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NOTE ON TOTAL CATEGORIES

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It is shown that, for a semi-topological functor $T : A \to X$, the category A is total, that is, the Yoneda embedding of A has a left adjoint, if X is total. In particular, monadic categories over Set (possibly without rank) are total, and full reflective subcategories of total categories are total.

1. Total and compact categories

A category A with small hom-classes is called *total* [6], if the Yoneda-embedding

$$Y_A : A \to \hat{A} = [A^{op}, Set] , \quad A \mapsto A(-, A) ,$$

has a left adjoint. It is known [6] that any full reflective subcategory of a functor category $[D, Set]$ with D being small is total. In particular, monadic categories over Set with rank and their full reflective subcategories are total.

A total category A is *compact* [3], that is, A has small hom-classes and any functor $U : A \to B$ preserving all existing colimits in A has a right adjoint, provided U is admissable [6] (that is, the hom-classes $B(UA, B)$ are small for all $A \in \text{Ob } A$, $B \in \text{Ob } B$). The reverse implication is false: it is proved in [2] that Adámek's [1] non-cocomplete (hence non-total) monadic category is compact.

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However, a cocompact category A (for example, any category satisfying the sufficient conditions of Freyd's Special Adjoint Functor Theorem) is total, provided A contains a generating set of objects. Namely, this last condition implies that Y_A is co-admissible whence, Y_A, preserving trivially all limits, has a left adjoint. Therefore, for categories which contain a generating set and a cogenerating set all notions total, cototal, compact, and cocompact coincide.

2. The general lifting technique

It is proved in [2] for a semi-topological [7] functor $T : A \to X$, that A is compact if X is. In the following we shall prove that a corresponding result holds for total categories. For this we consider a left adjoint $F : X \to A$ of T and a natural equivalence $\varphi : \hat{F} \circ Y_A \to Y_X \circ T$ (with $\hat{F} = [F^{op}, \mathcal{E}n\mathcal{s}]$):

(1)
$$\begin{array}{ccc} A & \xrightarrow{\ T\ } & X \\ {\scriptstyle Y_A}\downarrow & \overset{\varphi}{\Longrightarrow} & \downarrow{\scriptstyle Y_X} \\ \hat{A} & \xrightarrow{\ \hat{F}\ } & \hat{X} \end{array} .$$

Let Y_X have a left adjoint. According to the General Lifting Theorem 2.27 of [5], in order to prove right adjointness of Y_A it suffices to prove that *semi-initial factorizations of T-sources are locally respected* by the above diagram. This means: if the commutative diagram (2),

(2)
$$\begin{array}{ccccc} & & TA & & \\ & {\scriptstyle e}\nearrow & & \searrow{\scriptstyle Tm_i} & \\ X & & \xrightarrow[\ x_i\ ]{} & & TB_i \end{array}$$

has the property that for any $z : TC \to X$ and all $b_i : C \to B_i$ with $x_i z = Tb_i$ there is an $a : C \to A$ with $ez = Ta$ and, therefore, $m_i a = b_i$ ("*diagram* (2) *is T-semi-initial*"), then diagram (3),

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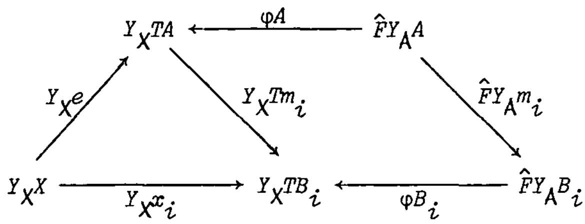
(3)



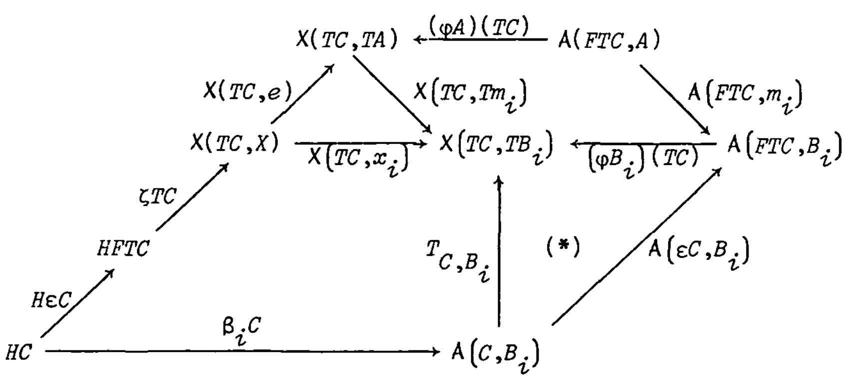
has the following property: for any $\zeta : \hat{F}H \to Y_X X$ ($H \in \mathrm{Ob}\ \hat{A}$) and all $\beta_i : H \to Y_A B_i$ with $(Y_X x_i)\zeta = (\varphi B_i)(\hat{F}\beta_i)$ there is an $\alpha : H \to Y_A A$ with $(Y_X e)\zeta = (\varphi A)(\hat{F}\alpha)$ and $(Y_A m_i)\alpha = \beta_i$ (*"diagram* (3) *is* $\hat{F}$*-semi-initial"*). Usually i ranges over a (proper) index class I. But the definition of semi-topological functors and the proof of 2.27 of [5] show that it does not matter, if I belongs to any higher universe. In the present situation, one takes I to be legitimate with respect to some universe for which $\hat{A}$ is legitimate.

3. The lifting theorem

THEOREM. *Let* $T : A \to X$ *be a semi-topological functor. Then* A *is total, if* X *is total.*

Proof. Let the T-semi-initial diagram (2) be given. In order to prove $\hat{F}$-semi-initiality of diagram (3) let ζ and β_i be as above. For each $C \in \mathrm{Ob}\ A$ one then obtains the commutative diagram (4):

(4)



Here ε denotes the co-unit of the adjoint pair (F, T) such that the triangle (*) commutes for T_{C,B_i} being the respective restriction of T.

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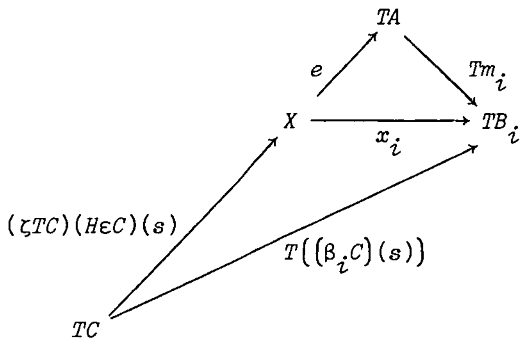
For each $s \in HC$ one now has the commutative diagram (5):

(5)



From the T-semi-initiality of (2) one therefore obtains a morphism $(\alpha C)(s) : C \to A$ with $T\big((\alpha C)(s)\big) = e\big((\zeta TC)(H\varepsilon C)(s)\big)$. It is easily checked that, in this way, a natural transformation $\alpha : H \to Y_{\mathrm{A}}A$ satisfying the needed equations is defined.

COROLLARY 1. *Any monadic category over the category of sets is total.*

COROLLARY 2. *A full reflective subcategory of a total category is total.*

4. Final remark

One is not able to prove a corresponding result as in the theorem for arbitrary monadic (instead of semi-topological) functors as Rattray [4] did in the case of compactness. To see this consider again the category of graphs over which Adámek [1] has constructed his non-total but monadic category: by the theorem, his base category, being semi-topological over sets, is total.

References

[1] Jiří Adámek, "Colimits of algebras revisited", *Bull. Austral. Math. Soc.* 17 (1977), 433-450.

[2] R. Börger, W. Tholen, M.B. Wischnewsky, H. Wolff, "Compactness and hypercompleteness", preprint.

https://doi.org/10.1017/S0004972700005992 Published online by Cambridge University Press

[3] John R. Isbell, "Small subcategories and completeness", *Math. Systems Theory* 2 (1968), 27-50.

[4] B.A. Rattray, "Adjoints to functors from categories of algebras", *Comm. Algebra* 3 (1975), 563-569.

[5] R. Street, W. Tholen, M.B. Wischnewsky, H. Wolff, "Semitopological functors III: lifting of monads and adjoint functors", *J. Pure Appl. Algebra* (to appear).

[6] Ross Street and Robert Walters, "Yoneda structures on 2-categories", *J. Algebra* 50 (1978), 350-379.

[7] Walter Tholen, "Semi-topological functors I", *J. Pure Appl. Algebra* 15 (1979), 53-73.

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