedge \neg\neg\varphi_2) \\
\longleftrightarrow \quad & \neg\neg(\neg\varphi_1 \vee \neg\varphi_2) \\
\underset{\text{[I.H.] } \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} \quad & \neg\neg(\neg\neg\varphi_1'' \vee \neg\neg\varphi_2'') \\
\longleftrightarrow \quad & \neg(\neg\varphi_1'' \wedge \neg\varphi_2'') \\
\longleftrightarrow \quad & \neg\neg(\varphi_1'' \vee \varphi_2'') \\
\underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} \quad & \neg\neg\varphi''.
\end{aligned}$$

The case of $\varphi_1 \vee \varphi_2$: For the second item, assume $\varphi_1 \vee \varphi_2 \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\varphi_1, \varphi_2 \in \mathrm{U}_k^+$. Then, by induction hypothesis, there exist $\rho_1(x_1), \rho_2(x_2) \in \Sigma_{k-1}$ such that $\mathsf{HA} + (\Pi_k \vee \Pi_k)$-DNE proves $\varphi_1 \leftrightarrow \forall x_1 \rho_1(x_1)$ and $\varphi_2 \leftrightarrow \forall x_2 \rho_2(x_2)$. By Lemma 5.2, $\mathsf{HA} + (\Pi_k \vee \Pi_k)$-DNE proves

$$\begin{aligned}
& \forall x_1 \rho_1(x_1) \vee \forall x_2 \rho_2(x_2) \\
\longrightarrow \quad & \forall x_1, x_2 \left(\rho_1(x_1) \vee \rho_2(x_2)\right) \\
\longrightarrow \quad & \neg\left(\exists x_1 \neg\rho_1(x_1) \wedge \exists x_2 \neg\rho_2(x_2)\right) \\
\longleftrightarrow \quad & \neg\left(\neg\neg\exists x_1 \neg\rho_1(x_1) \wedge \neg\neg\exists x_2 \neg\rho_2(x_2)\right) \\
\underset{\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} \quad & \neg\left(\neg\forall x_1 \rho_1(x_1) \wedge \neg\forall x_2 \rho_2(x_2)\right) \\
\longleftrightarrow \quad & \neg\neg\left(\forall x_1 \rho_1(x_1) \vee \forall x_2 \rho_2(x_2)\right) \\
\underset{(\Pi_k \vee \Pi_k)\text{-DNE}}{\longrightarrow} \quad & \forall x_1 \rho_1(x_1) \vee \forall x_2 \rho_2(x_2).
\end{aligned}$$

By Lemma 4.4, there exists $\xi(x_1, x_2) \in \Sigma_{k-1}$ such that $\mathsf{HA} \vdash \xi(x_1, x_2) \leftrightarrow \rho_1(x_1) \vee \rho_2(x_2)$. Then we have that $\mathsf{HA} + (\Pi_k \vee \Pi_k)$-DNE proves

$$\begin{aligned}
& \varphi_1 \vee \varphi_2 \\
\underset{\text{[I.H.] } (\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} \quad & \forall x_1 \rho_1(x_1) \vee \forall x_2 \rho_2(x_2) \\
\underset{(\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} \quad & \forall x_1, x_2 \left(\rho_1(x_1) \vee \rho_2(x_2)\right) \\
\longleftrightarrow \quad & \forall x_1, x_2\, \xi(x_1, x_2) \in \Pi_k.
\end{aligned}$$

For the first and third items, assume $\varphi_1 \vee \varphi_2 \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1, \varphi_2 \in \mathrm{E}_k^+$. By induction hypothesis, there exist $\rho_1(x_1), \rho_2(x_2) \in \Pi_{k-1}$ such that

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$\mathsf{HA} + \Sigma_k\text{-DNE} + \mathrm{U}_k^+\text{-DNS}$ proves $\varphi_1 \leftrightarrow \exists x_1 \rho_1(x_1)$ and $\varphi_2 \leftrightarrow \exists x_2 \rho_2(x_2)$. By the item 2 for $k-1$, there exists $\xi(x_1, x_2) \in \Pi_{k-1}$ such that

$$\mathsf{HA} + (\Pi_{k-1} \vee \Pi_{k-1})\text{-DNE} \vdash \xi(x_1, x_2) \leftrightarrow \rho_1(x_1) \vee \rho_2(x_2).$$

By Lemma 5.2, we have that $\mathsf{HA} + \Sigma_k\text{-DNE} + \mathrm{U}_k^+\text{-DNS}$ proves

$$\begin{aligned} & \varphi_1 \vee \varphi_2 \\ \underset{\text{[I.H.] } \Sigma_k\text{-DNE, } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} \quad & \exists x_1 \rho_1(x_1) \vee \exists x_2 \rho_2(x_2) \\ \longleftrightarrow \quad & \exists x_1, x_2 (\rho_1(x_1) \vee \rho_2(x_2)) \\ \underset{\text{[I.H.] } (\Pi_{k-1} \vee \Pi_{k-1})\text{-DNE}}{\longleftrightarrow} \quad & \exists x_1, x_2\, \xi(x_1, x_2). \end{aligned}$$

Thus we are done for the first item. For the third item, by induction hypothesis, there exist $\varphi_1'', \varphi_2'' \in \Pi_k$ such that $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}$ proves $\neg\varphi_1 \leftrightarrow \neg\neg\varphi_1''$ and $\neg\varphi_2 \leftrightarrow \neg\neg\varphi_2''$. In addition, by Lemma 4.3, there exists $\varphi'' \in \Pi_k$ such that $\mathsf{HA} \vdash \varphi'' \leftrightarrow \varphi_1'' \wedge \varphi_2''$. Then $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}$ proves

$$\begin{aligned} \neg(\varphi_1 \vee \varphi_2) \leftrightarrow \neg\varphi_1 \wedge \neg\varphi_2 \underset{\text{[I.H.] } \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} & \neg\neg\varphi_1'' \wedge \neg\neg\varphi_2'' \\ & \leftrightarrow \neg\neg(\varphi_1'' \wedge \varphi_2'') \leftrightarrow \neg\neg\varphi''. \end{aligned}$$

The case of $\varphi_1 \rightarrow \varphi_2$: For the second item, assume $\varphi_1 \rightarrow \varphi_2 \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\varphi_1 \in \mathrm{E}_k^+$ and $\varphi_2 \in \mathrm{U}_k^+$. By induction hypothesis, there exist $\rho_1(x_1), \rho_2(x_2) \in \Sigma_{k-1}$ such that $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE} \vdash \neg\varphi_1 \leftrightarrow \neg\neg\forall x_1 \rho_1(x_1)$ and $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \varphi_2 \leftrightarrow \forall x_2 \rho_2(x_2)$. By Lemma 4.5, we have that $\neg\rho_1(x_1) \rightarrow \rho_2(x_2)$ is in E_{k-1}^+. Then, by the item 1 for $k-1$, there exists $\xi(x_1, x_2) \in \Sigma_{k-1}$ such that

$$\mathsf{HA} + \Sigma_{k-1}\text{-DNE} + \mathrm{U}_{k-1}^+\text{-DNS} \vdash \xi(x_1, x_2) \leftrightarrow \left(\neg\rho_1(x_1) \rightarrow \rho_2(x_2)\right).$$

Then, using Lemma 5.2 and (3), we have that $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE}$ proves

$$\begin{aligned} & \varphi_1 \rightarrow \varphi_2 \\ \underset{\text{[I.H.] } (\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} \quad & \varphi_1 \rightarrow \forall x_2 \rho_2(x_2) \\ \underset{\Pi_k\text{-DNE}}{\longleftrightarrow} \quad & \varphi_1 \rightarrow \neg\neg\forall x_2 \rho_2(x_2) \\ \longleftrightarrow \quad & \neg\neg\varphi_1 \rightarrow \neg\neg\forall x_2 \rho_2(x_2) \\ \underset{\text{[I.H.] } \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} \quad & \neg\forall x_1 \rho_1(x_1) \rightarrow \neg\neg\forall x_2 \rho_2(x_2) \\ \underset{\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} \quad & \neg\neg\exists x_1 \neg\rho_1(x_1) \rightarrow \neg\neg\forall x_2 \rho_2(x_2) \\ \longleftrightarrow \quad & \neg\neg\forall x_1, x_2 \left(\neg\rho_1(x_1) \rightarrow \rho_2(x_2)\right) \\ \underset{\text{[I.H.] } \Sigma_{k-1}\text{-DNE, } \mathrm{U}_{k-1}^+\text{-DNS}}{\longleftrightarrow} \quad & \neg\neg\forall x_1, x_2\, \xi(x_1, x_2) \\ \underset{\Pi_k\text{-DNE}}{\longleftrightarrow} \quad & \forall x_1, x_2\, \xi(x_1, x_2) \in \Pi_k. \end{aligned}$$

For the first and third items, assume $\varphi_1 \rightarrow \varphi_2 \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1 \in \mathrm{U}_k^+$ and $\varphi_2 \in \mathrm{E}_k^+$. By induction hypothesis, there exists $\rho_2(x_2) \in \Pi_{k-1}$ such that $\mathsf{HA} + \Sigma_k\text{-DNE} + \mathrm{U}_k^+\text{-DNS} \vdash \varphi_2 \leftrightarrow \exists x_2 \rho_2(x_2)$. In addition, by Lemma 5.1, there exists $\rho_1(x_1) \in \Sigma_{k-1}$ such that $\mathsf{HA} + \mathrm{U}_k^+\text{-DNS} \vdash \neg\neg\varphi_1 \leftrightarrow \neg\neg\forall x_1 \rho_1(x_1)$. By Lemma 4.5, we have $\neg\rho_2(x_2) \rightarrow \neg\rho_1(x_1)$ is in U_{k-1}^+. Then, by the item 2 for $k-1$, there exists

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$\xi(x_1, x_2) \in \Pi_{k-1}$ such that

$$\mathsf{HA} + (\Pi_{k-1} \vee \Pi_{k-1})\text{-DNE} \vdash \xi(x_1, x_2) \leftrightarrow \big(\neg\rho_2(x_2) \to \neg\rho_1(x_1)\big).$$

Then, using Lemma 5.2, we have that $\mathsf{HA} + \Sigma_k\text{-DNE} + \mathrm{U}_k^+\text{-DNS}$ proves

$$\begin{array}{cl}
 & \varphi_1 \to \varphi_2 \\
\underset{\text{[I.H.] } \Sigma_k\text{-DNE, } \mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} & \varphi_1 \to \exists x_2 \rho_2(x_2) \\
\underset{\Sigma_k\text{-DNE}}{\longleftrightarrow} & \varphi_1 \to \neg\neg\exists x_2 \rho_2(x_2) \\
\longleftrightarrow & \neg\neg\varphi_1 \to \neg\neg\exists x_2 \rho_2(x_2) \\
\underset{\mathrm{U}_k^+\text{-DNS}}{\longleftrightarrow} & \neg\neg\forall x_1 \rho_1(x_1) \to \neg\neg\exists x_2 \rho_2(x_2) \\
\longleftrightarrow & \neg\neg\exists x_2 \big(\forall x_1 \rho_1(x_1) \to \rho_2(x_2)\big) \\
\underset{\Pi_{k-1}\text{-DNE}}{\longleftrightarrow} & \neg\neg\exists x_2 \big(\neg\rho_2(x_2) \to \neg\forall x_1 \rho_1(x_1)\big) \\
\underset{\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} & \neg\neg\exists x_2 \big(\neg\rho_2(x_2) \to \neg\neg\exists x_1 \neg\rho_1(x_1)\big) \\
\longleftrightarrow & \neg\neg\exists x_2 \neg\neg\exists x_1 \big(\neg\rho_2(x_2) \to \neg\rho_1(x_1)\big) \\
\underset{\text{[I.H.] } (\Pi_{k-1} \vee \Pi_{k-1})\text{-DNE}}{\longleftrightarrow} & \neg\neg\exists x_1, x_2\, \xi(x_1, x_2) \\
\underset{\Sigma_k\text{-DNE}}{\longleftrightarrow} & \exists x_1, x_2\, \xi(x_1, x_2) \in \Sigma_k.
\end{array}$$

Thus we are done for the first item. For the third item, by induction hypothesis, there exist $\varphi_1', \varphi_2' \in \Pi_k$ such that $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \varphi_1 \leftrightarrow \varphi_1'$ and $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE} \vdash \neg\varphi_2 \leftrightarrow \neg\neg\varphi_2'$. On the other hand, by Lemma 4.3, there exists $\xi' \in \Pi_k$ such that $\mathsf{HA} \vdash \varphi_1' \wedge \varphi_2' \leftrightarrow \xi'$. Then $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}$ proves

$$\neg(\varphi_1 \to \varphi_2) \leftrightarrow (\neg\neg\varphi_1 \wedge \neg\varphi_2) \underset{\text{[I.H.] } \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} \neg\neg\varphi_1' \wedge \neg\neg\varphi_2'$$
$$\leftrightarrow \neg\neg(\varphi_1' \wedge \varphi_2') \leftrightarrow \neg\neg\xi'.$$

The case of $\forall x \varphi_1(x)$: For the second item, assume $\forall x \varphi_1(x) \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\varphi_1(x) \in \mathrm{U}_k^+$. By induction hypothesis, there exists $\varphi_1'(x) \in \Pi_k$ such that $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \varphi_1(x) \leftrightarrow \varphi_1'(x)$. Then $\forall x \varphi_1(x)$ is equivalent to $\forall x \varphi_1'(x) \in \Pi_k$ over $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE}$.

For the first and third items, assume $\forall x \varphi_1(x) \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\forall x \varphi_1(x) \in \mathrm{U}_{k-1}^+$. Then, by the item 2 for $k-1$, there exists $\xi \in \Pi_{k-1} \subseteq \Sigma_k$ (see Remark 2.5) such that

$$\mathsf{HA} + (\Pi_{k-1} \vee \Pi_{k-1})\text{-DNE} \vdash \forall x \varphi_1(x) \leftrightarrow \xi. \tag{4}$$

By Lemma 5.2, we are done for the first item. For the third item, by Lemma 4.8, there exists $\xi' \in \Sigma_{k-1} \subseteq \Pi_k$ (see Remark 2.5) such that $\mathsf{HA} + \Sigma_{k-1}\text{-DNE} \vdash \neg\xi \leftrightarrow \xi'$. By Corollary 4.2, we have $\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-DNE} \vdash \neg\xi \leftrightarrow \neg\neg\xi'$. In addition,

$$\mathsf{HA} + \neg\neg(\Pi_{k-1} \vee \Pi_{k-1})\text{-DNE} \vdash \neg\neg\forall x \varphi_1(x) \leftrightarrow \neg\neg\xi$$

follows from (4). Then, by Lemma 5.2, we have that $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}$ proves

$$\neg\forall x \varphi_1(x) \underset{\text{[I.H.] } \neg\neg(\Pi_{k-1} \vee \Pi_{k-1})\text{-DNE}}{\longleftrightarrow} \neg\xi \underset{\neg\neg\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} \neg\neg\xi'.$$

Thus we have shown the third item.

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The case of $\exists x\varphi_1(x)$: For the second item, assume $\exists x\varphi_1(x) \in \mathrm{U}_k^+$. By Lemma 4.5, we have $\exists x\varphi_1(x) \in \mathrm{E}_{k-1}^+$. Then, by the item 1 for $k-1$, there exists $\xi \in \Sigma_{k-1} \subseteq \Pi_k$ (see Remark 2.5) such that $\mathsf{HA} + \Sigma_{k-1}\text{-DNE} + \mathrm{U}_{k-1}^+\text{-DNS} \vdash \exists x\varphi_1(x) \leftrightarrow \xi$. By Lemma 5.2 and (3), we have $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \exists x\varphi_1(x) \leftrightarrow \xi$.

For the first and third items, assume $\exists x\varphi_1(x) \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1(x) \in \mathrm{E}_k^+$. By induction hypothesis, there exist $\varphi_1'(x) \in \Sigma_k$ and $\varphi_1''(x) \in \Pi_k$ such that $\mathsf{HA} + \Sigma_k\text{-DNE} + \mathrm{U}_k^+\text{-DNS} \vdash \varphi_1(x) \leftrightarrow \varphi_1'(x)$ and $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE} \vdash \neg\varphi_1(x) \leftrightarrow \neg\neg\varphi_1''(x)$. Then $\exists x\varphi_1(x)$ is equivalent to $\exists x\varphi_1'(x) \in \Sigma_k$ over $\mathsf{HA} + \Sigma_k\text{-DNE} + \mathrm{U}_k^+\text{-DNS}$. Thus we are done for the first item. For the third item, since $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}$ proves

$$\neg\exists x\varphi_1(x) \leftrightarrow \forall x\neg\varphi_1(x) \underset{\text{[I.H.] } \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}}{\longleftrightarrow} \forall x\neg\neg\varphi_1''(x),$$

we have that $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}$ proves

$$\neg\exists x\varphi_1(x) \leftrightarrow \neg\neg\forall x\neg\neg\varphi_1''(x).$$

On the other hand, the latter is equivalent to $\neg\neg\forall x\varphi_1''(x)$ in the presence of $\neg\neg\Pi_k$-DNE. Thus we have $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE} \vdash \neg\exists x\varphi_1(x) \leftrightarrow \neg\neg\forall x\varphi_1''(x)$. ⊣

Corollary 5.4. $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE} \vdash \mathrm{U}_k\text{-DNS}$.

Proof. Since U_k-DNS is intuitionistically equivalent to $\neg\neg\mathrm{U}_k$-DNS (see Remark 2.8), it suffices to show $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \mathrm{U}_k\text{-DNS}$. By Theorem 5.3.(2), any formula $\varphi \in \mathrm{U}_k$ is equivalent to some $\varphi' \in \Pi_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ over $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE}$. Since $\mathsf{HA} + \Pi_k\text{-DNE} \vdash \Pi_k\text{-DNS}$, we have $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-DNE} \vdash \mathrm{U}_k\text{-DNS}$. ⊣

Remark 5.5. Corollary 5.4 shows that Lemma 5.1 (equivalent to the item 1 in the proof of Lemma 5.1) is a stronger statement of the item 3 in the proof of Theorem 5.3. On the other hand, it is still open whether $\mathsf{HA} + \mathrm{U}_k\text{-DNS}$ is a proper sub-theory of $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}$. In fact, in the proof of Theorem 5.3, Lemma 5.1 is only used for obtaining $\psi_1 \in \Pi_k$ such that $\neg\neg\psi_1 \leftrightarrow \neg\neg\varphi_1$ (where $\varphi_1 \in \mathrm{U}_k^+$) in the verification theory of the first item. Since this argument is available in $\mathsf{HA} + \neg\neg(\Pi_k \vee \Pi_k)\text{-DNE}$ with assuming the second item for φ_1, one can show the alternative assertions of Theorem 5.3 where U_k-DNS is replaced by $\neg\neg(\Pi_k \vee \Pi_k)$-DNE without using Lemma 5.1. Thus, if $\mathsf{HA} + \mathrm{U}_k\text{-DNS}$ proves $\neg\neg(\Pi_k \vee \Pi_k)$-DNE, we can simplify the proof of Theorem 5.3.

Remark 5.6. It follows from Theorem 5.3 and the results in §3 that $\mathsf{HA} + \Sigma_1\text{-DNE}$ does not prove U_1-DNS.

At the end of this section, we study the prenex normal form theorem for formulas which do not contain the disjunction $\vee$. In fact, the proof of Theorem 5.3 suggests that the unusual form $(\Pi_k \vee \Pi_k)$-DNE of the double negation elimination is caused by the argument especially in the case of $\varphi_1 \vee \varphi_2$. On the other hand, if a formula φ does not contain $\vee$, one can intuitionistically derive the original formula φ from a formula in prenex normal form which is classically equivalent to φ (cf. [10, Lemma 6.2.1]). Then the proof of the prenex normal form theorem for those formulas becomes to be fairly simple.

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THEOREM 5.7. *For each natural number k and a formula φ which does not contain $\vee$, the following hold*:

1. *If* $\varphi \in \mathrm{E}_k^+$, *then there exists* $\varphi' \in \Sigma_k$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and*

$$\mathsf{HA} + \Sigma_k\text{-DNE} \vdash \varphi \leftrightarrow \varphi'.$$

2. *If* $\varphi \in \mathrm{U}_k^+$, *then there exists* $\varphi' \in \Pi_k$ *such that* $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ *and*

$$\mathsf{HA} + \Sigma_{k-1}\text{-DNE } (\mathsf{HA} \textit{ if } k = 0) \vdash \varphi \leftrightarrow \varphi'.$$

PROOF. We mimic the proof of Theorem 5.3. Thus we first prepare the following auxiliary assertion (which is in fact a consequence of the item 2):

3. If $\varphi \in \mathrm{E}_k^+$, then there exists $\varphi' \in \Pi_k$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and

$$\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-DNE} \vdash \neg\varphi \leftrightarrow \neg\neg\varphi'.$$

Then we show the items 1–3 by induction on k simultaneously. The base case is trivial. Most of the parts for the induction step is the same as those for Theorem 5.3. The same proof works since each of the logical principles in the items 1 and 2 implies both of them for $k-1$ and the logical principle in the item 3 is the double negation of the logical principle in the item 2 as in Theorem 5.3.

Only the difference with the proof of Theorem 5.3 is in proving the item 1 for $\varphi :\equiv \varphi_1 \to \varphi_2 \in \mathrm{E}_k^+$, where we use Lemma 5.1. Here one can use the item 2 instead of Lemma 5.1. This is because Σ_k-DNE includes Σ_{k-1}-DNE while the verification theory of the item 1 in Theorem 5.3 contains the verification theory of Lemma 5.1. To be absolutely clear, we present the proof of this part: Let $\varphi_1 \to \varphi_2 \in \mathrm{E}_k^+$. By Lemma 4.5, we have $\varphi_1 \in \mathrm{U}_k^+$ and $\varphi_2 \in \mathrm{E}_k^+$. By induction hypothesis, there exists $\rho_1(x_1) \in \Sigma_{k-1}$ and $\rho_2(x_2) \in \Pi_{k-1}$ such that $\mathsf{HA} + \Sigma_{k-1}\text{-DNE} \vdash \varphi_1 \leftrightarrow \forall x_1 \rho_1(x_1)$ and $\mathsf{HA} + \Sigma_k\text{-DNE} \vdash \varphi_2 \leftrightarrow \exists x_2 \rho_2(x_2)$. By Lemma 4.5, we have $\neg\rho_2(x_2) \to \neg\rho_1(x_1)$ is in U_{k-1}^+. Then, by the item 2 for $k-1$, there exists $\xi(x_1, x_2) \in \Pi_{k-1}$ such that

$$\mathsf{HA} + \Sigma_{k-2}\text{-DNE} \vdash \xi(x_1, x_2) \leftrightarrow \big(\neg\rho_2(x_2) \to \neg\rho_1(x_1)\big).$$

Then $\mathsf{HA} + \Sigma_k$-DNE proves

$$\begin{array}{cl}
 & \varphi_1 \to \varphi_2 \\
\underset{[\text{I.H.}]\,\Sigma_k\text{-DNE}}{\longleftrightarrow} & \varphi_1 \to \exists x_2 \rho_2(x_2) \\
\underset{\Sigma_k\text{-DNE}}{\longleftrightarrow} & \varphi_1 \to \neg\neg\exists x_2 \rho_2(x_2) \\
\underset{[\text{I.H.}]\,\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} & \forall x_1 \rho_1(x_1) \to \neg\neg\exists x_2 \rho_2(x_2) \\
\longleftrightarrow & \neg\neg\exists x_2 \big(\forall x_1 \rho_1(x_1) \to \rho_2(x_2)\big) \\
\underset{\Pi_{k-1}\text{-DNE}}{\longleftrightarrow} & \neg\neg\exists x_2 \big(\neg\rho_2(x_2) \to \neg\forall x_1 \rho_1(x_1)\big) \\
\underset{\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} & \neg\neg\exists x_2 \big(\neg\rho_2(x_2) \to \neg\neg\exists x_1 \neg\rho_1(x_1)\big) \\
\longleftrightarrow & \neg\neg\exists x_2 \neg\neg\exists x_1 \big(\neg\rho_2(x_2) \to \neg\rho_1(x_1)\big) \\
\underset{[\text{I.H.}]\,\Sigma_{k-2}\text{-DNE}}{\longleftrightarrow} & \neg\neg\exists x_1, x_2\, \xi(x_1, x_2) \\
\underset{\Sigma_k\text{-DNE}}{\longleftrightarrow} & \exists x_1, x_2\, \xi(x_1, x_2) \in \Sigma_k.
\end{array}$$

⊣

REMARK 5.8. It follows from Theorem 5.3 and Corollary 5.4 that E_k-LEM, U_k-LEM and U_k-DNE are equivalent to Σ_k-LEM, Π_k-LEM and $(\Pi_k \vee \Pi_k)$-DNE respectively over HA (cf. [1, Corollary 2.9]). This may not be the case for E_k-DNE and Σ_k-DNE. On the other hand, Theorem 5.7 implies that $\mathsf{HA} + \Sigma_k$-DNE proves the double negation elimination for all formulas in E_k which do not contain $\vee$.

§6. A conservation theorem. In this section, we generalize a well-known fact that PA is Π_2-conservative over HA in the context of semi-classical arithmetic (see Theorem 6.14). The fact is normally shown by applying the negative translation followed by the Friedman A-translation (see e.g., [6, Chapter 14]). As for the negative translation, there are several equivalent forms (see [8, §1.10.1]). Here we employ Kuroda's negative translation among them.

Definition 6.1 (cf. [6, Definition 10.1]). Let φ be a HA-formula. Then its negative translation φ^N is defined as $\varphi^N :\equiv \neg\neg\varphi_N$, where φ_N is defined inductively as follows:

- $(\varphi_{\mathrm{p}})_N :\equiv \varphi_{\mathrm{p}}$ if φ_{p} is a prime formula;
- $(\varphi_1 \circ \varphi_2)_N :\equiv (\varphi_1)_N \circ (\varphi_2)_N$, where $\circ \in \{\wedge, \vee, \to\}$;
- $(\exists x \varphi_1)_N :\equiv \exists x (\varphi_1)_N$;
- $(\forall x \varphi_1)_N :\equiv \forall x \neg\neg(\varphi_1)_N$.

Remark 6.2. By induction on the structure of formulas, one can show $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi_N) = \mathrm{FV}(\varphi^N)$ for all formulas φ. When it is clear from the context, we suppress the argument on free variables.

Lemma 6.3. *For any* HA-*formula* φ *in prenex normal form,* HA *proves* $\varphi \to \varphi_N$.

Proof. By induction on the structure of formulas in prenex normal form. ⊣

Proposition 6.4. *For any* HA-*formula* φ, *if* $\mathsf{PA} \vdash \varphi$, *then* $\mathsf{HA} \vdash \varphi^N$.

Proof. By induction on the length of the derivations (see the proof of [6, Proposition 10.3]). ⊣

Lemma 6.5. *Let k be a natural number.*

1. *For any* HA-*formula* $\varphi \in \Sigma_k$, $\mathsf{HA} + \Sigma_k$-DNE *proves* $\varphi^N \leftrightarrow \varphi$.
2. *For any* HA-*formula* $\varphi \in \Pi_k$, $\mathsf{HA} + \Sigma_{k-1}$-DNE (HA *if* $k = 0$) *proves* $\varphi^N \leftrightarrow \varphi$.

Proof. By simultaneous induction on k. The base case is trivial. For the induction step, assume the items 1 and 2 for k to show those for $k+1$. For the first item, let $\exists x \varphi_1 \in \Sigma_{k+1}$ where $\varphi_1 \in \Pi_k$. We have that $\mathsf{HA} + \Sigma_{k+1}$-DNE proves

$$(\exists x \varphi_1)^N \equiv \neg\neg\exists x (\varphi_1)_N \leftrightarrow \neg\neg\exists x \neg\neg(\varphi_1)_N \underset{\text{[I.H.] } \Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} \neg\neg\exists x \varphi_1 \underset{\Sigma_{k+1}\text{-DNE}}{\longleftrightarrow} \exists x \varphi_1.$$

For the second item, let $\forall x \varphi_1 \in \Pi_{k+1}$ where $\varphi_1 \in \Sigma_k$. Since Π_{k+1}-DNE is derived from Σ_k-DNE (see Lemma 5.2.(4)), we have that $\mathsf{HA} + \Sigma_k$-DNE proves

$$(\forall x \varphi_1)^N \equiv \neg\neg\forall x \neg\neg(\varphi_1)_N \underset{\text{[I.H.] } \Sigma_k\text{-DNE}}{\longleftrightarrow} \neg\neg\forall x \varphi_1 \underset{\Pi_{k+1}\text{-DNE}}{\longleftrightarrow} \forall x \varphi_1.$$ ⊣

Let HA^* denote HA in the extended language where a predicate symbol $*$ of arity 0, which behaves as a "place holder", is added. In particular, HA^* has $\perp \to *$ as an axiom. To make our arguments absolutely clear, we prefer to add the distinguished new predicate $*$ rather than discussing about A-translation inside the original language as done in [2, 6].

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DEFINITION 6.6 (A-translation [2]). For a HA-formula φ, we define φ^* as a formula obtained from φ by replacing all the prime formulas φ_{p} in φ with $\varphi_{\mathrm{p}} \vee *$. Officially, φ^* is defined inductively as in Definition 6.1. In particular, $\perp^* :\equiv (\perp \vee *)$, which is equivalent to $*$ over HA^*. In what follows, $\neg_* \varphi$ denotes $\varphi \to *$.

REMARK 6.7. By induction on the structure of formulas, one can show $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi^*)$ for all HA-formulas φ.

PROPOSITION 6.8 (cf. [2, Lemma 2]). *For any* HA-*formula* φ, *if* $\mathsf{HA} \vdash \varphi$, *then* $\mathsf{HA}^* \vdash \varphi^*$.

PROOF. By induction on the length of the derivations. ⊣

REMARK 6.9. An analogous assertion of Proposition 6.8 holds for $\mathsf{HA} + \Sigma_1$-LEM and $\mathsf{HA}^* + \Sigma_1$-LEM instead of HA and HA^* respectively (see [7, Lemma 3.1]).

The following substitution result is important in the application of the A-translation:

LEMMA 6.10 (cf. [10, Theorem 6.2.4]). *Let X be a set of* HA-*sentences and φ be a* HA^*-*formula. If* $\mathsf{HA}^* + X \vdash \varphi$, *then* $\mathsf{HA} + X \vdash \varphi[\psi/*]$ *for any* HA-*formula ψ such that the free variables of ψ are not bounded in φ, where $\varphi[\psi/*]$ is the* HA-*formula obtained from φ by replacing all the occurrences of $*$ in φ with ψ.*

PROOF. Fix a set X of HA-sentences. By induction on k, one can show straightforwardly that for any k and any HA^*-formula φ, if $\mathsf{HA}^* + X \vdash \varphi$ with the proof of length k, then $\mathsf{HA} + X \vdash \varphi[\psi/*]$ for any HA-formula ψ such that the free variables of ψ is not bounded in φ. The variable condition is used to verify the case of the axioms and rules for quantifiers. ⊣

The following lemma is a key for our generalized conservation result.

LEMMA 6.11. *Let k be a natural number.*

1. *For any* HA-*formula* $\varphi \in \Sigma_k$, $\mathsf{HA}^* + \Sigma_{k-1}$-LEM ($\mathsf{HA}^*$ *if* $k = 0$) *proves* $(\varphi_N)^* \leftrightarrow \varphi_N \vee *$.
2. *For any* HA-*formula* $\varphi \in \Pi_k$, $\mathsf{HA}^* + \Sigma_k$-LEM *proves* $(\varphi_N)^* \leftrightarrow \varphi_N \vee *$.

PROOF. We show the items 1 and 2 simultaneously by induction on k.

The base case: Since every quantifier-free formula φ_{qf} such that $\mathrm{FV}(\varphi_{\mathrm{qf}}) = \{\overline{x}\}$ is equivalent to a prime formula $t(\overline{x}) = 0$ for some closed term t (see e.g., [6, Proposition 3.8]), by Proposition 6.8, it suffices to show the assertions only for prime formulas. Since $((\varphi_{\mathrm{p}})_N)^* \equiv {\varphi_{\mathrm{p}}}^* \equiv \varphi_{\mathrm{p}} \vee * \equiv (\varphi_{\mathrm{p}})_N \vee *$, we are done.

The induction step: Assume that the items 1 and 2 hold for k. We first show the item 1 for $k+1$. Let $\varphi_1 \in \Pi_k$. Since

$$((\exists x \varphi_1)_N)^* \equiv (\exists x\, (\varphi_1)_N)^* \equiv \exists x\, ((\varphi_1)_N)^*,$$

by induction hypothesis, we have

$$\mathsf{HA}^* + \Sigma_k\text{-LEM} \vdash ((\exists x \varphi_1)_N)^* \leftrightarrow \exists x\, ((\varphi_1)_N \vee *).$$

Since HA^* proves $\exists x\, ((\varphi_1)_N \vee *) \leftrightarrow (\exists x\, (\varphi_1)_N \vee *) \equiv ((\exists x \varphi_1)_N \vee *)$, we have

$$\mathsf{HA}^* + \Sigma_k\text{-LEM} \vdash ((\exists x \varphi_1)_N)^* \leftrightarrow ((\exists x \varphi_1)_N \vee *).$$

Thus we have shown the item 1 for $k+1$.

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Next, we show the item 2 for $k+1$. Let $\varphi_2 \in \Sigma_k$. We shall show that $\mathsf{HA}^* + \Sigma_k$-DNE (and hence, $\mathsf{HA}^* + \Sigma_{k+1}$-LEM) proves $(\forall x \varphi_2)_N \vee * \to \left((\forall x \varphi_2)_N\right)^*$. By Lemma 6.5.(1), we have

$$\mathsf{HA} + \Sigma_k\text{-DNE} \vdash \varphi_2 \leftrightarrow (\varphi_2)^N \equiv \neg\neg(\varphi_2)_N. \tag{5}$$

Then we have that $\mathsf{HA}^* + \Sigma_k$-DNE proves

$$(\forall x \varphi_2)_N \vee * \equiv \left(\forall x \neg\neg(\varphi_2)_N \vee *\right) \leftrightarrow \forall x \varphi_2 \vee *.$$

By Lemma 6.3, HA proves $\varphi_2 \to (\varphi_2)_N$. Then, using induction hypothesis and the fact that Σ_k-DNE derives Σ_{k-1}-LEM, we have that $\mathsf{HA}^* + \Sigma_k$-DNE proves

$$\begin{array}{rcl}
(\forall x \varphi_2)_N \vee * & \underset{\Sigma_k\text{-DNE}}{\longleftrightarrow} & \forall x \varphi_2 \vee * \\
& \longrightarrow & \forall x (\varphi_2)_N \vee * \\
& \longrightarrow & \forall x \left((\varphi_2)_N \vee *\right) \\
& \underset{[\text{I.H.}]\ \Sigma_{k-1}\text{-LEM}}{\longleftrightarrow} & \forall x \left((\varphi_2)_N\right)^* \\
& \longrightarrow & \forall x \left(\left(\left((\varphi_2)_N\right)^* \to *\right) \to *\right) \\
& \longleftrightarrow & \left((\forall x \varphi_2)_N\right)^*.
\end{array}$$

In the following, we show the converse direction:

$$\mathsf{HA}^* + \Sigma_{k+1}\text{-LEM} \vdash \left((\forall x \varphi_2)_N\right)^* \to (\forall x \varphi_2)_N \vee *. \tag{6}$$

Reason in $\mathsf{HA}^* + \Sigma_{k+1}$-LEM. Suppose $\left((\forall x \varphi_2)_N\right)^*$, equivalently,

$$\forall x \left(\left(\left((\varphi_2)_N\right)^* \to *\right) \to *\right). \tag{7}$$

By induction hypothesis, (7) is equivalent to $\forall x \left(\left((\varphi_2)_N \vee * \to *\right) \to *\right)$, which is intuitionistically equivalent to

$$\forall x \left(\left((\varphi_2)_N \to *\right) \to *\right).$$

Then we have

$$\exists x \neg(\varphi_2)_N \to *. \tag{8}$$

By Lemma 4.8.(2), there exists $\psi_2 \in \Pi_k$ such that $\mathrm{FV}(\varphi_2) = \mathrm{FV}(\psi_2)$ and $\neg\varphi_2$ is equivalent to ψ_2. Since $\exists x \psi_2 \in \Sigma_{k+1}$, by Σ_{k+1}-LEM, we have $\exists x \psi_2 \vee \neg\exists x \psi_2$, and hence,

$$\exists x \neg\varphi_2 \vee \forall x \neg\neg\varphi_2.$$

Then, by (5), we obtain

$$\exists x \neg(\varphi_2)_N \vee \forall x \neg\neg(\varphi_2)_N.$$

In the former case, we have $*$ by (8). In the latter case, we have $(\forall x \varphi_2)_N$. Thus we have shown (6). ⊣

Lemma 6.12. *Let φ be a HA^*-formula.*

1. $\mathsf{HA}^* \vdash \varphi \to \neg_*\neg_*\varphi$.
2. $\mathsf{HA}^* \vdash \forall x \neg_*\varphi \leftrightarrow \neg_*\exists x \varphi$.

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3. $\mathsf{HA}^* \vdash \neg_*\neg_*\neg_*\varphi \to \neg_*\varphi$.
4. $\mathsf{HA}^* \vdash \exists x\neg_*\neg_*\varphi \to \neg_*\neg_*\exists x\varphi$.

PROOF. (1)–(3) are immediate from the definition of $\neg_*$ (see Definition 6.6). (4) follows from (1)–(3). ⊣

LEMMA 6.13. *For any* HA-*formula* φ *in prenex normal form*, $\mathsf{HA}^* \vdash \varphi \to \left(\varphi^N\right)^*$.

PROOF. Since there exists a closed term t such that $\mathsf{HA} \vdash \varphi_{\mathrm{qf}}(x_1, \dots, x_k) \leftrightarrow t(x_1, \dots, x_k) = 0$ for each quantifier-free formula φ_{qf} such that $\mathrm{FV}\left(\varphi_{\mathrm{qf}}\right) = \{x_1, \dots, x_k\}$ (see e.g., [6, Proposition 3.8]), by Proposition 6.4 and Proposition 6.8, one can assume that formulas in prenex normal form consist of the formulas of form $Q_1 x_1 \dots Q_k x_k\, \varphi_{\mathrm{p}}$ where Q_is are quantifiers and φ_{p} is prime. We show our assertion by induction on the structure of formulas of this form.

For a prime formula φ_{p}, it is trivial to see that HA^* proves

$$\varphi_{\mathrm{p}} \to \varphi_{\mathrm{p}} \vee * \to \neg_*\neg_*\left(\varphi_{\mathrm{p}} \vee *\right) \leftrightarrow \left(\left(\varphi_{\mathrm{p}}\right)^N\right)^*.$$

Assume the assertion for φ. Then, using Lemma 6.12, HA^* proves

$$\begin{aligned}\exists x\varphi \underset{[\mathrm{I.H.}]}{\longrightarrow} \exists x\left(\varphi^N\right)^* \equiv \exists x\left(\neg\neg\varphi_N\right)^* \leftrightarrow \exists x\neg_*\neg_*\left(\varphi_N\right)^* \to \neg_*\neg_*\exists x\left(\varphi_N\right)^* \\ \leftrightarrow \left(\left(\exists x\varphi\right)^N\right)^*\end{aligned}$$

and

$$\begin{aligned}\forall x\varphi \underset{[\mathrm{I.H.}]}{\longrightarrow} \forall x\left(\varphi^N\right)^* \equiv \forall x\left(\neg\neg\varphi_N\right)^* \leftrightarrow \forall x\neg_*\neg_*\left(\varphi_N\right)^* \to \neg_*\neg_*\forall x\neg_*\neg_*\left(\varphi_N\right)^* \\ \leftrightarrow \left(\left(\forall x\varphi\right)^N\right)^*.\end{aligned}$$ ⊣

THEOREM 6.14. *Let k be a natural number. For any $\varphi \in \Pi_{k+2}$ and any ψ in prenex normal form, if* $\mathsf{PA} \vdash \psi \to \varphi$, *then* $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \varphi$.

PROOF. Let $\varphi :\equiv \forall x\exists y\varphi_1$ where $\varphi_1 \in \Pi_k$. Since one can freely replace the bound variables, assume that the free variables of $\exists y\varphi_1$ are not bounded in ψ and x does not occur in ψ without loss of generality.

Suppose $\mathsf{PA} \vdash \psi \to \forall x\exists y\varphi_1$. By Proposition 6.4, we have that HA proves $\neg\neg(\psi_N \to \forall x\neg\neg\exists y\left(\varphi_1\right)_N)$, which is intuitionistically equivalent to $\neg\neg\psi_N \to \forall x\neg\neg\exists y\left(\varphi_1\right)_N$, namely, $\psi^N \to \forall x\neg\neg\exists y\left(\varphi_1\right)_N$. Then we have

$$\mathsf{HA} \vdash \psi^N \to \neg\neg\exists y\left(\varphi_1\right)_N.$$

By Proposition 6.8, we have

$$\mathsf{HA}^* \vdash \left(\psi^N\right)^* \to \neg_*\neg_*\exists y\left(\left(\varphi_1\right)_N\right)^*,$$

and hence,

$$\mathsf{HA}^* \vdash \psi \to \neg_*\neg_*\exists y\left(\left(\varphi_1\right)_N\right)^*$$

by Lemma 6.13. Then, by Lemma 6.11.(2), we have that $\mathsf{HA}^* + \Sigma_k\text{-LEM}$ proves

$$\psi \to \neg_*\neg_*\exists y\left(\left(\varphi_1\right)_N \vee *\right),$$

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which is intuitionistically equivalent to

$$\psi \to \neg_*\neg_*\exists y\,(\varphi_1)_N.$$

Since the free variables of $\exists y\varphi_1$ are not bounded in ψ, using Lemma 6.10 with Remark 6.2, we have

$$\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \left((\exists y\,(\varphi_1)_N \to \exists y\varphi_1) \to \exists y\varphi_1\right). \tag{9}$$

On the other hand, by Lemma 6.5.(2) and the fact that Σ_k-LEM derives Σ_k-DNE, we have that $\mathsf{HA} + \Sigma_k$-LEM proves

$$(\varphi_1)_N \to \neg\neg(\varphi_1)_N \equiv (\varphi_1)^N \underset{\Sigma_{k-1}\text{-DNE}}{\longleftrightarrow} \varphi_1.$$

and hence, $\exists y\,(\varphi_1)_N \to \exists y\varphi_1$. Then, by (9), we have $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \exists y\varphi_1$. By our assumption, x does not occur in ψ, and hence, $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \forall x\exists y\varphi_1$ follows. ⊣

Corollary 6.15. $\mathsf{HA} + \Sigma_k$-LEM *is closed under the* Σ_{k+1}*-generalization of Markov's rule*: *If* $\mathsf{HA} + \Sigma_k$-LEM *proves* $\neg\neg\varphi$ *where* $\varphi \in \Sigma_{k+1}$, *then* $\mathsf{HA} + \Sigma_k$-LEM *proves* φ.

Corollary 6.15 is announced in [5, §4.4] without proof and the proof for $k = 1$ with using the soundness of the A-translation for $\mathsf{HA} + \Sigma_1$-LEM (cf. Remark 6.9) can be found in [7, Proposition 3.2]. In fact, by using the latter, Kohlenbach and Safarik essentially show an instance of Theorem 6.14 for $k = 1$ and $\psi \equiv 0 = 0$ in [7, Proposition 3.3].

In this paper, we have shown Theorem 6.14 in order to prove the optimality of our prenex normal form theorems in §5 (see §7). On the other hand, the conservation result on semi-classical arithmetic itself is interesting. This will be studied comprehensively in [4].

§7. Characterizations.

Notation 2. Let T be an extension of HA. Let Γ and Γ' be classes of HA-formulas. Then $\mathrm{PNFT}_T(\Gamma, \Gamma')$ denotes the following statement: for any $\varphi \in \Gamma$, there exists $\varphi' \in \Gamma'$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and $T \vdash \varphi \leftrightarrow \varphi'$.

Under this notation, Theorem 5.3 asserts (modulo Lemma 2.3) that for a semi-classical theory T containing $\mathsf{HA} + (\Pi_k \vee \Pi_k)$-DNE, $\mathrm{PNFT}_T(\mathrm{U}_{k'}, \Pi_{k'})$ holds for all $k' \leq k$ as well as the analogous assertion for E_k and Σ_k. It is natural to ask whether the verification theories are optimal. In this section, among other things (see Table 1), we show that this is exactly the case:

1. For a theory T in-between HA and PA, $T \vdash (\Pi_k \vee \Pi_k)$-DNE if and only if $\mathrm{PNFT}_T(\mathrm{U}_{k'}, \Pi_{k'})$ for all $k' \leq k$. (Theorem 7.3)
2. For a theory T in-between $\mathsf{HA} + \Pi_{k-1}$-LEM (HA if $k = 0$) and PA, $T \vdash \Sigma_k\text{-DNE} + \mathrm{U}_k\text{-DNS}$ if and only if $\mathrm{PNFT}_T(\mathrm{E}_{k'}, \Sigma_{k'})$ for all $k' \leq k$. (Theorem 7.11)

Lemma 7.1. *Let T be a theory in-between* $\mathsf{HA} + \Sigma_{k-2}$-LEM ($\mathsf{HA}$ *if* $k < 2$) *and* PA. *If* $\mathrm{PNFT}_T(\mathrm{U}_k, \Pi_k)$, *then* $T \vdash (\Pi_k \vee \Pi_k)$-DNE.

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P_k	(Γ_k, Δ_k)	Q_k	cf.
$\neg\neg\Sigma_{k-1}$-DNE	$\left(({\mathrm{U}_k}^{\mathrm{dn}})^{\mathrm{df}}, {\Pi_k}^{\mathrm{dn}}\right)$	$\neg\neg\Pi_{k-2}$-LEM	Thm. 7.18
U_k-DNS	$\left({\mathrm{U}_k}^{\mathrm{dn}}, {\Pi_k}^{\mathrm{dn}}\right)$	$\emptyset$	Thm. 7.6
Σ_{k-1}-DNE	$({\mathrm{U}_k}^{\mathrm{df}}, \Pi_k)$	Π_{k-2}-LEM	Thm. 7.16.(2)
Σ_{k-1}-DNE + U_k-DNS	$({\mathrm{U}_k}^{\mathrm{dn}}, \Pi_k)$	Π_{k-2}-LEM	Thm. 7.17
Σ_k-DNE	$({\mathrm{E}_k}^{\mathrm{df}}, \Sigma_k)$	Π_{k-1}-LEM	Thm. 7.16.(1)
Σ_k-DNE + U_k-DNS	(E_k, Σ_k)	Π_{k-1}-LEM	Thm. 7.11
$(\Pi_k \vee \Pi_k)$-DNE	(U_k, Π_k)	$\emptyset$	Thm. 7.3
Σ_k-DNE + $(\Pi_k \vee \Pi_k)$-DNE	(U_k, Π_k) & (E_k, Σ_k)	$\emptyset$	Cor. 7.14

TABLE 1. Characterizations of the prenex normal form theorems

PROOF. Fix an instance of $(\Pi_k \vee \Pi_k)$-DNE

$$\varphi :\equiv \forall x \left(\neg\neg\left(\varphi_1(x) \vee \varphi_2(x)\right) \to \varphi_1(x) \vee \varphi_2(x)\right),$$

where $\varphi_1(x), \varphi_2(x) \in \Pi_k(x)$. Since $\neg\neg\left(\varphi_1(x) \vee \varphi_2(x)\right)$ and $\varphi_1(x) \vee \varphi_2(x)$ are in U_k, by our assumption, there exist $\rho(x)$ and $\rho'(x)$ in $\Pi_k(x)$ such that T proves $\rho(x) \leftrightarrow \varphi_1(x) \vee \varphi_2(x)$ and $\rho'(x) \leftrightarrow \neg\neg\left(\varphi_1(x) \vee \varphi_2(x)\right)$. Since $\mathsf{PA} \vdash \varphi$ and PA is an extension of T, we have $\mathsf{PA} \vdash \rho'(x) \to \rho(x)$. By Theorem 6.14, we have that $\mathsf{HA} + \Sigma_{k-2}$-LEM proves $\rho'(x) \to \rho(x)$, and hence, $\forall x\left(\rho'(x) \to \rho(x)\right)$. Since T is an extension of $\mathsf{HA} + \Sigma_{k-2}$-LEM, we have $T \vdash \forall x\left(\rho'(x) \to \rho(x)\right)$, and hence, $T \vdash \varphi$. ⊣

LEMMA 7.2. *Let T be a theory in-between* HA *and* PA. *If* $\mathrm{PNFT}_T\,(\mathrm{U}_{k'}, \Pi_{k'})$ *for all $k' \leq k$, then $T \vdash \Sigma_{k-1}$-LEM.*

PROOF. By induction on k. The base case is trivial. For the induction step, assume $\mathrm{PNFT}_T\,(\mathrm{U}_{k'}, \Pi_{k'})$ for all $k' \leq k+1$. Then, by induction hypothesis, we have $T \vdash \Sigma_{k-1}$-LEM. Fix an instance of Σ_k-LEM

$$\varphi :\equiv \forall x(\varphi_1(x) \vee \neg\varphi_1(x)),$$

where $\varphi_1(x) \in \Sigma_k(x)$. Since $\varphi \in \mathrm{U}_{k+1}$, by our assumption, there exists a sentence $\varphi' \in \Pi_{k+1}$ such that $T \vdash \varphi \leftrightarrow \varphi'$. Since $\mathsf{PA} \vdash \varphi$, we have $\mathsf{PA} \vdash \varphi'$. Then, by Theorem 6.14, we have $\mathsf{HA} + \Sigma_{k-1}$-LEM $\vdash \varphi'$, and hence, $T \vdash \varphi'$. Thus we have $T \vdash \varphi$. ⊣

THEOREM 7.3. *Let T be a theory in-between* HA *and* PA. *Then* $T \vdash (\Pi_k \vee \Pi_k)$-DNE *if and only if* $\mathrm{PNFT}_T\,(\mathrm{U}_{k'}, \Pi_{k'})$ *for all $k' \leq k$.*

PROOF. The "only if" direction is immediate from Theorem 5.3.(2). We show the converse direction. Assume $\mathrm{PNFT}_T\,(\mathrm{U}_{k'}, \Pi_{k'})$ for all $k' \leq k$. Let $k > 0$ without loss of generality. By Lemma 7.2, $T \vdash \Sigma_{k-1}$-LEM. Then, by Lemma 7.1, we have $T \vdash (\Pi_k \vee \Pi_k)$-DNE. ⊣

Definition 7.4. Let Γ be a class of formulas. Then Γ^{dn} denotes the class of HA-formulas $\neg\neg\varphi$ where $\varphi \in \Gamma$, and Γ^{n} denotes that for $\neg\varphi$ where $\varphi \in \Gamma$.

Lemma 7.5. *Let T be a theory in-between* HA *and* PA. *If* $\mathrm{PNFT}_T\left({\mathrm{U}_{k'}}^{\mathrm{dn}}, {\Pi_{k'}}^{\mathrm{dn}}\right)$ *for all $k' \leq k$, then $T \vdash \neg\neg\Sigma_{k-1}$-LEM.*

Proof. By induction on k. The base case is trivial. For the induction step, assume $\mathrm{PNFT}_T\left({\mathrm{U}_{k'}}^{\mathrm{dn}}, {\Pi_{k'}}^{\mathrm{dn}}\right)$ for all $k' \leq k+1$. Then, by induction hypothesis, T proves $\neg\neg\Sigma_{k-1}$-LEM. Fix an instance of $\neg\neg\Sigma_k$-LEM

$$\varphi :\equiv \neg\neg\forall x(\varphi_1(x) \vee \neg\varphi_1(x)),$$

where $\varphi_1(x) \in \Sigma_k(x)$. Since $\varphi \in {\mathrm{U}_{k+1}}^{\mathrm{dn}}$, by our assumption, there exists a sentence $\varphi' \in \Pi_{k+1}$ such that $T \vdash \varphi \leftrightarrow \neg\neg\varphi'$. Since $\mathsf{PA} \vdash \forall x(\varphi_1(x) \vee \neg\varphi_1(x))$, we have $\mathsf{PA} \vdash \varphi'$. Then, by Theorem 6.14, we have $\mathsf{HA} + \Sigma_{k-1}$-LEM $\vdash \varphi'$, and hence, $\mathsf{HA} + \neg\neg\Sigma_{k-1}$-LEM $\vdash \neg\neg\varphi'$ by Lemma 4.1. Then $T \vdash \varphi$. ⊣

Theorem 7.6. *Let T be a theory in-between* HA *and* PA. *The following are pairwise equivalent*:

1. $\mathrm{PNFT}_T\left({\mathrm{E}_{k'}}^{\mathrm{n}}, {\Pi_{k'}}^{\mathrm{dn}}\right)$ *for all $k' \leq k$;*
2. $\mathrm{PNFT}_T\left({\mathrm{U}_{k'}}^{\mathrm{dn}}, {\Pi_{k'}}^{\mathrm{dn}}\right)$ *for all $k' \leq k$;*
3. $T \vdash \mathrm{U}_k$-DNS.

Proof. The equivalence of (1) and (2) is trivial (cf. the proof of Lemma 5.1). In addition, ($3 \to 2$) is immediate from Lemma 5.1 and Proposition 4.6. In what follows, we show ($2 \to 3$). Assume $\mathrm{PNFT}_T\left({\mathrm{U}_{k'}}^{\mathrm{dn}}, {\Pi_{k'}}^{\mathrm{dn}}\right)$ for all $k' \leq k$. Since U_k-DNS is intuitionistically equivalent to $\neg\neg\mathrm{U}_k$-DNS (See Remark 2.8), it suffices to show $T \vdash \neg\neg\mathrm{U}_k$-DNS. Let $k > 0$ without loss of generality. Fix an instance of $\neg\neg\mathrm{U}_k$-DNS

$$\varphi :\equiv \neg\neg\forall x\left(\forall y\neg\neg\varphi_1(x,y) \to \neg\neg\forall y\varphi_1(x,y)\right),$$

where $\varphi_1(x,y) \in \mathrm{U}_k(x,y)$. Since $i(s) \equiv -$ for all alternation paths s of $\forall y\neg\neg\varphi_1(x,y)$ and $\forall y\varphi_1(x,y)$, it is straightforward to show that $\forall y\neg\neg\varphi_1(x,y)$ and $\forall y\varphi_1(x,y)$ are in $\mathrm{U}_k(x)$. Then, by $\mathrm{PNFT}_T\left({\mathrm{U}_k}^{\mathrm{dn}}, {\Pi_k}^{\mathrm{dn}}\right)$, there exist $\rho(x), \rho'(x) \in \Pi_k(x)$ such that T proves $\neg\neg\rho(x) \leftrightarrow \neg\neg\forall y\varphi_1(x,y)$ and $\neg\neg\rho'(x) \leftrightarrow \neg\neg\forall y\neg\neg\varphi_1(x,y)$. Since PA is an extension of T and $\mathsf{PA} \vdash \varphi$, we have $\mathsf{PA} \vdash \rho'(x) \to \rho(x)$. Then, by Theorem 6.14, we have that $\mathsf{HA} + \Sigma_{k-2}$-LEM (HA if $k < 2$) proves $\rho'(x) \to \rho(x)$, and hence, $\forall x\left(\neg\neg\rho'(x) \to \neg\neg\rho(x)\right)$. By Lemma 4.1, we have

$$\mathsf{HA} + \neg\neg\Sigma_{k-2}\text{-LEM} \vdash \neg\neg\forall x\left(\neg\neg\rho'(x) \to \neg\neg\rho(x)\right).$$

On the other hand, by Lemma 7.5 and our assumption, we have $T \vdash \neg\neg\Sigma_{k-1}$-LEM. Then we have

$$T \vdash \neg\neg\forall x\left(\neg\neg\forall y\neg\neg\varphi_1(x,y) \to \neg\neg\forall y\varphi_1(x,y)\right),$$

and hence, $T \vdash \varphi$. ⊣

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Remark 7.7. Theorem 7.6 shows that the verification theory for Lemma 5.1 is optimal.

Definition 7.8. Let Γ be a class of HA-formulas. Γ^{df} denotes the class of formulas in Γ which do not contain $\vee$.

Lemma 7.9. *Let T be a theory in-between* $\mathsf{HA} + \Pi_{k-1}$-LEM (HA *if* $k = 0$) *and* PA. *If* $\mathrm{PNFT}_T\left(\mathrm{E}_{k'}{}^{\mathrm{df}}, \Sigma_{k'}\right)$ *for all* $k' \leq k$, *then* $T \vdash \Sigma_k$-DNE.

Proof. By induction on k. The base case is trivial. For the induction step, assume the assertion for k and let T be a theory in-between $\mathsf{HA} + \Pi_k$-LEM and PA. Assume also that $\mathrm{PNFT}_T\left(\mathrm{E}_{k'}{}^{\mathrm{df}}, \Sigma_{k'}\right)$ holds for all $k' \leq k+1$. Then, by induction hypothesis, T proves Σ_k-DNE. Since T contains $\mathsf{HA} + \Pi_k$-LEM, we have $T \vdash \Sigma_k$-LEM by [1, Theorem 3.1(ii)]. Fix an instance of Σ_{k+1}-DNE

$$\varphi :\equiv \forall x(\neg\neg\varphi_1(x) \to \varphi_1(x)),$$

where $\varphi_1(x) \in \Sigma_{k+1}(x)$. Without loss of generality, one can assume that $\varphi_1(x)$ does not contain $\vee$ (cf. [6, Proposition 3.8]). Since $\neg\neg\varphi_1(x) \in \mathrm{E}_{k+1}{}^{\mathrm{df}}$, By $\mathrm{PNFT}_T\left(\mathrm{E}_{k+1}{}^{\mathrm{df}}, \Sigma_{k+1}\right)$, there exists $\varphi_1'(x) \in \Sigma_{k+1}(x)$ such that $T \vdash \neg\neg\varphi_1(x) \leftrightarrow \varphi_1'(x)$. Since PA is an extension of T and $\mathsf{PA} \vdash \varphi$, we have $\mathsf{PA} \vdash \varphi_1'(x) \to \varphi_1(x)$. Then, by Theorem 6.14, we have that $\mathsf{HA} + \Sigma_k$-LEM proves $\varphi_1'(x) \to \varphi_1(x)$, and hence, $\forall x\left(\varphi_1'(x) \to \varphi_1(x)\right)$. Since T is an extension of $\mathsf{HA} + \Sigma_k$-LEM, we have $T \vdash \varphi$. ⊣

Lemma 7.10. *Let T be an extension of* $\mathsf{HA} + \neg\neg\Sigma_{k-1}$-DNE ($\mathsf{HA}$ *if* $k = 0$). *If* $\mathrm{PNFT}_T(\mathrm{E}_{k'}, \Sigma_{k'})$ *for all* $k' \leq k$, *then* $\mathrm{PNFT}_T\left(\mathrm{E}_{k'}{}^{\mathrm{n}}, \Pi_{k'}{}^{\mathrm{dn}}\right)$ *for all* $k' \leq k$.

Proof. Assume $\mathrm{PNFT}_T(\mathrm{E}_{k'}, \Sigma_{k'})$ for all $k' \leq k$. Fix $k' \leq k$ and $\varphi \in \mathrm{E}_{k'}$. By $\mathrm{PNFT}_T(\mathrm{E}_{k'}, \Sigma_{k'})$, there exists $\varphi' \in \Sigma_{k'}$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and $T \vdash \varphi \leftrightarrow \varphi'$. Then

$$T \vdash \neg\varphi \leftrightarrow \neg\varphi'.$$

On the other hand, by Lemma 4.8.(2), there exists $\varphi'' \in \Pi_{k'}$ such that $\mathrm{FV}(\varphi') = \mathrm{FV}(\varphi'')$ and $\mathsf{HA} + \Sigma_{k'-1}$-DNE $\vdash \neg\varphi' \leftrightarrow \varphi''$. Then, by Corollary 4.2, we have

$$\mathsf{HA} + \neg\neg\Sigma_{k'-1}\text{-DNE} \vdash \neg\varphi' \leftrightarrow \neg\neg\varphi''.$$

Then $\mathrm{FV}(\neg\varphi) = \mathrm{FV}(\neg\neg\varphi'')$ and $T \vdash \neg\varphi \leftrightarrow \neg\neg\varphi''$. Thus we have shown $\mathrm{PNFT}_T\left(\mathrm{E}_{k'}{}^{\mathrm{n}}, \Sigma_{k'}{}^{\mathrm{dn}}\right)$. ⊣

Theorem 7.11. *Let T be a theory in-between* $\mathsf{HA} + \Pi_{k-1}$-LEM (HA *if* $k = 0$) *and* PA. *Then* $T \vdash \Sigma_k$-DNE + U_k-DNS *if and only if* $\mathrm{PNFT}_T(\mathrm{E}_{k'}, \Sigma_{k'})$ *for all* $k' \leq k$.

Proof. The "only if" direction is immediate from Theorem 5.3.(1). We show the converse direction. Assume $\mathrm{PNFT}_T(\mathrm{E}_{k'}, \Sigma_{k'})$ for all $k' \leq k$. Let $k > 0$ without loss of generality. By Lemma 7.9, we have $T \vdash \Sigma_k$-DNE. Then, by Lemma 7.10 and Theorem 7.6, we have $T \vdash \mathrm{U}_k$-DNS. ⊣

Remark 7.12. It is still open whether the assumption that T contains Π_{k-1}-LEM can be omitted in Theorem 7.11.

REMARK 7.13. Akama et al. [1] shows that Π_k-LEM does not derive Σ_k-DNE and Σ_k-DNE does not derive $(\Pi_k \vee \Pi_k)$-DNE. Theorem 7.11 reveals that the prenex normal form theorem for E_k and Σ_k does not hold in $\mathsf{HA} + \Pi_k$-LEM, and Theorem 7.3 reveals that the prenex normal form theorem for U_k and Π_k does not hold in $\mathsf{HA} + \Sigma_k$-DNE.

COROLLARY 7.14. *Let T be a theory in-between* HA *and* PA. *Then* $T \vdash \Sigma_k\text{-DNE} + (\Pi_k \vee \Pi_k)\text{-DNE}$ *if and only if* $\mathrm{PNFT}_T\,(\mathrm{E}_{k'}, \Sigma_{k'})$ *and* $\mathrm{PNFT}_T\,(\mathrm{U}_{k'}, \Pi_{k'})$ *for all* $k' \leq k$.

PROOF. Let T be a theory in-between HA and PA. The "only if" direction follows from Theorem 5.3 and Corollary 5.4.

For the converse direction, assume that $\mathrm{PNFT}_T\,(\mathrm{E}_{k'}, \Sigma_{k'})$ and $\mathrm{PNFT}_T\,(\mathrm{U}_{k'}, \Pi_{k'})$ hold for all $k' \leq k$. By Theorems 7.3, we have $T \vdash (\Pi_k \vee \Pi_k)$-DNE. Since Π_{k-1}-LEM is derived from $(\Pi_k \vee \Pi_k)$-DNE (cf. [1, Theorem 3.1(1)]), by Theorem 7.11, we also have $T \vdash \Sigma_k$-DNE. ⊣

In the following, we show the optimality of Theorem 5.7 (see Theorem 7.16).

LEMMA 7.15. *Let T be a theory in-between* $\mathsf{HA} + \Pi_{k-2}$-LEM (HA *if* $k < 2$) *and* PA. *If* $\mathrm{PNFT}_T\left(\left(\mathrm{U}_{k'}{}^{\mathrm{dn}}\right)^{\mathrm{df}}, \Pi_{k'}\right)$ *for all* $k' \leq k$, *then* $T \vdash \Sigma_{k-1}$-DNE.

PROOF. By induction on k. The base case is trivial. For the induction step, assume the assertion for k and let T be a theory in-between $\mathsf{HA} + \Pi_{k-1}$-LEM and PA. Assume also that $\mathrm{PNFT}_T\left(\left(\mathrm{U}_{k'}{}^{\mathrm{dn}}\right)^{\mathrm{df}}, \Pi_{k'}\right)$ holds for all $k' \leq k+1$. Then, by induction hypothesis, T proves Σ_{k-1}-DNE. Since T contains $\mathsf{HA} + \Pi_{k-1}$-LEM, we have $T \vdash \Sigma_{k-1}$-LEM by [1, Theorem 3.1(ii)]. Fix an instance of Σ_k-DNE

$$\varphi :\equiv \forall x(\neg\neg\varphi_1(x) \to \varphi_1(x)),$$

where $\varphi_1(x) \in \Sigma_k(x)$. Without loss of generality, one can assume that $\varphi_1(x)$ does not contain $\vee$ (cf. [6, Proposition 3.8]). From the perspective of Remark 2.5, $\varphi_1(x)$ is in $\Pi_{k+1}(x)$. Then, by $\mathrm{PNFT}_T\left(\left(\mathrm{U}_{k+1}{}^{\mathrm{dn}}\right)^{\mathrm{df}}, \Pi_{k+1}\right)$, there exists $\varphi_1'(x) \in \Pi_k(x)$ such that $T \vdash \neg\neg\varphi_1(x) \leftrightarrow \varphi_1'(x)$. Since PA is an extension of T and $\mathsf{PA} \vdash \varphi$, we have $\mathsf{PA} \vdash \varphi_1'(x) \to \varphi_1(x)$. By Theorem 6.14, we have that $\mathsf{HA} + \Sigma_{k-1}$-LEM proves $\varphi_1'(x) \to \varphi_1(x)$, and hence, $\forall x\left(\varphi_1'(x) \to \varphi_1(x)\right)$. Since T is an extension of $\mathsf{HA} + \Sigma_{k-1}$-LEM, we have $T \vdash \varphi$. ⊣

THEOREM 7.16.

1. *Let T be a theory in-between* $\mathsf{HA} + \Pi_{k-1}$-LEM (HA *if* $k = 0$) *and* PA. *Then* $T \vdash \Sigma_k$-DNE *if and only if* $\mathrm{PNFT}_T(\mathrm{E}_{k'}{}^{\mathrm{df}}, \Sigma_{k'})$ *for all* $k' \leq k$.
2. *Let T be a theory in-between* $\mathsf{HA} + \Pi_{k-2}$-LEM (HA *if* $k < 2$) *and* PA. *Then* $T \vdash \Sigma_{k-1}$-DNE *if and only if* $\mathrm{PNFT}_T(\mathrm{U}_{k'}{}^{\mathrm{df}}, \Pi_{k'})$ *for all* $k' \leq k$.

PROOF. (1): The "only if" direction is by Theorem 5.7.(1). The converse direction is by Lemma 7.9.

(2): The "only if" direction is by Theorem 5.7.(2). Note that any formula in $\left(\mathrm{U}_{k'}{}^{\mathrm{dn}}\right)^{\mathrm{df}}$ is in $\mathrm{U}_{k'}{}^{\mathrm{df}}$. Then the converse direction follows from Lemma 7.15. ⊣

At the end of this section, we characterize some variants of prenex normal form theorems.

THEOREM 7.17. *Let T be a theory in-between* $\mathsf{HA} + \Pi_{k-2}$-LEM (HA *if* $k < 2$) *and* PA. *Then* $T \vdash \Sigma_{k-1}$-DNE $+$ U_k-DNS *if and only if* $\mathrm{PNFT}_T\left(\mathrm{U}_{k'}{}^{\mathrm{dn}}, \Pi_{k'}\right)$ *for all* $k' \leq k$.

PROOF. We first show the "only if" direction. Let $k > 0$ without loss of generality. Assume $T \vdash \Sigma_{k-1}$-DNE $+$ U_k-DNS and fix $k' \leq k$. Since $T \vdash \mathrm{U}_{k'}$-DNS (cf. Proposition 4.6), by Lemma 5.1, for any $\varphi \in \mathrm{U}_{k'}$, there exists $\varphi' \in \Pi_{k'}$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$ and

$$T \vdash \neg\neg\varphi \leftrightarrow \neg\neg\varphi'.$$

Since $T \vdash \Sigma_{k-1}$-DNE, by Lemma 5.2.(4), we have $T \vdash \neg\neg\varphi' \leftrightarrow \varphi'$, and hence, $T \vdash \neg\neg\varphi \leftrightarrow \varphi'$. Thus we have $\mathrm{PNFT}_T\left(\mathrm{U}_{k'}{}^{\mathrm{dn}}, \Pi_{k'}\right)$.

Next, we show the converse direction. Assume that $\mathrm{PNFT}_T\left(\mathrm{U}_{k'}{}^{\mathrm{dn}}, \Pi_{k'}\right)$ holds for all $k' \leq k$. By Lemma 7.15, we have $T \vdash \Sigma_{k-1}$-DNE. Then, by the assumption, we have $\mathrm{PNFT}_T\left(\mathrm{U}_{k'}{}^{\mathrm{dn}}, \Pi_{k'}{}^{\mathrm{dn}}\right)$ for all $k' \leq k$, and hence, $T \vdash \mathrm{U}_k$-DNS by Theorem 7.6. ⊣

THEOREM 7.18. *Let T be a theory in-between* $\mathsf{HA} + \neg\neg\Pi_{k-2}$-LEM ($\mathsf{HA}$ *if* $k < 2$) *and* PA. *The following are pairwise equivalent*:

1. $\mathrm{PNFT}_T\left((\mathrm{E}_{k'}{}^{\mathrm{n}})^{\mathrm{df}}, \Pi_{k'}{}^{\mathrm{dn}}\right)$ *for all* $k' \leq k$;
2. $\mathrm{PNFT}_T\left(\left(\mathrm{U}_{k'}{}^{\mathrm{dn}}\right)^{\mathrm{df}}, \Pi_{k'}{}^{\mathrm{dn}}\right)$ *for all* $k' \leq k$; *and*
3. $T \vdash \neg\neg\Sigma_{k-1}$-DNE.

PROOF. The equivalence of (1) and (2) is trivial (cf. the proof of Lemma 5.1). In addition, (3 → 1) is immediate from the item 3 in the proof of Theorem 5.7. Then it suffices to show (1 → 3). We show this by induction on k. The base case is trivial. For the induction step, assume the assertion for k and let T be a theory in-between $\mathsf{HA} + \neg\neg\Pi_{k-1}$-LEM and PA. Assume also that $\mathrm{PNFT}_T\left((\mathrm{E}_{k'}{}^{\mathrm{n}})^{\mathrm{df}}, \Pi_{k'}{}^{\mathrm{dn}}\right)$ holds for all $k' \leq k+1$. Then, by induction hypothesis, we have $T \vdash \neg\neg\Sigma_{k-1}$-DNE. Since T contains $\mathsf{HA} + \neg\neg\Pi_{k-1}$-LEM, we have $T \vdash \neg\neg\Sigma_{k-1}$-LEM by [1, Theorem 3.1(ii)]. Fix an instance of $\neg\neg\Sigma_k$-DNE

$$\varphi :\equiv \neg\neg\forall x(\neg\neg\varphi_1(x) \to \varphi_1(x)),$$

where $\varphi_1(x) \in \Sigma_k(x)$. Without loss of generality, one can assume that $\varphi_1(x)$ does not contain $\vee$ (cf. [6, Proposition 3.8]). Note $\forall x(\neg\neg\varphi_1(x) \to \varphi_1(x)) \in \mathrm{U}_{k+1}{}^{\mathrm{df}}$, and hence, $\varphi \in (\mathrm{E}_{k+1}{}^{\mathrm{n}})^{\mathrm{df}}$. By $\mathrm{PNFT}_T\left((\mathrm{E}_{k+1}{}^{\mathrm{n}})^{\mathrm{df}}, \Pi_{k+1}{}^{\mathrm{dn}}\right)$, there exists a sentence $\varphi' \in \Pi_{k+1}$ such that $T \vdash \varphi \leftrightarrow \neg\neg\varphi'$. Since PA is an extension of T and $\mathsf{PA} \vdash \varphi$, we have $\mathsf{PA} \vdash \varphi'$. By Theorem 6.14, we have $\mathsf{HA} + \Sigma_{k-1}$-LEM $\vdash \varphi'$. Then, by Lemma 4.1, we have

$$\mathsf{HA} + \neg\neg\Sigma_{k-1}\text{-LEM} \vdash \neg\neg\varphi'.$$

Since T is an extension of $\mathsf{HA} + \neg\neg\Sigma_{k-1}$-LEM, we have $T \vdash \varphi$. ⊣

All of our characterization results are of the following form: For any theory T in-between $\mathsf{HA} + \mathrm{Q}_k$ and PA, $T \vdash \mathrm{P}_k$ if and only if $\mathrm{PNFT}_T\,(\Gamma_{k'}, \Delta_{k'})$ holds for all $k' \leq k$, where $\mathrm{P}_k, \mathrm{Q}_k$ are logical principles and $\Gamma_{k'}, \Delta_{k'}$ are classes of formulas. Based on this representation, our results are summarized in Table 1.

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