ANALYTIC FUNCTIONS OF A PRESPECTRAL OPERATOR

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The purpose of this note is to present a unified treatment of the material contained in Chapter 10 of [**2**] on roots and logarithms of prespectral operators. Our main result gives a sufficient condition for an analytic function of a prespectral operator of class Γ to be prespectral of class Γ. A result in the opposite direction for spectral operators has been obtained by Apostol [**1**]. Terminology and notation in this paper are as in [**2**].

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The main result of the paper follows. Throughout X is a complex Banach space and $X \neq \{0\}$.

THEOREM 1. *Let A be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Let f be a function analytic on a region Ω, such that $\sigma(A) \subseteq f(\Omega)$ and $f'(\lambda) \neq 0$ for all λ in Ω. Then there is an operator T on X such that T is prespectral of class Γ, $\sigma(T) \subseteq \Omega$ and $f(T) = A$.*

Proof. Let $\lambda \in \sigma(A)$. Then there is a point ζ in Ω such that $f(\zeta) = \lambda$. By Theorem 10.34 of [**3**, p.217], there exist open neighbourhoods V_ζ and W_λ such that f is a one-to-one mapping of V_ζ onto W_λ. The set W_λ is open and $\lambda \in W_\lambda$. Hence we can find an open disc D_λ which is properly contained in W_λ and has λ as its centre. Let $\delta(\lambda)$ be the open disc with centre λ and radius half that of D_λ. As λ runs through $\sigma(A)$, the corresponding discs

$$\{\delta(\lambda) : \lambda \in \sigma(A)\}$$

cover $\sigma(A)$. Since $\sigma(A)$ is compact, there is a finite subcovering; that is

$$\sigma(A) \subseteq \bigcup_{r=1}^{n} \delta(\lambda_r).$$

For brevity, let δ_r denote $\delta(\lambda_r)$ and let W_r denote the open neighbourhood corresponding to λ_r. Let g_r be the inverse of f on W_r. Define

$$\tau_1 = \delta_1 \cap \sigma(A),$$

$$\tau_2 = (\delta_2 \setminus \delta_1) \cap \sigma(A),$$

$$\tau_n = \left(\delta_n \setminus \left[\bigcup_{r=1}^{n-1} \delta_r\right]\right) \cap \sigma(A).$$

Observe that $\{\tau_r : r=1,\ldots,n\}$ is a family of pairwise disjoint sets such that $\tau_r \in \Sigma_p$,

$$\tau_r \subseteq \bar{\delta}_r \subseteq W_r \qquad (r=1,\ldots,n),$$

$$\sigma(A) \subseteq \bigcup_{r=1}^{n} \tau_r.$$

Define

$$T = \bigoplus_{r=1}^{n} g_r(A \mid E(\tau_r)X).$$

We know that

$$\sigma(A \mid E(\tau_r)X) \subseteq \bar{\tau}_r \qquad (r=1,\ldots,n).$$

Also, by Theorem 14.2 of [**2**, pp 265–6], $A \mid E(\tau_r)X$ is a prespectral operator with resolution of the identity $E(\cdot) \mid E(\tau_r)X$ of class Γ.

Now, by Theorem 10.34 of [**3**, p. 217], g_r is analytic on W_r, and so it follows from Theorem 5.16 of [**2**, pp 130–1] that $g_r(A \mid E(\tau_r)X)$ is also prespectral of class Γ. Also, the resolution of the identity of class Γ for $g_r(A \mid E(\tau_r)X)$ is $F_r(\cdot)$, where

$$\begin{aligned} F_r(\delta) &= E(g_r^{-1}(\delta)) \mid E(\tau_r)X \\ &= E(g_r^{-1}(\delta) \cap \tau_r) \mid E(\tau_r)X \qquad (\delta \in \Sigma_p,\ r=1,\ldots,n). \end{aligned}$$

Define

$$F(\delta) = \sum_{r=1}^{n} F_r(\delta) \qquad (\delta \in \Sigma_p).$$

We wish to prove that T is a prespectral operator with resolution of the identity $F(\cdot)$ of class Γ and that $f(T) = A$.

Let $\delta_1, \delta_2 \in \Sigma_p$. Observe that

$$\begin{aligned} F(\delta_1 \cap \delta_2) &= \sum_{r=1}^{n} F_r(\delta_1 \cap \delta_2) \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\delta_1 \cap \delta_2) \cap \tau_r) \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\delta_1) \cap \tau_r \cap g_r^{-1}(\delta_2) \cap \tau_r) \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\delta_1) \cap \tau_r) E(g_r^{-1}(\delta_2) \cap \tau_r) \end{aligned}$$

It follows that

$$F(\delta_1 \cap \delta_2) = \sum_{r=1}^{n} F_r(\delta_1) F_r(\delta_2) = F(\delta_1) F(\delta_2) \qquad (\delta_1, \delta_2 \in \Sigma_p),$$

using the fact that $\tau_1, \ldots, \tau_n$ are pairwise disjoint. Also

$$F(\mathbf{C}) = \sum_{r=1}^{n} F_r(\mathbf{C}) = \sum_{r=1}^{n} E(g_r^{-1}(\mathbf{C}) \cap \tau_r)$$
$$= \sum_{r=1}^{n} E(\tau_r) = E(\sigma(A)) = I.$$

Now let $\delta \in \Sigma_p$. Observe that

$$\begin{aligned} F(\mathbf{C} \setminus \delta) &= \sum_{r=1}^{n} F_r(\mathbf{C} \setminus \delta) \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\mathbf{C} \setminus \delta) \cap \tau_r) \\ &= \sum_{r=1}^{n} [E(g_r^{-1}(\mathbf{C}) \cap \tau_r) - E(g_r^{-1}(\delta) \cap \tau_r)] \\ &= \sum_{r=1}^{n} E(g_r^{-1}(\mathbf{C}) \cap \tau_r) - \sum_{r=1}^{n} E(g_r^{-1}(\delta) \cap \tau_r) \\ &= I - \sum_{r=1}^{n} E(g_r^{-1}(\delta) \cap \tau_r) \\ &= I - F(\delta). \end{aligned}$$

Let $\delta_1, \delta_2 \in \Sigma_p$. Then

$$\begin{aligned} F(\delta_1 \cup \delta_2) &= F(\mathbf{C} \setminus (\mathbf{C} \setminus \delta_1) \cap (\mathbf{C} \setminus \delta_2)) \\ &= I - F((\mathbf{C} \setminus \delta_1) \cap (\mathbf{C} \setminus \delta_2)) \\ &= I - F(\mathbf{C} \setminus \delta_1) F(\mathbf{C} \setminus \delta_2) \\ &= I - (I - F(\delta_1))(I - F(\delta_2)) \\ &= F(\delta_1) + F(\delta_2) - F(\delta_1) F(\delta_2). \end{aligned}$$

Also if $\|E(\tau)\| \le M < \infty$, then clearly $\|F(\tau)\| \le nM < \infty$ $(\tau \in \Sigma_p)$.

We deduce that if $\{\delta_m\}$ is a pairwise disjoint sequence of sets in Σ_p, then

$$F\left(\bigcup_{m=1}^{k} \delta_m\right) = \sum_{m=1}^{k} F(\delta_m).$$

Hence if $x \in X$ and $y \in \Gamma$, then we obtain

$$\begin{aligned} \left\langle F\left(\bigcup_{m=1}^{k} \delta_m\right)x, y\right\rangle &= \sum_{m=1}^{k} \langle F(\delta_m)x, y\rangle \\ &= \sum_{m=1}^{k} \sum_{r=1}^{n} \langle F_r(\delta_m)x, y\rangle \\ &= \sum_{m=1}^{k} \sum_{r=1}^{n} \langle E(g_r^{-1}(\delta_m) \cap \tau_r)x, y\rangle \\ &= \sum_{r=1}^{n} \left\langle E\left(g_r^{-1}\left(\bigcup_{m=1}^{k} \delta_m\right) \cap \tau_r\right)x, y\right\rangle. \end{aligned}$$

Let $k \to \infty$. Using the properties of the inverse image and the countable additivity of $\langle E(\cdot)x, y\rangle$ on Σ_p for x in X and y in Γ, we deduce that

$$\text{(a)}\ \lim_{k\to\infty}\left\langle F\left(\bigcup_{m=1}^{k}\delta_m\right)x, y\right\rangle \text{ exists } (x\in X,\ y\in\Gamma)$$

$$\text{(b)}\ \lim_{k\to\infty}\left\langle F\left(\bigcup_{m=1}^{k}\delta_m\right)x, y\right\rangle = \left\langle F\left(\bigcup_{m=1}^{\infty}\delta_m\right)x, y\right\rangle\ (x\in X,\ y\in\Gamma).$$

This completes the proof that $F(\cdot)$ is a spectral measure of class (Σ_p, Γ).

Since g_r is analytic on a neighbourhood of $\bar{\tau}_r$,

$$g_r(A \mid E(\tau_r)X) = \frac{1}{2\pi i}\int_B g_r(\lambda)((\lambda I - A) \mid E(\tau_r)X)^{-1}\,d\lambda,$$

where B is a suitable finite family of contours in $\rho(A \mid E(\tau_r)X)$. Since A is prespectral, it follows that

$$AE(\tau) = E(\tau)A \qquad (\tau \in \Sigma_p)$$

and so $A \mid E(\tau_r)X$ commutes with $E(\cdot) \mid E(\tau_r)X$. We deduce readily from this that $F_r(\cdot)$ commutes with $g_r(A \mid E(\tau_r)X)$ and consequently

$$TF(\tau) = F(\tau)T \qquad (\tau \in \Sigma_p).$$

Also

$$F(\delta)X = \bigoplus_{r=1}^{n} E(g_r^{-1}(\delta) \cap \tau_r)X \qquad (\delta \in \Sigma_p).$$

Each of the subspaces on the right-hand side reduces T and so, by Proposition 1.37 of [**2**, pp 25–6]

$$\begin{aligned}\sigma(T \mid F(\delta)X) &= \bigcup_{r=1}^{n}\sigma(T \mid E(g_r^{-1}(\delta)\cap\tau_r)X \\ &= \bigcup_{r=1}^{n}\sigma((g_r(A \mid E(\tau_r)X) \mid E(g_r^{-1}(\delta)\cap\tau_r)X \\ &\subseteq \bar{\delta} \quad (\delta \in \Sigma_p).\end{aligned}$$

It follows that $F(\cdot)$ is a resolution of the identity for T. Finally

$$\begin{aligned}f(T) &= f\left(\bigoplus_{r=1}^{n} g_r(A \mid E(\tau_r)X)\right) \\ &= \bigoplus_{r=1}^{n}(f\circ g_r)(A \mid E(\tau_r)X) \\ &= \bigoplus_{r=1}^{n} A \mid E(\tau_r)X \\ &= A\end{aligned}$$

and so the proof is complete.

COROLLARY. *Let A, in $L(X)$, be a scalar-type operator of class Γ. Let Ω be a region and let f be a function analytic on Ω such that $\sigma(A) \subseteq f(\Omega)$ and $f'(\lambda) \neq 0$ for all λ in Ω. Then there is a scalar-type operator S of class Γ such that $\sigma(S) \subseteq \Omega$ and $f(S) = A$.*

Proof. In the notation of the proof of Theorem 1, $A \mid E(\tau_r)X$ is a scalar-type operator of class Γ for each $r = 1, \ldots, n$. It follows that $g_r(A \mid E(\tau_r)X)$ is also a scalar-type operator of class Γ and hence that T is a scalar-type operator of class Γ. This completes the proof.

Finally, we state the special case of Theorem 1 for spectral operators combined with a result of Apostol [**1**].

THEOREM 2. *Let A, in $L(X)$, be a spectral operator. Let Ω be a region and let f be a function analytic on Ω such that $\sigma(A) \subseteq f(\Omega)$ and $f'(\lambda) \neq 0$ for all λ in Ω. Then there is a spectral operator T_0 on X such that $\sigma(T_0) \subseteq \Omega$ and $f(T_0) = A$. Moreover, if $T \in L(X)$ and $f(T) = A$, then T is a spectral operator.*

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